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Some New Families of Fourth Order Methods

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Abstract. In this article, we derive four new familes of fourth order iterative methods using the binomial approximation in certain existing three families of methods. The convergence analysis of the methods is discussed. Per iteration, the new families of methods require two evaluations of the given function and one of its derivative. Thus each of the four families has computational efficiency 1.587 which is better than many of the existing iterative methods. Some numerical examples are tested to check the efficiency and performance of the new families of methods.

1. Introduction

Consider a nonlinear equation,

$$f(x)=0 \tag{1.1}$$

A non-linear equation (1.1) cannot be solved in general analytically. Consequently, a number of numerical methods have been developed to compute the approximate solution of non-linear equations. These methods have been derived using Quadrature formulas, Adomian Decomposition methods, etc. see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and references therein. Newton's method is the most significant and simple iterative method for solving non-linear equations.

In recent years, many fourth-order variations of the Newton's method have been developed and analyzed. In current study, we present and examine some new fourth-order families of methods for solving nonlinear equations, which are obtained as variants of King's fourth-order family [3], Variants of fourth order family due to Li Tac-fang [5] and new variants of fourth order family due to C. Chun and Y. Hum [3]. We have also proved that the new methods are fourth order convergent. The new families of methods require two evaluations of the function and one value of its derivative. Thus each of theses four families has computational efficiency 1.587 which is better than computational efficiency

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1.414 of Newton's method and many of other iterative methods in the literature. Numerous numerical results are given to demonstrate the performance of the methods presented in this paper and are compared with similar existing methods.

2. **Derivation of methods**

Throughout this paper, we consider

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.1}$$

Let us consider the third order iterative methods, namely,

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right) \tag{2.2}$$

and

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(y_n)]} \tag{2.3}$$

due to H.H.H. Homeier [6], and J.R. Sharma [10] respectively.

C. Chun [2], equated the correcting terms of the methods (2.2) and (2.3) and got the approximation:

$$f'(y_n) \approx \frac{f'(x_n)[f(x_n) - f(y_n)]}{f(x_n) + f(y_n)} \, . \tag{2.4}$$

Let us consider the fourth order convergent method namely,

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n) - \alpha f(y_n)} \, , \tag{2.5}$$

due to Li Tai-fang, Li De-sheng, Xu Zhao-di and Fang Yi-ling [5].

From (2.4) putting in (2.5), we get:

$$x_{n+1} = y_n - \frac{f(y_n)[f(x_n) + f(y_n)]}{f'(x_n)[f(x_n) - f(y_n)] - \alpha f(y_n)[f(x_n) + f(y_n)]} \, . \tag{2.6}$$

Thus, we have proposed a new family of methods (2.6) for any $\alpha \in \mathbb{R}$ and y_n defined by (2.1). Now, consider King's fourth-order family of methods [7] given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)} \tag{2.7}$$

which can also be written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \beta f(y_n)}{f(x_n)\left[1 + (\beta - 2)\dfrac{f(y_n)}{f(x_n)}\right]} \frac{f(y_n)}{f'(x_n)} \, . \tag{2.8}$$

Using first-order binomial approximation, we get:

$$x_{n+1} = y_n - \frac{[f(x_n) + \beta f(y_n)][f(x_n) - (\beta - 2) f(y_n)]}{f^2(x_n)} \frac{f(y_n)}{f'(x_n)} \, . \tag{2.9}$$

Using the second-order binomial approximation, we obtain:

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} \frac{\begin{pmatrix} [f(x_n) + \beta f(y_n)] \\ \times [f^2(x_n) - (\beta-2) f(x_n) f(y_n) + (\beta-2)^2 f^2(y_n)] \end{pmatrix}}{f^3(x_n)}. \tag{2.10}$$

Thus, we have two new families of methods (2.9) and (2.10) for any $\beta \in \mathbb{R}$ and y_n defined by (2.1).

Now, consider a new fourth-order method proposed by C. Chun and Y. Ham [3], a variant of King's fourth order methods, namely :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) + (2\beta - 1) f(y_n)}{2f(x_n) + (2\beta - 5) f(y_n)} \frac{f(y_n)}{f'(x_n)}. \tag{2.11}$$

This can also be written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) + (2\beta - 1) f(y_n)}{2f(x_n)\left[1 + \frac{(2\beta-5)}{2} \frac{f(y_n)}{f(x_n)}\right]} \frac{f(y_n)}{f'(x_n)}. \tag{2.12}$$

Using first-order binomial approximation, we get:

$$x_{n+1} = y_n - \frac{[2f(x_n) + (2\beta - 1) f(y_n)][2f(x_n) - (2\beta - 5) f(y_n)]}{4f^2(x_n)} \frac{f(y_n)}{f'(x_n)}. \tag{2.13}$$

Thus, we have other new family of methods for any $\beta \in \mathbb{R}$ and y_n defined by (2.1).

