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Research Article

Laceability in Hanoi Graphs

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Abstract. The topological structure of an interconnection network can be modelled by a connected, simple and undirected graph $G=(V,E)$ where V represents the set of processors and E represents the set of communication links. Interconnection networks are used to interconnect the processors of data centres and cluster computers. The study of Hamiltonicity and the related areas such as Hamiltonian laceability and Hamiltonian connectedness has lot of significance in computer networks. A network (graph) is Hamiltonian connected if it contains a Hamiltonian path between two distinct nodes (vertices). In this paper we shall study the laceability properties associated with Hanoi graphs H_n. To be more specific we shall explore Hamiltonian-t^*-connectedness of Hanoi graphs H_n for $n \geq 3$.

Keywords. Hamiltonian Laceability; Hamiltonian connected

MSC. 05C45

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1. Introduction

The Hanoi graph H_n corresponds to the allowed moves in the tower of Hanoi problem. The tower of Hanoi puzzle invented in 1883 by the French mathematician Edouard Lucas, has become a classic example in the analysis of algorithms and discrete mathematical structures. The Hanoi graph can be constructed by taking the vertices to be the odd binomial coefficients of Pascal's triangle computed on the integers from 0 to $2^n - 1$ and drawing an edge whenever coefficients are adjacent diagonally or horizontally. The graph H_n has $3n$ vertices and $\frac{3(3^n)-1}{2}$ edges. The diameter of H_n is $2^n - 1$. Each Hanoi graph has exactly two distinct directed

Hamiltonian cycles. H_n has $3^n - 1$ small triangles, each of which can contain atmost one vertex in an independent vertex set. But the triangles are arranged in a plane in such a way that choosing the apex of each, gives a maximum independent vertex set. Hanoi graphs are perfect. Klavzar and Milutinovic [15], and Aumann et al. have explored [14], some classical numbers of Hanoi graphs. Xuemiao [12] has introduced the notion of Hanoi graph for the towers of Hanoi puzzle which greately helps investigate the problem. In [13] Kumar and Maheshwari have discussed about the Matrix representation of Hanoi graphs. The tower of Hanoi problem on path graphs have been discussed by Berend et al. [5]. Also, they have discussed about the diameter of Hanoi graphs. Hamiltonian-t-laceability in the brick product of even cycles was studied by Shenoy and Murali [11]. In [7] Girisha and Murali have studied Hamiltonian-t^*-laceability of 4 regular graphs.

A connected graph G is termed as Hamiltonian t_0^* (t_e^*) connected if there exists in G, a Hamiltonian path between atleast one pair of its vertices u and v with the property $d(u,v) = t$ for all odd (even) t such that $1 \le t \le \text{diam}(G)$ where $\text{diam}(G)$ is the diameter of a graph G. A connected graph G is termed as Hamiltonian-t^*-connected if there exists in G, a Hamiltonian path between atleast one pair of its vertices u and v with the property $d(u,v) = t$ for all t such that $1 \le t \le \text{diam}(G)$. The Hanoi graph H_3 is illustrated in Figure 1.

