I-Convergence and Summability in Topological Group

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Abstract. In this article we introduce the $I$-convergence of sequences in topological groups and give certain characterizations of $I$-convergent sequences in topological groups and prove some fundamental theorems for topological groups.

1. Introduction

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [7], independently for the real sequences. Later on it was further investigated from sequence point of view and linked with the summability theory by Fridy [8] and many others. The idea is based on the notion of natural density of subsets of $\mathbb{N}$, the set of positive integers, which is defined as follows: The natural density of a subset of $\mathbb{N}$ is denoted by $\delta(E)$ and is defined by $\delta(E) = \lim_{n \to \infty} \frac{|\{k \in E : k \leq n\}|}{n}$, where the vertical bar denotes the cardinality of the respective set. This notion was used by Cakalli [5] to extend to topological Hausdroff groups.

The notion of $I$-convergence ($I$ denotes the ideal of subsets of $\mathbb{N}$, the set of positive integers), which is a generalization of statistical convergence, was introduced by Kastyrko, Salat and Wilczynski [9] and further studied by many other authors. Later on it was further investigated from sequence space point of view and linked with summability theory by Salat, Tripathy and Ziman [11, 12], Tripathy and Hazarika [13, 14, 15, 16], Hazarika [17], Hazarika and Savas [18] and many other authors.

The purpose of this article is to give certain characterizations of $I$-convergent sequences in topological groups and to obtain fundamental theorems in topological groups.

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2. Definitions and preliminaries

Definition 2.1. Let $S$ be a non-empty set. A non-empty family of sets $I \subseteq \mathcal{P}(S)$ (power set of $S$) is called an ideal in $S$ if (i) for each $A, B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$.

Definition 2.2. Let $S$ be a non-empty set. A family $F \subseteq \mathcal{P}(S)$ (power set of $S$) is called a filter on $S$ if (i) $\phi \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; (iii) for each $A \in F$ and $B \supseteq A$, we have $B \in F$.

Definition 2.3. An ideal $I$ is called non-trivial if $I \neq \phi$ and $S \notin I$. It is clear that $I \subseteq \mathcal{P}(S)$ is a non-trivial ideal if and only if the class $F = F(I) = \{ S - A : A \in I \}$ is a filter on $S$.

The filter $F(I)$ is called the filter associated with the ideal $I$.

Definition 2.4. A non-trivial ideal $I \subseteq \mathcal{P}(S)$ is called an admissible ideal in $S$ if it contains all singletons, i.e., if it contains $\{ \{ x \} : x \in S \}$.

Definition 2.5. A sequence $(x_k)$ of points in $X$ is said to be $I$-convergent to an element $x_0$ of $X$ if for each neighbourhood $V$ of 0 such that the set

$$\{ k \in \mathbb{N} : x_k - x_0 \notin V \} \in I$$

and it is denoted by $I\text{-}\lim_{k \to \infty} x_k = x_0$.

Definition 2.6. A sequence $(x_k)$ of points in $X$ is said to be $I$-Cauchy in $X$ if for each neighbourhood $V$ of 0, there is an integer $n(V)$ such that the set

$$\{ k \in \mathbb{N} : x_k - x_{n(V)} \notin V \} \in I$$

Definition 2.7. Let $A \subset X$ and $x_0 \in X$. Then $x_0$ is in the $I$-sequential closure of $A$ if there is a sequence $(x_k)$ of points in $A$ such that $I\text{-}\lim_{k \to \infty} x_k = x_0$. We denote $I$-sequential closure of a set $A$ by $\bar{A}^I$. We say that a set is $I$-sequentially closed if it contains all of the points in its $I$-sequential closure.

Throughout the article $s(X)$, $c^I(X)$ and $C^I(X)$ denote the set of all $X$-valued sequences, the set of all $X$-valued $I$-convergent sequences and the set of all $X$-valued $I$-Cauchy sequences in $X$, respectively.

By a method of sequential convergence, we mean an additive function $B$ defined on a subgroup of $s(X)$, denoted by $c^I_B(X)$ into $X$.

Definition 2.8. A sequence $x = (x_k)$ is said to be $B$-convergent to $x_0$ if $x \in c^I_B(X)$ and $B(x) = x_0$.

