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Research Article

Integrating Algorithms for Complete Bipartite Graph in Network Analysis

K. Rama Thulasi*[1,2] and G. Shobhalatha[1]

[1] *Department of Mathematics, Sri Krishnadevaraya University, Ananthapuramu 515003, Andhra Pradesh, India*
[2] *Department of Mathematics, Malla Reddy (MR) Deemed to be University, Hyderabad 500100, Telangana, India*
***Corresponding author:** ramathulasik82@gmail.com

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Abstract. This study aims to provide a comprehensive overview of network analysis, emphasizing the construction and application of complete bipartite graphs within organizational networks. The approach integrates both statistical and algorithmic perspectives to explore the network relationships among components in a system. In this framework, complete bipartite graphs serve as a powerful structure for implementing classical graph theory algorithms to address key problems such as game theory valuation, *minimum spanning tree* (MST) construction, maximum flow determination, and maximum weight matching. To address these problems, various well-established techniques are employed: the graphical method is used to solve the value of the game, Prim's algorithm for finding the MST, the Ford-Fulkerson Algorithm for computing the maximum flow, and the Hungarian Algorithm for identifying the maximum weight matching in a complete bipartite graph. These methodologies are demonstrated through suitable numerical examples within a unified network design framework. The complete bipartite graph structure, characterized by its full interconnectedness across vertex sets, enables greater flexibility and precision in handling complex network-related problems typically encountered in organizational and operational contexts. The findings are essential for optimizing human resource allocations across departments, improving supply chain logistics by matching suppliers to distributors, balancing workloads in distributed computing systems, and enhancing communication flow in hierarchical structures. Furthermore, this model is particularly beneficial in scenarios such as task scheduling, transportation planning, and decision-making processes where interdependent entities must be efficiently paired or evaluated.

Keywords. Complete bipartite graph, Maximum flow, maximum matching, Minimum spanning tree, Network optimization

Mathematics Subject Classification (2020). 05C30, 68R10

1. Introduction

Graph theory and game theory have developed as effective and indispensable analytical and problem-solving tools in a variety of areas, including economics, computer science, cybersecurity, artificial intelligence, network analysis, and cryptography. Game theory offers a mathematical framework for modelling strategic interactions between rational actors, whereas graph theory provides a strong structure for representing complicated networks and relationships. Recently, there has been an increase interest in using graph-theoretical models and game-theoretic tactics in cryptography and secure communication systems to improve both the efficiency and security of modern digital infrastructures. The graph $G=(V,E)$ is an ordered pair, where V represents a set of vertices and E represents a set of edges that are pairs of vertices. A pair of vertices in a graph are said to be adjacent if they are connected by an edge.

A complete bipartite graph is a pair $G=(V,E)$ where $V=\{R_1,R_2,C_1,C_2,C_3,C_4\}$, i.e., $V_1=\{R_1,R_2\}$, $V_2=\{C_1,C_2,C_3,C_4\}$ and E is a subset of 2-element subsets of V with this property that for each number R_1 and R_2 there are at most four other numbers C_1, C_2, C_3 and C_4 such that $\{\{R_1,C_1\},\{R_1,C_2\},\{R_1,C_3\},\{R_1,C_4\},\{R_2,C_1\},\{R_2,C_2\},\{R_2,C_3\},\{R_2,C_4\}\}$.

Recent studies present a structured mathematical method for generating balanced $k_{n,n}$ complete bipartite graphs with efficient connectivity, applicable in network design, data structures, and parallel computing. Zhang *et al*. [17] proposed which integrates sample latent cluster bipartite graphs to form a unified graph with clearer cluster structures. The method model tasks and computing nodes as a weighted bipartite graph and applies optimal complete matching using the Hungarian algorithm to achieve balanced and efficient task allocation. Gurjar and Krishna [7] have discussed balanced bipartite trees are structured data models that enable efficient storage, retrieval, and traversal by maintaining balance between two distinct node sets. Zhanga *et al*. [16] in their innovative approach that enhances missing data imputation by integrating a bipartite graph to represent observation-feature interactions with a complete directed graph to capture complex dependencies among features. Laube and Nebel [9] examined the performance of the Ford-Fulkerson method on special graph structures, providing probabilistic insights into its convergence behaviour. Hemanth and Vijayan [8] proposes an enhanced Ford-Fulkerson algorithm optimized for large-scale mesh networks, leveraging Hadoop's distributed processing to efficiently compute maximum flow. Turchina and Gulko [13] explored the application of the Ford-Fulkerson algorithm to detect and quantify excess information flow within networks, using surplus capacity analysis to identify potential data overload or redundancy. Fedrigo [6] investigated the end behaviour and equilibrium dynamics of the Threshold Protocol Game when played on complete and bipartite graphs, revealing how network structure influences convergence to stable thresholds. Sun *et al*. [11] presented an optimal bipartite graph matching method to select hindsight goals that align closely with the original goal distribution. Allouch *et al*. [1] described a novel approach based on cooperative game theory that takes into account the peer effects of worker productivity represented by a complete bipartite network of interactions.