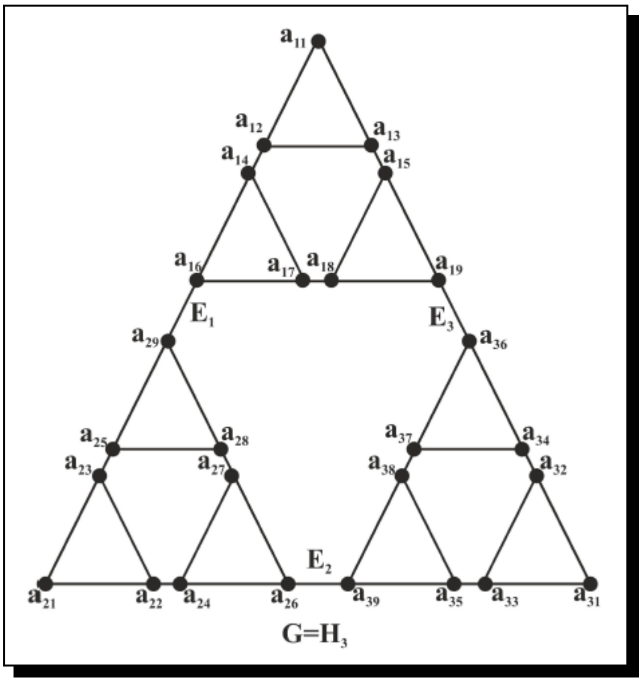

Figure 1

Definition 1.1. Let P_α and P_β respectively be the paths from a_i to a_j and a_j to a_k in G. Then the path $P_\alpha \cup P_\beta$ is the path obtained by extending the path P_α from a_i to a_j and from a_j to a_k through the common vertex a_j. That is if $P_\alpha : a_i \to a_j$ and $P_\beta : a_j \to a_k$ then $P_\alpha \cup P_\beta : a_i \to a_j \to a_k$.

2. Results

Lemma 2.1. *The graph H_3 is (t_o^*) connected for odd t such that $1 \le t \le \operatorname{diam} G$.*

Proof. Let $G = H_n$, $n = 3$. Clearly, $\operatorname{diam}(G) = 2^3 - 1 = 7$. Let the vertex set of H_3 be $V(H_3) = \{a_{ij}; i = 1; 1 \le j \le 3^{n-1}\} \cup \{a_{ij}; i = 2; 1 \le j \le 3^{n-1}\} \cup \{a_{ij}; i = 3; 1 \le j \le 3^{n-1}\}$ and the edge set of H_3 be $E(H_3) = \left\{b_i, 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \left\{b'_i, 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \left\{b'', 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \{E_3\}$ where $\{E_3\} = \{e_1, e_2, e_3\}$ such that $e_1 = (a_{(i)(j-m)} - a_{(i+1)(j)})$, $e_2 = (a_{(i+1)(j-m)} - a_{(i+2)(j)})$, $e_3 = (a_{(i+2)(j-m)} - a_{(i)(j)})$ and $m = 2^{(n-1)} - 1$.

Now to establish the result we consider the following cases.

In each case, we take $i = 1$, $j = 3^{n-1}$.

Case (i): Let $t = 1$. In G, $d(a_{12}, a_{11}) = 1$ and the Hamiltonian path between the two vertices a_{12} and a_{11} is $P_1 : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_3\}$, where $P_1 : a_{(i)(j-7)} - a_{(i)(j-5)} - a_{(i)(j-2)} - a_{(i)(j-3)}$, $P2 : a_{(i+1)(j)} - a_{(i+1)(j-1)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-2)} - a_{(i+1)(j-3)}$, $P_3 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_4 : a_{(i)(j)} - a_{(i)(j-1)} - a_{(i)(j-4)} - a_{(i)(j-6)} - a_{(i)(j-8)}$ and $\{E_3\} = \{e_1, e_2, e_3\}$.

Case (ii): In G, $d(a_{17}, a_{11}) = 3$ and the Hamiltonian path between the vertices a_{17} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_3\}$ where $P_1 : a_{(i)(j-2)} - a_{(i)(j-5)} - a_{(i)(j-3)}$, $P_2 : a_{(i+1)(j)} - a_{(i+1)(j-1)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-2)} - a_{(i+1)(j-3)}$, $P_3 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_4 : a_{(i)(j)} - a_{(i)(j-1)} - a_{(i)(j-4)} - a_{(i)(j-6)} - a_{(i)(j-7)} - a_{(i)(j-8)}$ and $\{E_3\} = \{e_1, e_2, e_3\}$.

Case (iii): Let $t = 5$. In G, $d(a_{28}, a_{11}) = 5$ and the Hamiltonian path between the vertices a_{28} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_2\}$ where $P_1 : a_{(i+1)(j-1)} - a_{(i+1)(j)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-2)} - a_{(i+1)(j-3)}$, $P_2 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_3 : a_{(i)(j)} - a_{(i)(j-4)} - a_{(i)(j-1)} - a_{(i)(j-2)} - a_{(i)(j-3)} - a_{(i)(j-5)} - a_{(i)(j-7)} - a_{(i)(j-6)} - a_{(i)(j-8)}$ and $\{E_2\} = \{e_2, e_3\}$.