Definition 2.9. A method $B$ is called regular if every convergent sequence $x = (x_k)$ is $B$-convergent with $B(x) = \lim x$. 
Definition 2.10. A point $x_0$ is called a $B$-sequential accumulation point of $A$ (or is in the $B$-sequential derived set) if there is a sequence $x = (x_k)$ of points in $A - \{x_0\}$ such that $B(x) = x_0$.

Definition 2.11. A subset $A$ of $X$ is called $B$-sequentially countably compact if any infinite subset $A$ has at least one $B$-sequentially accumulation point in $A$.

Definition 2.12. A subset $A$ of $X$ is called $B$-sequentially compact if $x = (x_k)$ is a sequence of points of $A$, there is a subsequence $y = (y_{k_n})$ of $x$ with $B(y) = x_0$.

3. Main results

Theorem 3.1. A sequence $(x_k)$ is $I$-convergent if and only if for each neighbourhood $V$ of 0 there exists a subsequence $(x_{k(r)})$ of $(x_k)$ such that $\lim_{r \to \infty} x_{k(r)} = x_0$ and

$$\{k \in N : x_k - x_{k(r)} \notin V\} \in I.$$ 

Proof. Let $x = (x_k)$ be a sequence in $X$ such that $\lim_{k \to \infty} x_k = x_0$. Let $\{V_n\}$ be a sequence of nested base of neighbourhoods of 0. We write $E^{(i)} = \{k \in N : x_k - x_0 \notin V_i\}$ for any positive integer $i$. Then for each $i$, we have $E^{(i+1)} \subseteq E^{(i)}$ and $E^{(i)} \not\subseteq F(I)$.

Choose $n(1)$ such that $k > n(1)$, then $E^{(1)} \not\subseteq \phi$. Then for each positive integer $r$ such that $n(p + 1) \leq r < n(2)$, choose $k'(r) \in E^{(i)}$, i.e., $x_{k'(r)} - x_0 \in V_i$. In general, choose $n(p + 1) > n(p)$ such that $r > n(p + 1)$, then $E^{(p+1)} \not\subseteq \phi$. Then for all $r$ satisfying $n(p) \leq r < n(p + 1)$, choose $k'(r) \in E^{(i)}$, i.e., $x_{k'(r)} - x_0 \in V_i$. Also for every neighbourhood $V$ of 0, there is a symmetric neighbourhood $W$ of 0 such that $W \cup W \subset V$. Then we get

$$\{k \in N : x_k - x_{k(r)} \notin V\} \subseteq \{k \in N : x_k - x_0 \notin W\} \cup \{r \in N : x_{k'(r)} - x_0 \notin W\}.$$ 

Since $\lim_{k \to \infty} x_k = x_0$, therefore there is a neighbourhood $W$ of 0 such that

$$\{k \in N : x_k - x_0 \notin W\} \not\subseteq I$$

and $\lim_{r \to \infty} x_{k(r)} = x_0$ implies $\{r \in N : x_{k'(r)} - x_0 \notin W\} \not\subseteq I$.

Thus we have

$$\{k \in N : x_k - x_0 \notin V\} \not\subseteq I$$

Next suppose for each neighbourhood $V$ of 0 there exists a subsequence $(x_{k(r)})$ of $(x_k)$ such that $\lim_{r \to \infty} x_{k(r)} = x_0$ and $\{k \in N : x_k - x_{k(r)} \notin V\} \in I$.

Since $V$ is a neighbourhood of 0, we may choose a symmetric neighbourhood $W$ of 0 such that $W \cup W \subset V$. Then we have

$$\{k \in N : x_k - x_0 \notin V\} \subseteq \{k \in N : x_k - x_{k(r)} \notin W\} \cup \{r \in N : x_{k'(r)} - x_0 \notin W\}.$$ 

Since both the sets on the right hand side of the above relation belongs to $I$. Therefore $\{k \in N : x_k - x_0 \notin V\} \in I$.

This completes the proof. $\square$
Theorem 3.2. Any B-sequentially closed subset of a B-sequentially compact subset of \( X \) is B-sequentially compact.

