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Research Article

Applications of the Tachibana Operator for Invariant Pseudoparallel Submanifold in Kenmotsu Manifolds Equipped With a General Connection

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Abstract. The present paper aims to study invariant pseudoparallel submanifolds of a Kenmotsu manifold admitting general connection, obtain necessary and sufficient conditions for an invariant pseudoparallel submanifold to be totally geodesic under some conditions. Furthermore, we investigate the conditions on the general connection of an invariant submanifold of a Kenmotsu manifold.

Keywords. Kenmotsu manifold, Pseudoparallel and Ricci pseudoparallel submanifolds, Ricci-generalized pseudoparallel and 2-Pseudoparallel submanifolds, General connection

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1. Introduction

Kenmotsu manifolds are a class of Riemannian manifolds characterized by their unique geometric properties. Kenmotsu manifolds have developed from a specific geometric study into a rich area of research with connections to various mathematical disciplines and applications. Their unique properties continue to inspire further exploration in both pure and applied mathematics (Mert and Atçeken [6]).

A connection on a manifold provides a way to differentiate vector fields along curves. More formally, a connection allows the definition of a derivative of a vector field along another vector field, facilitating the study of how vectors change in a manifold's curved geometry. Levi-Civita connection is the most common type of connection, uniquely determined for a Riemannian manifold. It is compatible with the metric and is torsion-free, meaning the connection does not introduce any twisting in the vectors (Zamkovoy [11]).

General connection, often referred to as a connection on a differentiable manifold, is a fundamental concept in differential geometry and plays a crucial role in the study of curved spaces. General connections are a powerful tool in understanding the geometric structure of manifolds. They provide the framework for defining differentiation in curved spaces and have significant implications in both mathematics and physics. The study of connections continues to be an active area of research, leading to deeper insights into the geometry and topology of manifolds (Biswas and Baishya [3]).

The aim of the present paper is to study invariant pseudoparallel submanifolds of a Kenmotsu manifold with respect to general connection, obtain necessary and sufficient conditions for an invariant pseudoparallel submanifold to be totally geodesic under some conditions. Furthermore, we investigate the conditions on the general connection of an invariant pseudoparallel submanifold of a Kenmotsu manifold.

2. Preliminary

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, called the structure vector field ξ, and a 1-form η-satisfying;

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where I denote the identity mapping of tangent space of each point at $\widetilde{M}$. From (2.1), it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \text{rank}(\phi) = 2n. \tag{2.2}$$

In this case, $\widetilde{M}^{2n+1}$ is said to be almost contact manifold (Mert and Atçeken [6]). An almost contact manifold $\widetilde{M}^{2n+1}$ is called an almost contact metric manifold if a Riemannian metric tensor g satisfies

$$\begin{cases} g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \\ \eta(X) = g(X,\xi),\ g(\phi X, Y) + g(X, \phi Y) = 0, \end{cases} \tag{2.3}$$

for any vector fields X, Y on $\widetilde{M}^{2n+1}$. The structure (ϕ,ξ,η,g) on $\widetilde{M}^{2n+1}$ is said to be almost contact metric structure. In an almost contact metric structure (ϕ,ξ,η,g), the Nijenhuis tensor and the fundamental form are, respectively, defined by

$$N_\phi(X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

and

$$\Phi(X,Y) = g(X, \phi Y),$$

for all $X, Y \in \Gamma(T\widetilde{M})$. An almost contact metric structure (φ,ξ,η,g) is said to be normal if

$$N_\phi(X,Y) + 2d\eta(X,Y)\xi = 0.$$

A normal contact metric manifold is called a Kenmotsu manifold, and it is shown as $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. It is well known that Kenmotsu manifolds can be also characterized by

$$(\widetilde{\nabla}_X\phi)Y = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

for all $X,Y \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. It follows that

$$\widetilde{\nabla}_X\xi = -\phi^2 X. \tag{2.5}$$

On the other hand, by $\widetilde{R}$ and $\widetilde{S}$ we denote the Riemannian curvature and Ricci tensors of $\widetilde{\nabla}$ in Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, respectively, then we have

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$\widetilde{R}(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.7}$$

$$\widetilde{S}(X,\xi) = -2n\eta(X). \tag{2.8}$$

Recently, Biswas and Baishya [2, 3] introduced a new connection which is called general connection in the set of contact geometry as

$$\nabla^G_X Y = \widetilde{\nabla}_X Y + \lambda_1[(\widetilde{\nabla}_X\eta)(Y)\xi - \eta(Y)\widetilde{\nabla}_X\xi] + \lambda_2\eta(X)\phi Y, \tag{2.9}$$

for all $X,Y \in \Gamma(T\widetilde{M})$, where λ_1 and λ_2 are real constants.