Case (iv): Let $t = 7$. In G, $d(a_{22}, a_{11}) = 7$ and the Hamiltonian path between the vertices a_{22} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_2\}$ where $P_1 : a_{(i+1)(j-7)} - a_{(i+1)(j-8)} - a_{(i+1)(j-6)} - a_{(i+1)(j-4)} - a_{(i+1)(j)} - a_{(i+1)(j-1)} - a_{(i+1)(j-2)} - a_{(i+1)(j-5)} - a_{(i+1)(j-3)}$, $P_2 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P3 : a_{(i)(j)} - a_{(i)(j-4)} - a_{(i)(j-1)} - a_{(i)(j-2)} - a_{(i)(j-3)} - a_{(i)(j-5)} - a_{(i)(j-7)} - a_{(i)(j-6)} - a_{(i)(j-8)}$ and $\{E_2\} = \{e_2, e_3\}$.

From the above cases, it is obvious that the graph of H_3 is (t_o^*) connected. □

Lemma 2.2. *The graph H_3 is (t_e^*) connected for even t such that $2 \le t \le \operatorname{diam}(G)$.*

Proof. Let $G = H_n$, $n = 3$. Clearly, $\operatorname{diam}(G) = 2^3 - 1 = 7$. The vertex set and the edge set of H_3 are same as in Lemma 2.1. We have the following cases. In each case, we take $i = 1$, $j = 3^{n-1}$.

Case (i): Let $t = 2$. In G, $d(a_{14}, a_{11}) = 2$ and the Hamiltonian path between the two vertices a_{14} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_3\}$ where $P_1 : a_{(i)(j-5)} - a_{(i)(j-2)} - a_{(i)(j-3)}$, $P_2 : a_{(i+1)(j)} - a_{(i+1)(j-1)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-2)} - a_{(i+1)(j-3)}$, $P_3 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_4 : a_{(i)(j)} - a_{(i)(j-1)} - a_{(i)(j-4)} - a_{(i)(j-6)} - a_{(i)(j-7)} - a_{(i)(j-8)}$ and $\{E_3\} = \{e_1, e_2, e_3\}$.

Case (ii): Let $t = 4$. In G, $d(a_{29}, a_{11}) = 4$ and the Hamiltonian path between the vertices a_{29} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_2\}$ where $P_1 : a_{(i+1)(j)} - a_{(i+1)(j-1)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-2)} - a_{(i+1)(j-3)}$, $P_2 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_3 : a_{(i)(j)} - a_{(i)(j-4)} - a_{(i)(j-1)} - a_{(i)(j-2)} - a_{(i)(j-3)} - a_{(i)(j-5)} - a_{(i)(j-7)} - a_{(i)(j-6)} - a_{(i)(j-8)}$ and $\{E_2\} = \{e_2, e_3\}$.

Case (iii): Let $t = 6$. In G, $d(a_{27}, a_{11}) = 6$ and the Hamiltonian path between the vertices a_{27} and a_{11} is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_2\}$ where $P_1 : a_{(i+1)(j-2)} - a_{(i+1)(j-1)} - a_{(i+1)(j-4)} - a_{(i+1)(j-6)} - a_{(i+1)(j-8)} - a_{(i+1)(j-7)} - a_{(i+1)(j-5)} - a_{(i+1)(j-3)}$, $P_2 : a_{(i+2)(j)} - a_{(i+2)(j-1)} - a_{(i+2)(j-4)} - a_{(i+2)(j-6)} - a_{(i+2)(j-8)} - a_{(i+2)(j-7)} - a_{(i+2)(j-5)} - a_{(i+2)(j-2)} - a_{(i+2)(j-3)}$, $P_3 : a_{(i)(j)} - a_{(i)(j-4)} - a_{(i)(j-1)} - a_{(i)(j-2)} - a_{(i)(j-3)} - a_{(i)(j-5)} - a_{(i)(j-7)} - a_{(i)(j-6)} - a_{(i)(j-8)}$ and $\{E_2\} = \{e_2, e_3\}$.