Proof. Let \( A \) be a B-sequentially compact subset of \( X \) and \( E \) be a B-sequentially closed subset of \( A \). Let \( x = (x_k) \) be a sequence of points in \( E \). Then \( x \) is a sequence of points in \( A \). Since \( A \) is B-sequentially compact, there exists a subsequence \( y = (y_r) = (x_k) \) of the sequence \( (x_k) \) such that \( B(y) \in A \). The subsequence \( (y_r) \) is also a sequence of points in \( E \) and \( E \) is B-sequentially closed, therefore \( B(y) \in E \). Thus \( x = (x_k) \) has a B-convergent subsequence with \( B(y) \in E \), so \( E \) is B-sequentially compact. \( \square \)

Theorem 3.3. Let \( B \) be a regular subsequential method. Any B-sequentially compact subset of \( X \) is B-sequentially closed.

Proof. Let \( A \) be any B-sequentially compact subset of \( X \). For any \( x_0 \in \overline{A}^0 \), there exists a sequence \( x = (x_k) \) be a sequence of points in \( A \) such that \( B(x) = x_0 \). Since \( B \) is a subsequential method, there exists a subsequence \( y = (y_r) = (x_k) \) of the sequence \( x = (x_k) \) such that \( I\lim x_k = x_0 \). Since \( B \) is regular, \( B(y) = x_0 \). Since \( A \) is B-sequentially compact, there is a subsequence \( z = (z_r) \) of the subsequence \( y = (y_r) \) such that \( B(z) = y_0 \in A \). Since \( I\lim z = x_0 \) and \( B \) is regular, \( B(z) = x_0 \). Thus \( x_0 = y_0 \) and hence \( x_0 \in A \). Thus \( A \) is B-sequentially closed. \( \square \)

Theorem 3.4. Let \( B \) be a regular subsequential method. Then a subset of \( X \) is B-sequentially compact if and only if it is B-sequentially countably compact.

Proof. Let \( A \) be any B-sequentially compact subset of \( X \) and \( E \) be an infinite subset of \( A \). Let \( x = (x_k) \) be a sequence of different points of \( E \). Since \( A \) is B-sequentially compact, \( x \) has a convergent subsequence \( y = (y_r) = (x_k) \) with \( B(y) = x_0 \). Since \( B \) is subsequential method, \( y \) has a convergent subsequence \( z = (z_r) \) of the subsequence \( y \) with \( I\lim z_r = x_0 \). Since \( B \) is regular, we obtain that \( x_0 \) is a B-sequentially accumulation point of \( E \). Then \( A \) is B-sequentially countably compact.

Next suppose \( A \) is any B-sequentially countably compact subset of \( X \). Let \( x = (x_k) \) be a sequence of different points in \( A \). Put \( G = \{x_k : k \in N\} \). If \( G \) is finite, then there is nothing to prove. If \( G \) is infinite, then \( G \) has a B-sequentially accumulation point in \( A \). Also each set \( G_n = \{x_n : n \geq k\} \), for each positive integer \( n \), has a B-sequentially accumulation point in \( A \). Therefore \( \bigcap_{n=1}^{\infty} \overline{G_n}^B \neq \phi \). So there is an element \( x_0 \in A \) such that \( x_0 \in \bigcap_{n=1}^{\infty} \overline{G_n}^B \). Since \( B \) is a regular subsequential method, \( x_0 \in \bigcap_{n=1}^{\infty} \overline{G_n} \). Then there exists a subsequence \( z = (z_r) \) of the sequence \( x = (x_k) \) with \( B(z) \in A \). This completes the proof. \( \square \)
Theorem 3.5. The $B$-sequential continuous image of any $B$-sequentially compact subset of $X$ is $B$-sequentially compact.

Proof. Let $f$ be any $B$-sequentially continuous function on $X$ and $A$ be any $B$-sequentially compact subset of $X$. Let $y = (y_k) = (f(x_k))$ be a sequence of points in $f(A)$. Since $A$ is $B$-sequentially compact, there exists a subsequence $z = (z_r) = (x_{k_r})$ of the sequence $x = (x_k)$ with $B(z) \in A$. Then the sequence $f(z) = (f(z_r)) = (f(x_{k_r}))$ is a subsequence of the sequence $y$. Since $f$ is $B$-sequentially continuous, $B(f(z)) = f(x) \in f(A)$. Then $f(A)$ is $B$-sequentially compact. \qed

References


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