The general connection ∇^G can be seen generalization other connections. Namely,

(i) Quater symmetric connection metric connection for $(\lambda_1,\lambda_2) = (0,-1)$ (Biswas *et al.* [4], Golab [5]).

(ii) Schouten-van Kampen connection for $(\lambda_1,\lambda_2) = (1,0)$ (Schouten and Van Kampen[9]).

(iii) Tanaka-Webster connection for $(\lambda_1,\lambda_2) = (1,-1)$ (Tanno [10]).

(iv) Zamkovoy connection for $(\lambda_1,\lambda_2) = (1,1)$ (Zamkovoy [11]).

In Kenmotsu manifold, making use of (2.4) and (2.9), the general connection ∇^G is given by

$$\nabla^G_X Y = \widetilde{\nabla}_X Y + \lambda_1\{g(X,Y)\xi - \eta(Y)X\} + \lambda_2\eta(X)\phi Y, \tag{2.10}$$

$$\nabla^G_X\xi = (\lambda_1 - 1)\phi^2 X. \tag{2.11}$$

Now, we will calculate the covariant derivative of ϕ with respect to general connection ∇^G. By using (2.11), we have

$$\begin{aligned}(\nabla^G_X\phi)Y &= \nabla^G_X\phi Y - \phi\nabla^G_X Y\\ &= \widetilde{\nabla}_X\phi Y + \lambda_1\{g(X,\phi Y)\xi - \eta(\phi Y)X\} + \lambda_2\eta(X)\phi^2 Y\\ &\quad - \phi\{\widetilde{\nabla}_X Y + \lambda_1[g(X,Y)\xi - \eta(Y)X] + \lambda_2\eta(X)\phi Y\}\\ &= (\widetilde{\nabla}_X\phi)Y + \lambda_1 g(\phi X,\phi Y)\xi + \lambda_1\eta(Y)\phi X\\ &= (\lambda_1 - 1)\{g(X,\phi Y)\xi - \eta(Y)\phi X\},\end{aligned} \tag{2.12}$$

for all $X,Y \in \Gamma(T\widetilde{M})$. Furthermore, the Riemannian curvature and Ricci tensors R^G and S^G with respect to general connection ∇^G are given by

$$\begin{aligned}R^G(X,Y)Z &= \widetilde{R}(X,Y)Z + \lambda_2(\lambda_1 - 1)\{\eta(Y)g(\phi Z,X)\xi - \eta(X)g(Y,\phi Z)\xi\}\\ &\quad - \lambda_2(\lambda_1 - 1)\{\eta(Y)\eta(Z)\phi X - \eta(X)\eta(Z)\phi Y\} + \lambda_1(1-\lambda_1)\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}\\ &\quad + \lambda_1(2-\lambda_1)\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{2.13}$$

for all $X,Y \in \Gamma(T\widetilde{M})$ (Baishya and Biswas [2]).

Furthermore, by using (2.6) and (2.12), we observe

$$R^G(X,Y)\xi = (\lambda_1-1)^2\{\eta(X)Y-\eta(Y)X\}+\lambda_2(\lambda_1-1)\{\eta(Y)\phi X-\eta(X)\phi Y\}, \tag{2.14}$$

$$R^G(\xi,X)Y = (1-\lambda_1)^2\eta(Y)X-(1-\lambda_1)g(X,Y)\xi-\lambda_2(\lambda_1-1)\{g(X,\phi Y)\xi+\eta(Y)\phi X\} + \lambda_1(1-\lambda_1)\eta(X)\eta(Y)\xi, \tag{2.15}$$

$$R^G(X,\xi)Y = (1-\lambda_1)g(X,Y)\xi-(1-\lambda_1)^2\eta(Y)X+\lambda_2(\lambda_1-1)\{g(X,\phi Y)\xi+\eta(Y)\phi X\} - \lambda_1(1-\lambda_1)\eta(X)\eta(Y)\xi. \tag{2.16}$$

Eqs. (2.8) and (2.15) require

$$S^G(X,Y) = \widetilde{S}(X,Y)+\lambda_2(1-\lambda_1)g(X,\phi Y)+\lambda_1(1-\lambda_1)\eta(X)\eta(Y) + \{2(2n+1)\lambda_1-(2n+1)\lambda_1^2-3\lambda_1+2\lambda_1^2\}g(X,Y). \tag{2.17}$$

Now, let M be an immersed submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. Then, the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y+\sigma(X,Y) \tag{2.18}$$

and

$$\widetilde{\nabla}_X V = -A_V X+\nabla_X^{\perp}V, \tag{2.19}$$

for all $X,Y\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are induced connections on M, $\Gamma(T^{\perp}M)$ and σ, A denote the second fundamental form, shape operator of M, respectively.