Naturally for various values of α and $\beta \in \mathbb{R}$, we can get many particular cases of fourth order convergent families of methods (2.6), (2.9)-(2.10) and (2.13).

3. **Convergence analysis**

Here we prove that each of the families (2.6), (2.9)-(2.10) and (2.13) has convergence order four.

Theorem 1. *Assume that the sufficiently differentiable function $f : D \subset R \to R$ in an open interval D has a simple root $x^* \in D$. Then the method* (2.6) *is fourth-order convergent and for any $\alpha \in \mathbb{R}$, its error equation is given by:*

$$e_{n+1} = (3c_2^3 - c_2 c_3 - \alpha c_2^2) e_n^4 + O(e_n^5). \tag{3.1}$$

Proof. Let x^* be a simple root of f. Since f is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about x^*, we get:

$$\begin{aligned} f(x_n) &= f'(x^*)(e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + \dots) \\ f'(x_n) &= f'(x^*)(1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + \dots) \end{aligned} \tag{3.2}$$

where $e_n = x_n - x^*$, $c_k = \frac{1}{k} \frac{f^{(k)}(x^*)}{f'(x^*)}$ and $k = 2, 3, \dots$.

By (3.1) and (3.2), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4) e_n^4 + \dots. \tag{3.3}$$

From (2.1) and (3.3), we have

$$y_n = x^* + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 - \dots . \tag{3.4}$$

From (3.4), we get:

$$\begin{aligned} f(y_n) &= f'(x_n)[y_n - x^* + c_2(y_n - x^*)^2 + c_3(y_n - x^*)^3 + \dots] \\ &= f'(x^*)(c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + \dots . \end{aligned} \tag{3.5}$$

Thus, we have

$$\begin{aligned} &\frac{f(y_n)[f(x_n) + f(y_n)]}{f'(x_n)[f(x_n) - f(y_n)] - \alpha f(y_n)[f(x_n) + f(y_n)]} \\ &\qquad = c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 + \alpha c_2^2 - 6c_2c_3 + c_2^3)e_n^4 + \dots . \end{aligned} \tag{3.6}$$

From (2.6), (3.4) and (3.6), we have

$$e_{n+1} = (3c_2^3 - c_2c_3 - \alpha c_2^2)e_n^4 + O(e_n^5). \tag{3.7}$$

This shows that the Method defined by (2.6) has fourth order convergence. □

Theorem 2. *Assume that the sufficiently differentiable function $f : D \subset R \to R$ in an open iterval D has a simple root $x^* \in D$. Then the method* (2.9) *is fourth-order convergent and for any $\beta \in \mathbb{R}$, its error equation is given by:*

$$e_{n+1} = (5c_2^3 - c_2c_3 - 2\beta c_2^3 + \beta^2 c_2^3)e_n^4 + O(e_n^5).$$

Proof. Proof is similar to the Theorem 1 and is omitted. □

Theorem 3. *Assume that the sufficiently differentiable function $f : D \subset R \to R$ in an open iterval D has a simple root $x^* \in D$. Then the method* (2.10) *is fourth-order convergent and for any $\beta \in \mathbb{R}$, its error equation is given by:*

$$e_{n+1} = (c_2^3 - c_2c_3 - 2\beta c_2^3)e_n^4 + O(e_n^5).$$

Proof. Proof is similar to the Theorem 1 and is omitted. □

Theorem 4. *Assume that the sufficiently differentiable function $f : D \subset R \to R$ in an open iterval D has a simple root $x^* \in D$. Then the method* (2.13) *is fourth-order convergent and for any $\beta \in \mathbb{R}$, its error equation is given by:*

$$e_{n+1} = \left(\frac{25}{2}c_2^3 - c_2c_3 - 3\beta c_2^3 + \beta^2 c_2^3\right)e_n^4 + O(e_n^5).$$

Proof. Proof is similar to the Theorem 1 and is omitted. □

4. **Numerical examples**

MAPLE 7.0 is used to do all computations using 64 digits floating point arithmetics. We take $\epsilon = 10^{-15}$ as tolerance. We use the following stopping criteria:

(1) $|x_{n+1} - x_n| < \epsilon$
(2) $|f(x_{n+1})| < \epsilon$

We present here some test examples for iterative schemes. We compare Newton's method (NM) (2.1), King's fourth order method (2.7) with $\beta = 3$ (KM) [3], fourth order Traub's-Ostrowski's method (TM) [11], defined by:

$$x_{n+1} = x_n - \frac{f(y_n) - f(x_n)}{2f(y_n) - f'(x_n)} \frac{f(x_n)}{f'(x_n)} . \tag{4.1}$$