From the above cases it is obvious that the graph of H_3 is (t_e^*) connected. □

Lemma 2.1 and Lemma 2.2 leads to the following theorem.

Theorem 2.3. *The graph H_3 is t^* connected for every t such that $1 \le t \le \operatorname{diam}(G)$.*

Lemma 2.4. *The graph H_4 is (t_o^*) connected for odd t such that $1 \le t \le \operatorname{diam}(G)$.*

Proof. Let $G = H_n$, $n = 4$. Clearly, $\operatorname{diam}(G) = 2^4 - 1 = 15$. The graph of H_4 contains three copies of H_3 namely C_1, C_2 and C_3 connected to each other by three unique edges $\{E_{C_d}\}$ for $d = 1,2,3$. The edge $\{E_{C_1}\}$ connects C_1 and C_2, $\{E_{C_2}\}$ connects C_2 and C_3 and $\{E_{C_3}\}$ connects C_3 and C_1. Let the vertex set of H_4 be $V(H_4) = \{x_k, 1 \le k \le q\} \cup \{y_k, 1 \le k \le q\} \cup \{z_k, 1 \le k \le q\}$ and the edge set of H_4 be $E(H_4) = \left\{E_g, 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \left\{E'_g, 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \left\{E''_g, 1 \le i \le \frac{3^{n+1}-9}{6}\right\} \cup \{E_{C_d}\}$ where $d = 1,2,3$ such that $\{E_{C_1}\} = (x_p, y_q)$, $\{E_{C_2}\} = (y_p, z_q)$, $\{E_{C_3}\} = (z_p, x_q)$ for $p = 3^{n-1} - m$, $q = 3^{n-1}$, $m = 2^{n-1} - 1$, $r = s = 2^{n-2} - 1$.

Case (i): For $(x_i, x_j) \in C_1$, $1 < i < q$, $j = 1$, the Hamiltonian path between x_i and x_j for $d(x_i, x_j) = t$ where $t = 1,3,5,7$ is $H_p : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_{C_d}\}$. Here $P_1 : (x_i \to x_{p+r} \to x_p)$, $P_2 : (y_q \to y_i \to y_p)$, $P_3 : (z_q \to z_j \to y_p)$, $P_4 : (x_q \to x_{q-s} \to x_j)$ and (E_{C_d}) where $d = 1,2,3$ such that $\{E_{C_1}\} = (x_p, y_q)\{E_{C_2}\} = (y_p, z_q)$, $\{E_{C_3}\} = (z_p, x_q)$.

Case (ii): Let $y_i \in C_2$ and $x_j \in C_1$ where $1 \le i \le q$, $j = 1$. Then the Hamiltonian path between (y_i, x_j) for $d(y_i, x_j) = t$ where $t = 9,11,13,15$ is $H_p : \{P_1 \cup P_2 \cup P_3\} \cup \{E_{C_d}\}$ where $P_1 : (y_i \to y_{q-s} \to y_{p+r} \to y_p)$, $P_2 : (z_q \to z_{q-s} \to z_{p+r} \to z_p)$, $P_3 : (x_q \to x_{q-s} \to x_{p+r} \to x_j)$, and (E_{C_d}) where $d = 2,3$ such that $\{E_{C_2}\} = (y_p, z_q)$, $\{E_{C_3}\} = (z_p, x_q)$.

