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Research Article

Exponential Stability Analysis of Difference Equation for Impulsive System

Elizabeth S. and Nirmal Veena S.*

Department of Mathematics, Auxilium College, Gandhi Nagar, Vellore 632006, Tamil Nadu, India
***Corresponding author:** nirmalveena1510@gmail.com

Abstract. In this paper, we study the exponential stability of impulsive difference equations with exponential decay and the uniformity of the stability is obtained by using Lyapunov functions. Theorems on exponential and uniform exponential stability are obtained, which shows that certain impulsive perturbations may make unstable systems exponentially stable.

Keywords. Difference equation; Uniform exponential stability; Lyapunov functions; Impulsive system

MSC. 11-xx

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1. Introduction

Impulsive difference equation systems are an excellent source of models to simulate processes and phenomena observed in control theory, physics, chemistry, population dynamics, biotechnology, industrial robotics, economics, etc. In recent years, impulsive difference equation systems have become a very active area of research. Many classical results have been extended to impulsive systems. Recently, impulsive systems with exponential decay play a frequent role in engineering, biology, economy, and other disciplines, so it is necessary to study the properties of these systems. However, not much has been developed in the direction of the stability theory of impulsive difference equation systems. There are a number of difficulties that one must face in developing the corresponding theory of impulsive functional differential systems with exponential decay. Therefore, it is an interesting and complicated problem to study the stability theory for impulsive difference equation systems with exponential decay. Research have been done, by using Lyapunov functions and Razumikhin techniques, some Razumikhin-type theorems on stability and uniform stability are obtained for a class of impulsive functional differential systems with infinite delays.

On the other hand, several papers devoted to the study of exponential stability of impulsive delay systems have appeared during the past years. In [7], the authors have investigated exponential stability of impulsive systems with finite delay by using the method of Lyapunov functions and Razumikhin techniques. In [13], the authors have studied exponential stability, by using fundamental function and inequalities, for linear impulsive differential equations. However, very little is known about the exponential stability of impulsive difference equation systems. In this paper, we consider impulsive difference equation systems involving exponential decay. By using Lyapunov functions, we establish uniform exponential stability of the impulsive system.

This paper is organized as follows. In section 2, we give some important definitions, system formulation and assumptions. Section 3, we obtain the exponential stability behavior of the homogeneous impulsive difference equation involving decay and the uniform exponential stability by suitable assumptions of difference system. Our conclusions are in Section 4.

2. Preliminaries

Basic Definitions

The zero solution of a system is said to be:

Stable **(S)** if given $\varepsilon > 0$ and $n_0 = 0$ there exists $\delta = \delta(\varepsilon,\ n_0)$ such that$\|x_0\| < \delta$ implies $\|x(n,n_0,x_0)\| < \epsilon$ for all $n = n_0$.

Uniformly stable **(US)** if δ may be chosen independent of n_0, ***unstable*** if it is not stable.

Attracting **(A)** if there exists $\mu = \mu(n_0)$ such that $\|x_0\| < \mu$ implies

$$\lim_{n\to\infty} x(n,n_0,x_0) = 0.$$

Uniformly Attracting **(UA)** if the choice of μ is independent of n_0. The condition for ***uniform attractivity*** may be paraphrased by saying that there exists $\mu > 0$ such that for every ε and n_0

there exists $N = N(\varepsilon)$ independent of n_0 such that

$$\|x(n,n_0,x_0)\| < \epsilon$$

for all $n \ge n_0 + N$ whenever $\|x_0\| < \mu$.

Asymptotically stable **(AS)** if it is stable and attracting, and ***uniformly asymptotically stable*** **(UAS)** if it is uniformly stable and uniformly attracting.

Exponentially stable **(ES)** if there exist $\delta > 0$, $M > 0$, and $\eta \in (0,1)$ such that

$$\|x(n,n_0,x_0)\| < M\|x_0\|\eta^{n-n_0}$$

whenever $\|x_0 - x^*\| < \delta$.

And is ***uniformly exponentially*** stable if M is independent of n_0.

A solution $x(n,n_0,x_0)$ is bounded if for some positive constant M, $\|x(n,n_0,x_0)\| = M$ for all $n = n_0$, where M may depend on each solution.

Mathematical Formulation

To begin with, we introduce some assumptions.

Let B be a Banach space, we will use $|\cdot|$ for the norm in this space and for the induced norm in the space of bounded operators in B, while $\|\cdot\|$ will be used for the operator norm in some space of function or sequence.

