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Research Article

A Note on Circular Distance Energy and Circular Distance Laplacian Energy

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Abstract. The circular distance energy of a simple connected graph G is defined as the sum of the absolute values of its eigen values of the circular distance matrix of G. In this paper, the bounds for circular distance energy is obtained. Also the circular distance energy and the circular distance laplacian energy of certain graphs via circular distance energy are derived.

Keywords. Circular distance matrix; Circular distance energy; Circular distance laplacian energy

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1. Introduction

Let G be a connected graph of order n, with vertex set $V(G) = \{v_1, v_2, v_3, v_n\}$. Let $A = [a_{ij}]_{n\times n}$ be the adjacency matrix of G. The eigen values $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$ of A assumed to be in non increasing order, are the eigen values of G. The Energy $E(G)$ of G is defined to be the sum of the absolute values of its eigen values of G [8, 14, 15].

The distance matrix of a graph G is defined as a square matrix $D = D(G) = [d_{ij}]$; where d_{ij} is the distance between the vertices v_i and v_j in G. The eigen values of $D(G)$ are denoted by $\mu_1, \mu_2, \mu_3, \dots, \mu_n$ and are said to be the D-eigen values of G. The distance energy $E_D = E_D(G)$ of a graph G is defined as the sum of the absolute values of μ_i [10, 11, 19].

The detour distance matrix of a graph G is the $n \times n$ matrix defined by $DD(G) = DD_{ij}$, where DD_{ij} is the longest distance between the vertices v_i and v_j in G. The eigen values $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ are said to be the DD eigen values of G. The detour distance energy $E_{DD} = E_{DD}(G)$ of a graph G is defined as

$$E_{DD} = E_{DD}(G) = \sum_{i=1}^{n} |\gamma_i|.$$

The circular distance matrix of a graph G is defined by

$$CD(G) = [d_{ij}^0],$$

where $d_{ij}^0 = DD(v_i, v_j) + d(v_i, v_j)$. Let $\phi_{CD}(\rho)$ denotes the characteristic polynomial of $CD(G)$. The eigen values of the circular distance matrix $CD(G)$ are denoted by $\rho_1, \rho_2, \rho_3, \dots, \rho_n$ are said to be the CD eigen values of G. Since the circular distance matrix is symmetric, its eigen values are real and it can be ordered as $\rho_1 \ge \rho_2 \ge \rho_3 \dots \ge \rho_n$. The eigen values $\rho_1, \rho_2, \rho_3, \dots, \rho_n$ form the CD spectrum $spec_{CD}(G)$. The circular distance energy $E_{CD} = E_{CD}(G)$ of a graph G is defined as

$$E_{CD} = E_{CD}(G) = \sum_{i=1}^{n} |\rho_i|.$$

The circular distance laplacian matrix of a connected graph G is defined as

$$CDL(G) = \text{diag}(T_r) - CD,$$

where $\text{diag}(T_r)$ denotes the diagonal matrix of the vertex transmissions in G. The eigen values of $CDL(G)$ are $\rho_1^L, \rho_2^L \rho_3^L, \dots, \rho_n^L$ are the circular distance laplacian eigen values of G derived from the circular distance eigen values. The circular distance laplacian eigen values $CDL(G)$ form the CDL spectrum $spec_{CDL}(G)$. The circular distance laplacian energy is defined as the sum of the absolute values of ρ_i^L [16, 20].

In this paper, we give bounds for the circular distance energy. Further the circular distance energy of some graphs and circular distance laplacian energy derived from circular distance energy are computed.

Definition 1.1. The crown graph S_n^0 for an integer $n \ge 2$ is the graph with vertex set $\{u_1, u_2, u_3, \dots, u_n, v_1, v_2, v_3, \dots, v_n\}$ and edge set $\{u_i v_j : 1 \le i, j \le n, i \ne j\}$.

Definition 1.2. The cocktail party graph is denoted by $K_{n\times 2}$, is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set $E = \{u_i u_j, v_i v_j, : i \ne j\} \cup \{u_i v_j, v_i u_j, : 1 \le i < j \le n\}$.

2. Bounded for Circular Distance Energy

Theorem 2.1. *Let G be a connected (n, m) graph and let $\rho_1, \rho_2, \rho_3, \dots, \rho_n$ be its circular distance eigen values. Then* $\sum_{i=1}^{n} \rho_i = 0$ *and* $\sum_{i=1}^{n} (\rho_i)^2 = 2 \sum_{1 \le i < j \le n} (d_{ij}^0)^2$.

Proof. For the Circular Distance Matrix CD,

$$\sum_{i=1}^{n} \rho_i = Trace(CD(G)) = \sum_{i=1}^{n} (d_{ij}^0) = 0.$$

For $i = 1,2,3,\dots,n$, the (i,j) entry of $(CD(G))^2$ is equal to $\sum_{i=1}^{n} (d_{ij}^0)^2$.