Figures 2 and 3 depicts the Hamiltonian paths between x_i and x_j for $d(x_i,x_j)=t$ where $t=1,3,5,7$ and y_i and x_j for $d(y_i,x_j)=t$ where $t=9,11,13,15$. □

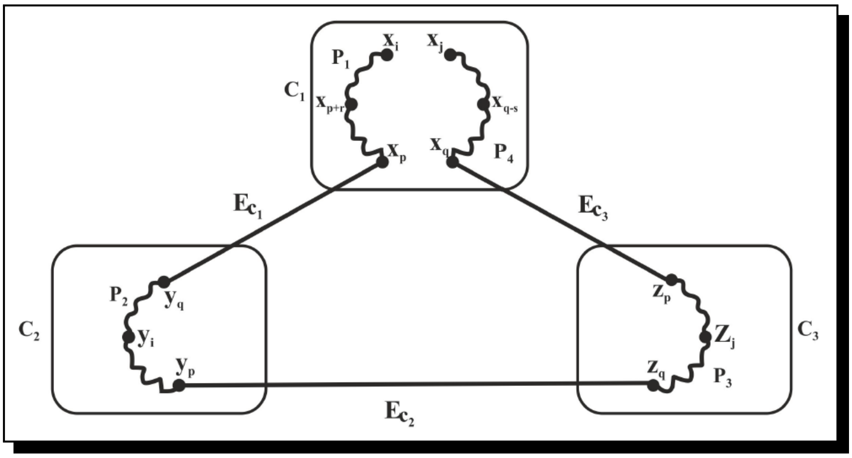

Figure 2

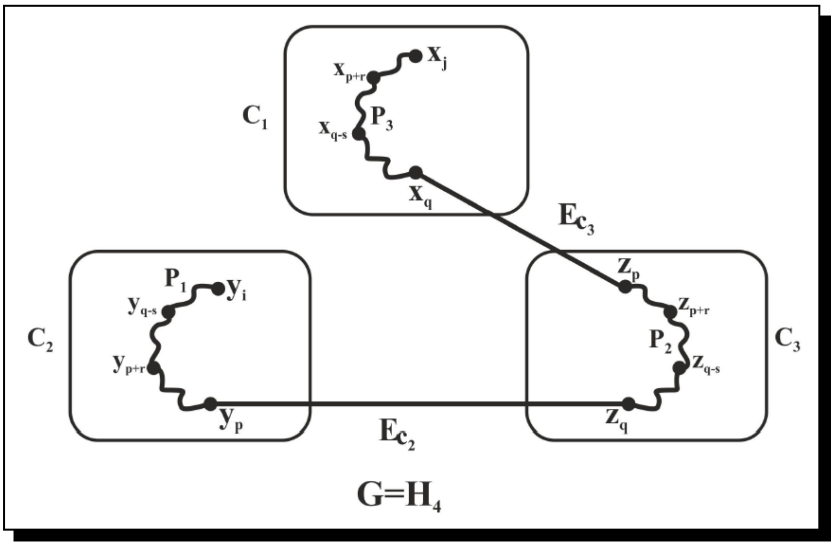

Figure 3

Lemma 2.5. *The graph H_4 is (t_e^*) connected for even t such that $2 \le t \le \operatorname{diam}(G)$.*

Proof. **Case (i):** Let $(x_i;x_j) \in C_1$, $1<i<q$, $j=1$. Then the Hamiltonian path between the two vertices x_i and x_j for $d(x_i,x_j)=t$ where $t=2,4,6$ is $P_1:(x_i \to x_{p+r} \to x_p) \cup P_2:(y_q \to y_i \to y_p) \cup P_3:(z_q \to z_j \to y_p) \cup P_4:(x_q \to x_{q-s} \to x_j) \cup (E_{C_d})$ where $d=1,2,3$ such that $\{E_{C_1}\}=(x_p,y_q)$, $\{E_{C_2}\}=(y_p,z_q)$, $\{E_{C_3}\}=(z_p,x_q)$ for $p=3^{n-1}-m$, $q=3^{n-1}$, $m=2^{n-1}-1$, $r=s=2^{n-2}-1$.

Case (ii): Let $y_i \in C_2$ and $x_j \in C_1$ where $1 \le i \le q$, $j=1$. Then the Hamiltonian path between (y_i,x_j) for $d(y_i,x_j)=t$ where $t=8,10,12,14$ is $P_1:(y_i \to y_{q-s} \to y_{p+r} \to y_p) \cup P_2:(z_q \to z_{q-s} \to z_{p+r} \to z_p) \cup P_3:(x_q \to x_{q-s} \to x_{p+r} \to x_j) \cup (E_{C_d})$ where $d=2,3$ such that $\{E_{C_2}\}=(y_p,z_q)$,

$\{E_{C_3}\} = (z_p, x_q)$ for $p = 3^{n-1} - m$, $q = 3^{n-1}$, $m = 2^{n-1} - 1$, $r = s = 2^{n-2} - 1$. This proves that the graph of H_4 is (t_e^*) connected. □

Lemma 2.4 and Lemma 2.5 leads to the following theorem.