Let us consider the sequence $\alpha = \{\alpha_k | k = 1,2,\dots\}$ with $\|\alpha\| = \sup\limits_{k\ge 1} |\alpha_k| < \infty$, and $\alpha = \{\alpha_k | k = 1,2,\dots\} \subset B$ with $\|\alpha\| = \sum_{k=1}^{\infty} |\alpha|^p$, $1 \le p \le \infty$ [3].

In order to clarify the concept of exponential decay, we introduce the weighted x_k spaces as follows. For $x = (x_1, x_2, x_3, \dots, x_s) \in B^s$. For a positive integer, B^s denotes the s-dimensional Euclidean space. Suppose u_k is a sequence on B^s, or $t \ge 0$ and for $1 \le q < \infty$ and set

$$\|u_k\| = \left(\sum_{\alpha=1}^{\infty} |u_k(\alpha)e^{t|\alpha|}|^q\right)^{\frac{1}{q}}.$$

For $t \ge 0$, let $\|u\|$ be the supremum of the sequence $|u_k(\alpha)e^{t|\alpha|}|$ and $\alpha \in B^s$.

System Formulation

Let $\{x_k | k = 1,2,3\dots\}$ and $\{\gamma_k | k = 1,2,3,\dots\}$ be sequences in B and $A_k : B \to B$ be a bounded operator for $k = 0,1,2,3\dots$. Now consider a class of difference equations in R^k shown as follows

$$x_{k+1} = A_k x_k, \quad k \ge 0 \tag{2.1}$$

where $x^* \in R^k$ is the unique equilibrium point and $T : R^k \to R^k$ is a continuous differentiable nonlinear operator in R^k, and

$$\lim_{\|x\|\to\infty} \sup \|T'(x)\| = \lambda < 1$$

holds for a positive constant λ, $0 < \lambda < 1$. The related non homogenous difference equation

$$x_{k+1} = A_k x_k + \gamma_{k+1}, \quad k \geq 0 \tag{2.2}$$

and the homogeneous difference equation with an arbitrary initial point n_0 will be [9]

$$x_{k+1} = A_k x_k, \quad k \geq n_0, \; x_{n_0} = I \tag{2.3}$$

where I is the identity operator which is called as **fundamental function** of any of the equation (2.1) and (2.2).

3. Main Results

Our main problem is to find the behavior of the exponential stability behavior, which we can deduce by the theorem, and also the impulse ratio is considered between the systems of difference equation. Let us recall some same kind of recent results [2, 10, 13].

Lemma 3.1. *Let $\psi_{k,n}$ be a fundamental function of* (2.1)*, then the solution of* (2.2) *can be presented as*

$$x_k = \psi_{k,0} x_0 + \sum_{s=0}^{k-1} \psi_{n,s+1} \gamma_{s+1} \tag{3.1}$$

Lemma 3.2 ([5]). *Suppose* $\sup_k |A_k| < \infty$

(i) *Let $1 \leq q \leq \infty$. Then* (2.1) *is exponentially stable if and only if for any $\{\gamma_k\} \in I^q$, the solution of* (2.2)*, with $x_0 = 0$, belongs to the same space $\{x_k\} \in I^q$.*

(ii) *If $1 \leq q \leq \infty$ and the solution of* (2.2)*, with $x_0 = 0$, is bounded $\{x_k\} \in I^\infty$ for any $\{\gamma_k\} \in I^q$ then* (2.1) *is exponentially stable.*

(iii) *If the solution of* (2.2)*, with $x_0 = 0$, is bounded for any $\{\gamma_k\} \in I^1$ then* (2.1) *is uniformly stable.*

Theorem 3.3. *If* (2.1) *is a fundamental function, then*

(i) *If $\{\gamma_k\} \in I^\infty$, then the solution is bounded.*

(ii) *If $\lim_{k\to\infty} \gamma_k = \infty$, then the solution of $\{x_k\}$ of* (2.2) *satisfies $\lim_{k\to\infty} x_k = 0$*

(iii) *If there exist $M_0 > 0$, $\lambda > 0$ such that $|\gamma_k| \leq M_0 e^{-\lambda k}$, then there exist $M_1 > 0$, $\eta_1 > 0$ such that the solution $\{x_K\}$ of* (2.2) *satisfies $|x_k| \leq M_1 e^{-\eta_1 k}$, which leads the exponential stability.*

Proof. Since $\psi_{k,n}$ is the fundamental solution of the problem (2.1), $\psi_{k,n} x_n$ is a solution of (2.3) for any $x_n \in B$. Exponential stability of the equation (2.1) implies that $|\psi_{k,l} x_k| \leq M e^{-\eta(n-k)} |x_k|$ for any $x_k \in B$. Thus the fundamental function has the exponential estimation

$$|\psi_{k,n}| \leq M e^{-\eta(k-n)} \quad \text{for any } n = 0, 1, 2, \ldots, k \tag{3.2}$$