Hence

$$\sum_{i=1}^{n} (\rho_i)^2 = trace(CD(G))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} (d_{ij}^0)^2$$

$$\sum_{i=1}^{n} (\rho_i)^2 = 2 \sum_{1\le i<j\le n} (d_{ij}^0)^2.$$

□

Theorem 2.2. *If G is a connected (n,m) graph, then*

$$\sqrt{2 \sum_{1\le i<j\le n} (d_{ij}^0)^2} \le E_{\mathrm{CD}}(G) \le \sqrt{2n \sum_{1\le i<j\le n} (d_{ij}^0)^2}.$$

Proof. Consider the Cauchy-Schwartz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Let us choose $a_i = 1$ and $b_i = |\rho_i|$, we get

$$\left(\sum_{i=1}^{n} |\rho_i|\right)^2 \le n\left(\sum_{i=1}^{n} |\rho_i^2|\right)$$

$$E_{CD}(G)^2 \le 2n \sum_{1\le i<j\le n} (d_{ij}^0)^2$$

Let us the upper bound for $E_{CD}(G)$.

$$E_{CD}(G)^2 = \left(\sum_{i=1}^{n} |\rho_i|\right)^2 \ge \left(\sum_{i=1}^{n} |\rho_i^2|\right)$$

$$= 2 \sum_{1\le i<j\le n} (d_{ij}^0)^2.$$

This is the lower bound for $E_{CD}(G)$.

Theorem 2.3. *If G is a connected (n,m) graph, then $E_{\mathrm{CD}}(G) \ge n\sqrt{n(n-1)}$.*

Proof. Since $d_{ij} \ge n$ for $i \ne j$ and there are $n(n-1)/2$ pairs of vertices in G. From the lower bound of Theorem 2.2,

$$E_{CD}(G) \ge \sqrt{2 \sum_{1\le i<j\le n} (d_{ij}^0)^2}$$

$$\ge \sqrt{\frac{2n^2 \times n(n-1)}{2}}$$

$$\ge n\sqrt{n(n-1)}.$$

□

3. Circular Distance Energy of k_n, $k_{n,n}$ and Some Special Graphs

Theorem 3.1. *If G is a complete graph of order n, then the circular distance energy of G is* $E_{\text{CD}}(G) = 2n(n-1)$.

Proof. In G, the circular distance between two adjacency vertices is n. The circular distance matrix $CD(G) = n[J - I]$, where J is the matrix of order n, whose entries are one.

The characteristic polynomial of $CD(G)$ is

$$\phi_{CD}(\rho) = (\rho - n)^{n-1}(\rho - n(n-1)).$$

Circular distance laplacian spectra is

$$spec_{CD}(G) = \begin{pmatrix} n & n(n-1) \\ n-1 & 1 \end{pmatrix}.$$

Hence $E_{CD}(G) = 2n(n-1)$.

Corollary 3.2. *The circular distance laplacian energy of* $k_{n,n}$ *and* c_n $(n > 4)$ *is same as complete graph.*

Theorem 3.3. *If G is a Crown graph* $n \geq 4$, *then the circular distance energy of G is* $E_{\text{CD}}(G) = 2n^2 - 2n + 4$.

Proof. Let $V(G) = U_i \cup V_j$. In S_n^0, the circular distance of any two vertices in U_i and in V_j is n, $i = j = 1, 2, 3, \ldots, n/2$ and the circular distance of any vertex to itself is 0. The circular distance between the vertices U_i and V_j, V_j and U_i is n, for $i \neq j$ and $n+2$, for $i = j$.

Then the circular distance matrix,

$$CD(G) = \begin{bmatrix} n[J-I] & (n+2)I + n[J-I] \\ (n+2)I + n[J-I] & n[J-I] \end{bmatrix}$$

where J is the matrix of order n, whose entries are one.

The characteristic polynomial of $CD(G)$ is,

$$\phi_{CD}(\rho) = (\rho + (n+2))^{\frac{n}{2}}(\rho + (n-2))^{(\frac{n}{2}-1)}(\rho + (n^2 - n + 2)).$$

Circular distance spectra is

$$spec_{CD}(G) = \begin{pmatrix} -(n+2) & -(n-2) & -(n^2-n+2) \\ \frac{n}{2} & \frac{n}{2}-1 & 1 \end{pmatrix}.$$

Hence $E_{CD}(G) = 2n^2 - 2n + 4$. □

Theorem 3.4. *If G is a Cocktail party graph, then the circular distance energy of G is* $E_{\text{CD}}(G) = 2n^2 - 2n + 2$.

Proof. Let $V(G) = U_i \cup V_j$. In G, the circular distance of any two vertices in U_i and in V_j is n, $i = j = 1, 2, 3, \ldots, n/2$ and the circular distance of any vertex to itself is 0. The circular distance between the vertices U_i and V_j, V_j and U_i is n, for $i \neq j$ and $n+1$, for $i = j$.