By applying the Hungarian algorithm that solved the assignment problems, it effective in addressing matching problems in bipartite graphs. Also, it applies to optimize structural index reduction in high-index differential algebraic equation systems by framing the problem as a bipartite matching task via combinatorial relaxation theory which was discussed by Zeng *et al*. [15]. Tang *et al*. [12] proposed a Hungarian algorithm in network theory that estimates the complete matching of this weighted bipartite graph. Bougleux *et al*. [4] secured to computing the *Graph Edit Distance* (GED) between two graphs, a central measure in pattern recognition and graph-based matching. Yamamoto *et al*. [14] presents a specialized tabu search algorithm for reconfiguring radial distribution systems to minimize power losses. It uses Prim's algorithm to generate a high-quality initial solution and avoids cycles by storing all visited configurations. In this paper, Fathima *et al*. [5] proposed an adaptive differential protection system uses a hybrid graph algorithm to enhance fault detection in LVDC microgrids. Specifically, the Bidirectional Dijkstra algorithm is employed to accurately determine the shortest distance from the fault location to the nearest distributed generation source, enabling faster and more precise protection response. Arogundade *et al*. [2] explored the Prim's algorithm offers a calculated approach to creating effective, affordable local access networks in rural areas. By creating a minimum spanning tree, it aims to connect several sites with the least amount of overall wire expense. (Medak [10]) explains the Prim's algorithm that is used to design and analyse a *Minimum Spanning Tree* (MST), emphasizing its graphical depiction and methodical procedure. The goal of the study is to show how Prim's method effectively joins every vertex with the least amount of edge weight overall. Bakar *et al*. [3] describes a method for creating cost-effective local area networks utilizing Prim's Minimum Spanning Tree (MST) algorithm. By reducing overall cable length, the method assures optimal connectivity across all network nodes.

2. Preliminaries

In this section, we outline the essential mathematical definitions and notations relevant to the present study, with particular emphasis on the properties and operations associated with graph representations. The preliminaries presented here serve as foundational tools for the theoretical development and analytical results.

In road network modelling, a $2 \times n$ matrix is commonly used to represent the spatial configuration of a set of n nodes (points or locations) within a two-dimensional plane. Each node is defined by its Cartesian coordinates (x, y) indicating its position in space. The matrix provides a structured and compact format to store and process these coordinates efficiently, which is essential for tasks such as routing, distance calculation, and network visualization.

In road network modelling, a bipartite graph provides an effective structure for representing relationships between different types of entities within the network. When using a $2 \times n$ matrix to define the spatial configuration of n nodes, each node is represented by Cartesian coordinates (x, y), this matrix can be naturally interpreted in terms of a bipartite structure.

3. Network Analysis

In this section we explain different optimization network analysis by using different methods with their corresponding algorithms.

Let $M = \begin{bmatrix} 8 & 6 & 9 & 7 \\ 5 & 7 & 4 & 6 \end{bmatrix}$ be a matrix.

The above framed matrix explains a complete bipartite network graph is depicted in Figure 1.