(i): The solution $\{x_k\}$ of (2.2) satisfies (3.1), therefore

$$|x_k| \leq |\psi_{k,0}| \cdot |x_0| + \sum_{s=0}^{k-1} |\psi_{k,s+1}| \cdot |\gamma_{s+1}| \tag{3.3}$$

Since $\{\gamma_k\} \in I^\infty$ and estimation (3.3) holds, then we get

$$|x_k| \le Me^{-\eta k}|x_0| + \sum_{s=0}^{k-1} Me^{-\eta(k-n-1)}|\gamma_{s+1}|$$

$$< M\left(|x_0| + \|\{\gamma_k\}\| \cdot \sum_{s=0}^{\infty} e^{-\eta s}\right)$$

$$= M\left(|x_0| + \frac{\|\{\gamma_k\}\|}{1-e^{-\eta}}\right).$$

This is true for any positive integer k. Thus any solution $\{x_k\}$ of (2.2) is bounded.

(ii): Let us pick any $x_0 \in B$, any $\epsilon > 0$. By the assumption of (2.1), the sequence $\{\gamma_k\}$ converges to zero solution, which implies that $\{\gamma_k\} \in I^\infty$ and there exists $k_1 > 0$ such that

$$|\gamma_k| < \epsilon \frac{1-e^{-\eta}}{3M} \quad \text{for any } k \ge k_1.$$

Since $e^{-\eta(k-n_1-1)} \to 0$ as $k \to \infty$, there exists a positive integer k_2 such that

$$e^{-\eta(k-k_1-1)} < \epsilon \frac{1-e^{-\eta}}{3M\|\{\gamma_k\}\|} \quad \text{for any } k \ge k_2.$$

Finally, for a given x_0 we can always find positive integer k_3 such that

$$e^{-\eta k} < \frac{\epsilon}{3M|x_0|} \quad \text{for any } k \ge k_3.$$

Let us suppose that $k > \max\{k_1, k_2, k_3\}$. Since the solution $\{x_k\}$ of (2.2) satisfies (3.1), we get

$$|x_k| \le |\psi_{k,0}| \cdot |x_0| + \sum_{s=0}^{k-1} |\psi_{k,s+1}| \cdot |\gamma_{s+1}|$$

$$= |\psi_{k,0}| \cdot |x_0| + \sum_{s=0}^{k_1} |\psi_{k,s+1}| \cdot |\gamma_{s+1}| + \sum_{s=k_1+1}^{k-1} |\psi_{k,s+1}| \cdot |\gamma_{s+1}|.$$

Applying (3.3) in the above equation, we get

$$|x_k| \le Me^{-\eta k}\,|x_0| + \sum_{s=0}^{k_1} Me^{-\eta(k-s-1)}\,|\gamma_{s+1}| \;+ \sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s-1)}\,|\gamma_{s+1}|. \tag{3.4}$$

Let us consider each component of the sum (3.3). Since $k \ge k_3$, we obtain

$$Me^{-\eta k}\,|x_0| < M\frac{\epsilon}{3M|x_0|}|x_0| = \frac{\epsilon}{3}.$$

If $k \ge k_2$ implies,

$$\sum_{s=0}^{k_1} Me^{-\eta(n-s-1)}|\gamma_{s+1}| = Me^{\eta}\|\{\gamma_k\}\| \sum_{s=0}^{k_1} e^{-\eta(k-s)}$$

$$= Me^{\eta}\|\{\gamma_k\}\| \sum_{s=k-k_1}^{k} e^{-\eta s}$$

$$< Me^{\eta}\|\{\gamma_k\}\| \sum_{s=k-k_1}^{\infty} e^{-\eta s}$$

$$< Me^{\eta}\|\{\gamma_k\}\| e^{-\eta(k-k_1)}\frac{1}{1-e^{-\eta}}$$

$$< M\|\{\gamma_k\}\| e^{-\eta(k-k_1-1)}\frac{1}{1-e^{-\eta}}$$

$$< M\|\{\gamma_k\}\| \epsilon \frac{1-e^{-\eta}}{3M\|\{\gamma_k\}\|}\frac{1}{1-e^{-\eta}}$$