Then the circular distance matrix,

$$CD(G) = \begin{bmatrix} n[J-I] & (n+1)I + n[J-I] \\ (n+1)I + n[J-I] & n[J-I] \end{bmatrix}$$

where J is the matrix of order n, whose entries are one.

The characteristic polynomial of $CD(G)$ is

$$\phi_{CD}(\rho) = (\rho + (n+1))^{\frac{n}{2}}(\rho + (n-1))^{(\frac{n}{2}-1)}(\rho + (n^2 - n + 1)).$$

Circular distance spectra is

$$spec_{CD}(G) = \begin{pmatrix} -(n+1) & -(n-1) & -(n^2-n+1) \\ \frac{n}{2} & \frac{n}{2}-1 & 1 \end{pmatrix}.$$

Hence $E_{CD}(G) = 2n^2 - 2n + 2$. □

4. Circular Laplacian Spectra

Theorem 4.1. *For any connected graph G, if ρ_n be its largest circular distance eigen value, then $\rho_n - \rho_n, \rho_n - \rho_{n-1}, \rho_n - \rho_{n-2}, \dots, \rho_n - \rho_1$ are the circular distance laplacian eigen values of G.*

Theorem 4.2. *If G is the complete graph or order n, then the circular distance laplacian energy of G is $E_{\text{CDL}}(G) = n^2(n-1)$.*

Proof. From Theorem 3.1,

$$CD(G) = n[J - I].$$

It follows that $\text{diag}(T_r) = n(n-1)$.

The circular distance laplacian matrix $CDL(G) = n(n-1) - CD(G)$.

The largest circular distance eigen value of G is $n(n-1)$ (by Theorem 3.1).

Hence the circular distance laplacian eigen values are $n(n-1) - n(n-1)$, $n(n-1) + n$, $(n-1)$ times that is 0, n^2, $(n-1)$ times.

Circular distance laplacian spectra is

$$spec_{CD}(G) = \begin{pmatrix} 0 & n^2 \\ 1 & (n-1) \end{pmatrix}.$$

Hence $E_{CD}(G) = n^2(n-1)$. □

Corollary 4.3. *The circular distance laplacian energy of $k_{n,n}$ and c_n $(n > 4)$ is same as complete graph.*

Theorem 4.4. *If G is a Crown graph S_n^0, $n \geq 4$, then the circular distance laplacian energy of G is $E_{\text{CDL}}(G) = n(n^2 - 2n + 2)$.*

Proof. From Theorem 3.3

$$CD(G) = \begin{bmatrix} n[J-I] & (n+2)I + n[J-I] \\ (n+2)I + n[J-I] & n[J-I] \end{bmatrix}.$$

It follows that $\text{diag}(T_r) = (n^2 - n + 2) - CD(G)$.

The largest circular distance eigen value of G is $(n^2 - n + 2)$.

Hence the circular distance laplacian eigen values are $(n^2 - n + 2) - (n^2 - n + 2), (n^2 - n + 2) + (n - 2), (\frac{n}{2} - 1)$ times, $(n^2 - n + 2) + (n + 2), \frac{n}{2}$ times that is 0, n^2, $\frac{n}{2} - 1$ times, $n^2 + 4$, $\frac{n}{2}$ times.

Circular distance laplacian spectra is

$$spec_{CD}(G) = \begin{pmatrix} 0 & n^2 & n^2+4 \\ 1 & (\frac{n}{2}-1) & \frac{n}{2} \end{pmatrix}.$$

Hence $E_{CD}(G) = n(n^2 - n + 2)$. □

Theorem 4.5. *If G is a Cocktail party graph, then the circular distance laplacian energy of G is* $E_{\mathrm{CD}}(G) = n(n^2 - n + 2)$.

Proof. From Theorem 3.4

$$CD(G) = \begin{bmatrix} n[J-I] & (n+1)I + n[J-I] \\ (n+1)I + n[J-I] & n[J-I] \end{bmatrix}.$$

It follows that $\mathrm{diag}(T_r) = (n^2 - n + 2) - CD(G)$.

The largest circular distance eigen value of G is $(n^2 - n + 1)$.

Hence the circular distance laplacian eigen values are $(n^2-n+1)-(n^2-n+1)$, $(n^2-n+1)+(n-1), (\frac{n}{2}-1)$ times, $(n^2-n+1)+(n+1), \frac{n}{2}$ times that is 0, $n^2, \frac{n}{2}-1$ times, n^2+2, $\frac{n}{2}$ times.

Circular distance laplacian spectra is

$$spec_{CD}(G) = \begin{pmatrix} 0 & n^2 & n^2+2 \\ 1 & (\frac{n}{2}-1) & \frac{n}{2} \end{pmatrix}.$$

Hence $E_{CD}(G) = n(n^2 - n + 1)$. □

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Competing Interests

The authorS declare that They have no competing interests.

Authors' Contributions

The authors wrote, read and approved the final manuscript.

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