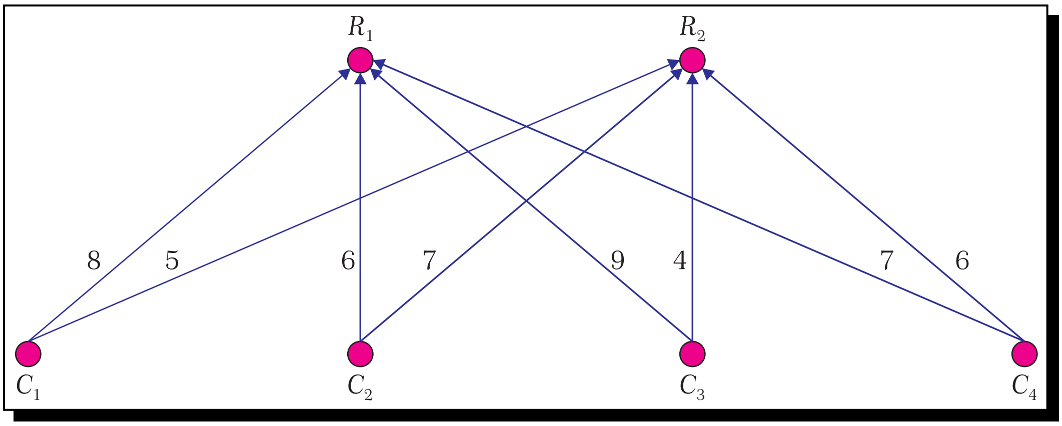

Figure 1. Network diagram

From Figure 1 the matrix M is the corresponding adjacent matrix of the graph, here R_1, R_2 are rows of the matrix and C_1, C_2, C_3, C_4 are the columns of the matrix. Algorithms for optimization network analysis, one of the algorithm to find maximum flow in a network is Ford-Fulkerson algorithm.

3.1 Maximal Flow Algorithm: Ford-Fulkerson Algorithm

The Ford-Fulkerson method is an algorithm for computing the maximum flow in a flow network, which is a directed graph with capacities on edges. It works by repeatedly finding augmenting paths from the source to the sink in the residual graph and increasing the flow along those paths. This process continues until no more augmenting paths exist, indicating that the maximum flow has been achieved.

3.2 Value of the Game (Min Cost): Graphical Method

In game theory, the value of the game is the expected payoff a player can guarantee themselves, assuming both players play optimally. The graphical method is used to find this value in two-person zero-sum games, particularly when one player has two strategies and the other has two or more. It involves plotting the expected payoffs and identifying the minimum of the maximum losses (or maximum of the minimum gains) to determine the game's value and optimal strategies.

3.3 Minimum Spanning Tree: Prim's Algorithm

Prim's algorithm is a greedy algorithm used to find a *Minimum Spanning Tree* (MST) of a connected, weighted, undirected graph. It starts from an arbitrary vertex and grows the spanning tree by repeatedly adding the smallest edge that connects a vertex inside the tree to a vertex outside the tree, until all vertices are included.

3.4 Matching's: Hungarian Method

Matrix-based models are widely used in applied graph theory and operations research to depict numerical correlations between two different sets of elements. The $2 \times n$ matrix is a popular structure that uses two sources (e.g., people, suppliers, or machines) and n targets (e.g., tasks, destinations, or outputs) to record relationships like cost, preference, or capacity. Even though matrix notation provides a concise and understandable representation, it does not necessarily give the most computationally efficient or intuitive framework for resolving intricate optimization issues. To improve analytical capabilities, the $2 \times n$ matrix can be systematically turned into a weighted bipartite graph, with two rows representing one set of vertices and n columns representing the other. Each matrix element represents a weighted edge from a source node to a target node. This graph-theoretic form permits the use of sophisticated combinatorial optimization techniques, including the Hungarian algorithm for assignment problems, the transportation simplex approach, and maximum matching or flow-based algorithms. Converting from a matrix to a graph framework is useful in real-world problems including logistics, scheduling, network routing, and resource allocation, where decisions rely on pairwise relationships and must be optimized within restrictions.

4. Numerical Example

The assignment problem is modelled from Figure 1 and solved by using Hungarian method.

Hungarian Method

$$\begin{bmatrix} 8 & 6 & 9 & 7 \\ 5 & 7 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 3 & 1 \\ 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & [0] & 3 & 1 \\ 1 & 3 & [0] & 2 \\ [0] & 0 & 0 & 0 \\ 0 & 0 & 0 & [0] \end{bmatrix}$$

The Hungarian method was employed to solve the assignment problem, yielding an optimal solution with a total matching cost of 10.

Total edge cost of matching by Hungarian method = 10.

Prim's Algorithm

The minimum spanning tree is obtained by applying the Prim's algorithm to Figure 1 is exhibited in Figure 2.