Jaratt's fourth order method (JM) [1] given by:

$$x_{n+1} = x_n - \left[1 - \frac{3}{2} \frac{f'(z_n) - f'(x_n)}{3f'(z_n) - 5f'(x_n)}\right] \frac{f(x_n)}{f'(x_n)} , \tag{4.2}$$

where

$$z_n = x_n - \frac{2f(x_n)}{3f'(x_n)}$$

and the fourth order methods, namely:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f^2(x_n) + 6f(x_n)f(y_n) + 3f^2(y_n)}{4f^2(x_n) - 2f(x_n)f(y_n) - f^2(y_n)} \frac{f(y_n)}{f'(x_n)} \tag{4.3}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)} \tag{4.4}$$

by C. Chun, Y. Ham [3], abbrevated by CM1 and CM2 respectively with our four families of methods defined by (2.6), (2.9)-(2.10) and (2.13) and denoted by (MSH1), (MSH2), (MSH3) and (MSH4) respectively.

We use the following test examples [3]:

Examples	**Zeros**
$f_1(x) = x^3 + 4x^2 - 10,$	$\alpha = 1.365230013414096845760806829 0$
$f_2(x) = x^2 - e^x - 3x + 2,$	$\alpha = 0.2575302854398607604553673049 4$
$f_3(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5,$	$\alpha = -1.207647827130918927009416758 4$
$f_4(x) = \sin x e^x + \ln(x^2 + 1),$	$\alpha = 0$
$f_5(x) = (x-1)^3 - 2,$	$\alpha = 2.259921049894873164767210607 3$
$f_6(x) = (x+2)e^x - 1,$	$\alpha = -0.442854401002388583141328000 00$
$f_7(x) = \sin^2 x - x^2 + 1,$	$\alpha = 1.404491648215341226035086817 8$

Displayed in Table 1 are the number of iterations (IT) to approximate the zero. It is to be noted that for most of the functions, the new family of methods MSH1 to MSH4 have at least equal performance as compared to the other well-known methods of the same order as given in the Table 1.

Table 1. Numerical Comparison of Newton's Method and various Fourth Order Methods

$f(x)$	NM	JM	TM	KM	CM1	CM2	MSH1 $\alpha=\frac{1}{8}$	MSH2 $\beta=2$	MSH2 $\beta=\frac{-1}{2}$	MSH3 $\beta=\frac{5}{2}$	MSH4 $\beta=0$	MSH4 $\beta=\frac{-1}{8}$
$f_1, x_0=-0.3$	55	46	46	49	9	44	4	25	10	10	10	11
$f_1, x_0=1$	6	4	4	4	4	4	4	4	4	4	4	4
$f_2, x_0=0$	5	3	3	3	3	3	3	3	3	3	3	3
$f_2, x_0=1$	5	3	3	3	3	3	3	3	3	4	3	3
$f_3, x_0=-1$	6	4	4	5	4	4	4	4	4	5	4	4
$f_3, x_0=-2$	9	5	5	6	6	6	5	5	6	6	6	6
$f_4, x_0=2$	6	4	4	6	4	4	5	5	5	6	5	5
$f_4, x_0=-5$	8	5	5	5	5	5	5	7	5	5	5	6
$f_5, x_0=3$	7	4	4	4	4	4	4	4	4	4	4	4
$f_5, x_0=4$	8	5	5	5	5	4	5	5	5	5	5	5
$f_6, x_0=2$	9	5	5	6	6	4	5	6	6	6	6	6
$f_6, x_0=3.5$	11	6	6	7	7	5	6	7	7	7	7	7
$f_7, x_0=1$	7	4	4	8	4	4	4	5	5	6	5	5
$f_7, x_0=2$	6	4	4	4	4	4	4	4	4	5	4	4

5. **Conclusion**

In this work, we derived four new families of fourth order methods using a very simple approach from the existing fourth order families, namely King's fourth order family, a family of fourth order methods due to C. Chun and Y. Hum [3] and variants of fourth order family due to Li Ta-fang et al. [5]. The four obtained families of methods require two function evaluations and one value of its derivative. Thus each of these families for constants α and β belonging to R has computational efficiency 1.587 which is better than computational efficiency of many other iterative methods in the literature. We observe that our methods are compareable with the existing methods and have at least equal performance. In particular, we observe that the family MSH1 (for $\alpha = 1/8$) gives over all better numerical results as compared to other methods in the Table 1. However, the families MSH3 (forβ =5/2) and MSH4 (for $\beta = 0$, $\beta = -1/8$) give at least better performance as compared to other similar methods in the Table 1. Using this simple approach, we can derive many other higher order iterative methods. Our results can be considered as an extension to the existing iterative methods.

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