Theorem 2.6. *The graph H_4 is t^* connected for every t such that $1 \le t \le \operatorname{diam}(G)$.*

Theorem 2.7. *The graph of H_n is t^* connected for all $n > 4$ with $d(u,v) = t$ such that $1 \le t \le \operatorname{diam}(G)$.*

Proof. Let $G = H_n$, $n > 4$ be the Hanoi graph where n is a positive integer. Clearly, $\operatorname{diam}(G) = 2^n - 1$. The graph of H_n contains three copies of $H_n - 1$ namely C_1, C_2 and C_3 connected to each other by three unique edges $\{E_{C_d}\}$ for $d = 1,2,3$. The edge $\{E_{C_1}\}$ connects C_1 and C_2, $\{E_{C_2}\}$ connects C_2 and C_3 and $\{E_{C_3}\}$ connects C_3 and C_1. The vertex set of H_n is $V(H_n) = \{x_k \cup y_k \cup z_k\}$ where $x_k \in C_1$, $y_k \in C_2$, $z_k \in C_3$ and $1 \le k \le q$, $q = 3^{n-1}$. The edge set of H_n is $E(H_n) = \left\{E_g/1 \le g \le \frac{3^{n+1}-9}{6} \cup E_{g'}/1 \le g' \le \frac{3^{n+1}-9}{6} \cup E_{g''}/1 \le g'' \le \frac{3^{n+1}-9}{6}\right\} \cup \{E_{C_d}\}$ where $d = 1,2,3$ such that $\{E_{C_1}\} = (x_p, y_q)$, $\{E_{C_2}\} = (y_p, z_q)$, $\{E_{C_3}\} = (z_p, x_q)$ for $p = 3^{n-1} - m$, $q = 3^{n-1}$, $m = 2^{n-1} - 1$, $r = s = 2^{n-2} - 1$.

To establish the result, we consider the following cases.

Case (i): For the vertices $(x_i, x_j) \in C_1$ where $1 < i < q$, $j = 1$ and $d(x_i, x_j) = t$ for every odd t, $1 \le t \le \operatorname{diam}(G)$, the Hamiltonian path is $P : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_{C_d}\}$ as in Case (i) of Lemma 2.4.

Case (ii): For the vertices $y_i \in C_2$ and $x_j \in C_1$ where $1 \le i \le q$, $j = 1$ and $d(x_j, y_i) = t$ for every odd t, $1 \le t \le \operatorname{diam}(G)$, the Hamiltonian path is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_{C_d}\}$ as in Case (ii) of Lemma 2.4.

Case (iii): For the vertices $(x_i, x_j) \in C_1$ where $1 < i < q$, $j = 1$ and $d(x_i, x_j) = t$ for every even t, $2 \le t \le \operatorname{diam}(G)$, the Hamiltonian path is $P : \{P_1 \cup P_2 \cup P_3 \cup P_4\} \cup \{E_{C_d}\}$ as in Case (i) of Lemma 2.5.

Case (iv): For the vertices $y_i \in C_2$ and $x_j \in C_1$ where $1 \le i \le q$, $j = 1$ and $d(x_j, y_i) = t$ for every even t, $2 \le t \le \operatorname{diam}(G)$, the Hamiltonian path is $P : \{P_1 \cup P_2 \cup P_3\} \cup \{E_{C_d}\}$ as in Case (ii) of Lemma 2.5. □

3. Conclusion

In this paper we have shown that the Hanoi graphs H_n are Hamiltonian t^* connected for all $n \ge 3$. This concludes that the existence of Hamiltonian path in such networks (graphs) suffice to solve data communication problems.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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