$$= \frac{\epsilon}{3}.$$

Finally, $k \geq K_1$ provides,

$$\sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s)}|\gamma_{s+1}| < \sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s)}\epsilon\frac{1-e^{-\eta}}{3M}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta}) \sum_{s=k_1+1}^{k-1} e^{-\eta(k-s-1)}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta}) \sum_{s=k_1+1}^{k-1} e^{-\eta(k-s-1)}$$

$$= \frac{\epsilon}{3}(1-e^{-\eta}) \sum_{s=0}^{k-k_1} e^{-\eta s}$$

$$< \frac{\epsilon}{3}(1-e^{-\eta}) \sum_{s=0}^{k_1} e^{-\eta s}$$

$$< \frac{\epsilon}{3}(1-e^{-\eta})\frac{1}{1-e^{-\eta}}$$

$$= \frac{\epsilon}{3}.$$

From (3.4), we can get the following,

$$|x_k| \leq Me^{-\eta k}|x_0| + \sum_{s=0}^{k_1} Me^{-\eta(k-s-1)}|\gamma_{s+1}| + \sum_{s=k_1+1}^{k-1} Me^{-\eta(k-s-1)}|\gamma_{s+1}|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

For any $k > \max\{k_1, k_2, k_3\}$ and $\lim\limits_{k\to\infty} x_k = 0$.

(iii): The solution $\{x_k\}$ of the equation (2.1) satisfies (3.1) therefore

$$|x_k| \leq |\psi_{k,0}| \cdot |x_0| + \sum_{s=0}^{k-1} |\psi_{k,s+1}| \cdot |\gamma_{s+1}|.$$

The assumption of Part 3 of this Theorem and (3.3) together impose,

$$|x_k| \leq Me^{-\eta k}|x_0| + \sum_{s=0}^{k-1} Me^{-\eta(k-s-1)} M_0 e^{-\lambda(s+1)}$$

$$\leq Me^{-\eta k}|x_0| + MM_0 e^{\eta-\lambda} e^{-\eta k} \sum_{s=0}^{k-1} e^{s(\eta-\lambda)}$$

without loss of generality we can assume that $\eta \neq \lambda$.

Therefore,

$$\sum_{s=0}^{k-1} e^{s(\eta-\lambda)} = \frac{e^{k(\eta-\lambda)}-1}{e^{(\eta-\lambda)}-1}.$$

So,

$$\begin{aligned}
Me^{-\eta k}|x_0| + MM_0 e^{\eta-\lambda} e^{-\eta k} \sum_{s=0}^{k-1} e^{s(\eta-\lambda)} &= Me^{-\eta k}\ |x_0| + MM_0 e^{\eta-\lambda} e^{-\eta k} \frac{e^{k(\eta-\lambda)}-1}{e^{(\eta-\lambda)}-1} \\
&\le Me^{-\eta k}\ |x_0| + MM_0 e^{\eta-\lambda} e^{-\eta k} \frac{e^{k(\eta-\lambda)}}{e^{(\eta-\lambda)}-1} \\
&\le Me^{-\eta k}\ |x_0| + MM_0 \frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1} e^{-\lambda k}.
\end{aligned}$$

So, for $\eta_1 = \min\{\eta, \lambda\}$ and $N_3 = M|x_0| + MM_0 \frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1}$.

This implies,

$$|x_k| \le Me^{-\eta k}|x_0| + MM_0 \frac{e^{\eta-\lambda}}{e^{(\eta-\lambda)}-1} e^{-\lambda k} \tag{3.5}$$

for any $k = 0, 1, 2, 3, \dots$ and hence the solution of (2.1) is exponential stable.

This completes the proof of the theorem. □

Theorem 3.4. *Let a be a constant with $a > 1$. Let $B \subset \mathbb{R}^k$ be an open set containing the origin, and let $V(x_k): \mathbb{R}^k \to \mathbb{R}$ be a given function satisfying*

$$\lambda_1 ||x_k||^p \le V(x_k) \le \lambda_2 \|x_k\|^q \tag{3.6}$$

and

$$\Delta V(x_k) \le -\lambda_3 \|x_k\|^r + Na^{-\delta n} \tag{3.7}$$

for some positive constants $\lambda_1, \lambda_2, \lambda_3, p, q, r, k$ and δ. Moreover, if for some positive constants α and γ,

$$0 < \frac{\lambda_3}{\lambda_2{}^{r/q}} \le \alpha < 1 \tag{3.8}$$

such that

$$V(x_k) - V^{r/q}\ (x_k) \le \gamma a^{-\delta n} \tag{3.9}$$

with

$$\delta \ge -\frac{\ln(1-\lambda_3/\lambda_2{}^{r/q})}{\ln a} \tag{3.10}$$

then the zero solution of (2.2) is uniformly exponentially stable.