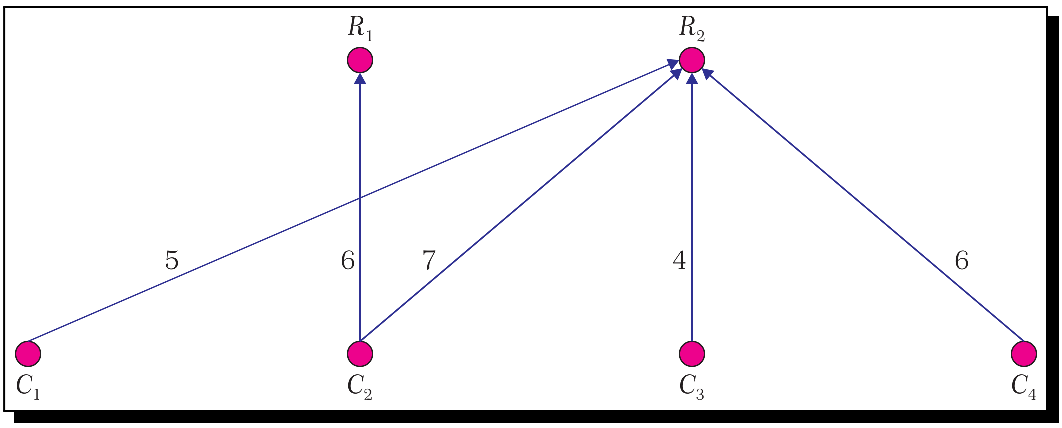

Figure 2. Minimum Spanning Tree

Table 1. Prim's algorithm table

Edge	Weight	Included in the spanning tree (or) not	If not included circuit formed
C_3-R_2	4	Yes	
R_2-C_4	6	Yes	
R_2-C_1	5	Yes	
R_2-C_2	7	Yes	
C_2-R_1	6	Yes	

The minimum spanning tree is obtained by applying the Prim's algorithm, i.e., $5+6+7+4+6 = 28$.

Graphical Method

The graphical method to solve a two-player zero-sum game represented by a 2×4 payoff matrix. In such games, the gain of one player is exactly the loss of the other, and the objective is to determine the optimal strategies and the value of the game.

Let Player A (row player) have two pure strategies (R_1 and R_2), and Player B (column player) have four pure strategies (C_1, C_2, C_3 and C_4). The payoff matrix is defined as follows:

$$\begin{bmatrix} 8 & 8 & 9 & 7 \\ 5 & 7 & 4 & 6 \end{bmatrix}$$

We consider two vertical axes corresponding to two rows from the graph. We find that the upper point of the lower region is H which is the intersection of C_2 and C_3. Therefore, C_2 and C_3 are the strategies of Player B this leads to the 2×2 payoff matrix, i.e.,

$$\begin{bmatrix} 6 & 9 \\ 7 & 4 \end{bmatrix}$$

From the obtained 2×2 pay off matrix, Player A chooses the strategies as R_1 and R_2 with probabilities p_1 and p_2 (i.e., $p_2 = 1 - p_1$). Similarly, Player B chooses the strategies as C_2 and C_3 with probabilities q_2 and q_3 (i.e., $q_3 = 1 - q_2$).

The expected payoff for each of Player B's strategies is then calculated as a linear function of P. The equations are plotted over the domain $0 \le P \le 1$, for each value of P; Player B will choose the strategy that minimizes Player A's expected payoff. Therefore, the maximum of these four equations at each P is taken, and the minimum of this upper envelope gives the value of the game is 6.5.

Ford-Fulkerson Algorithm

By applying the Ford-Fulkerson's Algorithm to Figure 1, the Maximum flow is obtained.

Table 2. Ford-Fulkerson's algorithm table

Augmenting path	Bottleneck capacity
C_1-R_2-C_4	5
C_1-R_1-C_2-R_2-C_4	1
C_1-R_1-C_4	7

The maximum flow that the network permits from the source to the sink corresponds to the total flow emerging from the source node. Based on the model, this cumulative flow is calculated as $5+1+7=13$. Therefore, the maximum flow in the network is 13.

5. Conclusion

This study integrates several algorithmic approaches namely the Graphical Method, Prim's algorithm, the Ford-Fulkerson algorithm, and the Hungarian method to analyse key network characteristics such as the value of the game, *minimum spanning tree* (MST), maximum flow, and maximum matching, all within a unified graph-based framework. Supporting matrix values is utilized to illustrate the numerical implementation of these methods. In this context, any selected pair of nodes or values can be interpreted as forming part of a graph, for which an optimal solution can be determined using the appropriate algorithm. Among these methods, the Graphical Method is employed to determine the value of the game and optimal strategies in two-person zero-sum games. Prim's algorithm constructs the minimum spanning tree of a connected, weighted graph by iteratively selecting the minimum weight edge that connects the growing tree to a new vertex. The Ford-Fulkerson algorithm is applied to compute the maximum flow in a flow network, while the Hungarian method is used to determine the maximum matching in bipartite graphs, ensuring optimal pairing between nodes of two disjoint sets. These graph-theoretical approaches collectively contribute to a comprehensive evaluation of the network's structural and operational properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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