Proof. First note that in view of (3.9), the constant δ; which is given by (3.10) is positive. Taking the difference of the function $V(x_k)a^{M(k-n_0)}$ with $M = -\frac{\ln(1-\lambda_3/\lambda_2{}^{r/q})}{\ln a}$ we have

$$\Delta(V(x_k)a^{M(k-n_0)}) = [V(x_{k+1})a^M - V(x_k)]a^{M(k-n_0)}.$$

For $x_k \in B$, using (3.7) we get

$$\Delta(V(x_k)a^{M(k-n_0)}) = [-\lambda_3 \|x_k\|^r a^M + V(x_k)a^M + Na^{-\delta k} a^M - V(x_k)]a^{M(k-n_0)}. \tag{3.11}$$

From condition (3.6) we have $\|x_k\|^q \ge V(x_k)/\lambda_2$. Consequently, -$\|x_k\|^r \le -\left[\frac{V(x_k)}{\lambda_2}\right]^{r/q}$.

Thus, inequality (3.11) becomes

$$\Delta(V(x_k)a^{M(k-n_0)}) \le [-(\lambda_3/\lambda_2^{r/q})V^{r/q}(x_k)a^M + V(x_k)a^M + Na^{-\delta k}a^M - V(x_k)]a^{M(k-n_0)}$$
$$= [-(\lambda_3/\lambda_2^{r/q})V^{r/q}(x_k)a^M + (a^m - 1)V(x_k) + Na^{-\delta k}a^M - V(x_k)]a^{M(k-n_0)}.$$

Since $M = -\frac{\ln(1-\lambda_3/\lambda_2^{r/q})}{\ln a}$, we have $a^M - 1 = a^M(\lambda_3/\lambda_2^{r/q})$. Thus, the above inequality reduces to

$$\Delta(V(x_k)a^{M(k-n_0)}) \le [V(x_k) - V^{r/q}(x_k))(a^M - 1) + Na^{-\delta k}a^M]a^{M(k-n_0)}. \tag{3.12}$$

By invoking condition (3.9), inequality (3.12) takes the form

$$\Delta(V(x_k)a^{M(k-n_0)}) \le [\gamma(a^M - 1) + Na^M\ ]a^{-\delta k}a^{M(k-n_0)}$$
$$\le [\gamma(a^M - 1) + Na^M\ ]a^{-\delta k + \delta n_0}a^{M(k-n_0)} \le La^{(M-\delta)(k-n_0)}$$

where $L = \gamma(a^M - 1) + Na^M$ Summing the above inequality from n_0 to $k-1$ we obtain,

$$V(x_k)a^{M(k-n_0)} - V(x_{n_0}) \le La^{-(M-\delta)n_0}\sum_{s=n_0}^{k-1} a^{(M-\delta)s}$$
$$= \frac{La^{-(M-\delta)n_0}}{a^{(M-\delta)} - 1}[a^{(M-\delta)k} - a^{(M-\delta)n_0}]$$
$$= \frac{L}{a^{(M-\delta)} - 1}[a^{(M-\delta)(k-n_0)} - 1].$$

Since $M < \delta$ and $V(x_{n_0}) \le \lambda_2\|x_0\|^q$, the above inequality reduces to

$$V(x_k)a^{M(k-n_0)} \le \lambda_2\|x_0\|^q + \frac{L}{1 - a^{(M-\delta)}}.$$

Set $B(\|x_0\|) = \lambda_2\|x_0\|^q + \frac{L}{1-a^{(M-\delta)}}$. Then

$$V(x_k) \le B(\|x_0 = |)a^{-M(k-n_0)}. \tag{3.13}$$

From condition (3.6), we have $\lambda_1\|x\|^p \le V(x_k)$, which implies that

$$\|x_k\| \le \left\{\frac{V(x_k)}{\lambda_1}\right\}^{\frac{1}{p}}. \tag{3.14}$$

Combining (3.13) and (3.14), we arrive at

$$\|x_k\| \le \left\{\frac{B(\|x_0\|)}{\lambda_1}\right\}^{\frac{1}{p}} a^{-M/p(k-n_0)}.$$

Hence, the zero solution of (2.2) is uniformly exponentially stable. □

4. Conclusion

In this paper, a impulsive difference equation system with exponential decay has been considered, and our objective of attaining the behavior of exponential stability has been obtained by establishing some new necessary conditions, to the system of impulsive difference equations with distributed decays. In this paper special Lyapunov function was utilized to obtain the desired uniform stability for the system of impulsive difference equations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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