For a submanifold M of a semi-Riemannian manifold $(\widetilde{M},g)$, the Gauss and Weingarten equations are given by

$$\widetilde{R}(X,Y)Z = R(X,Y)Z+A_{\sigma(X,Z)}Y-A_{\sigma(Y,Z)}X+(\widetilde{\nabla}_X\sigma)(Y,Z)-(\widetilde{\nabla}_Y\sigma)(X,Z) \tag{2.20}$$

and

$$g(\widetilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V)+g([A_V,A_U]X,Y), \tag{2.21}$$

for all $X,Y\in\Gamma(TM)$ and $U,V\in\Gamma(T^{\perp}M)$, where R and $R^{\perp}$ are the Riemannian curvature tensors of M and $\Gamma(T^{\perp}M)$, respectively.

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}\sigma(Y,Z)-\sigma(\nabla_X Y,Z)-\sigma(Y,\nabla_X Z), \tag{2.22}$$

for all $X,Y,Z\in\Gamma(TM)$.

For a $(0,k)$-type tensor field T, $k\geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-Tachibana operator is defined by

$$Q(A,T)(X_1,X_2,\ldots,X_k;X,Y) = -T((X\wedge_A Y)X_1,X_2,\ldots,X_k)-\ldots - T(X_1,X_2,\ldots,X_{k-1},(X\wedge_A Y)X_k), \tag{2.23}$$

for all $X_1,X_2,\ldots,X_k,X,Y\in\Gamma(TM)$, where the endomorphism $\wedge_A$ is defined by

$$(X\wedge_A Y)Z = A(Y,Z)X-A(X,Z)Y. \tag{2.24}$$

3. Invariant Pseudoparallel Submanifolds in Kenmotsu Manifolds Admitting a General Connection

The geometry of submanifolds of a contact metric manifold depends on the behavior of contact metric structure ϕ. A submanifold M of a contact manifold is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is also tangent to M for any vector field X tangent to M at each point of M. In other words, $\phi(TM) \subset TM$ at each point of M.

Theorem 3.1. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then, the following equalities are satisfied:*

(i) $\sigma(\phi X,Y) = \sigma(X,\phi Y) = \phi\sigma(X,Y)$,

(ii) $\sigma(X,\xi) = 0$ *and* $A_V\xi = 0$,

(iii) *the second fundamental forms σ and σ^G of M with respect to $\widetilde{\nabla}$ and ∇^G are equal,*

(iv) $R^G(X,Y)\xi = R^g(X,Y)\xi$,

where R^g denote the Riemannian curvature tensor of submanifold M with respect to ∇^G.

Proof. Since the proof is results of direct calculations, we give not to it.

In the rest of this paper, we will assume that M an invariant submanifold of a contact metric manifold $\widetilde{M}$ unless stated otherwise.

A submanifold M of a semi-Riemannian manifold $(\widetilde{M},g)$ is called Chaki-pseudo parallel if its second fundamental form σ satisfies

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = 2\gamma(X)\sigma(Y,Z) + \gamma(Y)\sigma(X,Z) + \gamma(Z)\sigma(X,Y), \tag{3.1}$$

for all $X,Y,Z \in \Gamma(TM)$ and γ is a nowhere vanishing 1-form.

In particular, if $\gamma = 0$ then M is said to be parallel submanifold of $\widetilde{M}$ (Atçeken *et al*. [1], Mert and Atçeken [7], Mert *et al*. [8]).

For a submanifold M of a semi-Riemannian manifold $(\widetilde{M},g)$, the Gauss and Weingarten equations are, respectively, given by

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z), \tag{3.2}$$

and

$$g(\widetilde{R}(X,Y)U,V) = g(R^{\perp}(X,Y)U,V) + g([A_V,A_U]X,Y), \tag{3.3}$$

for all $X,Y \in \Gamma(TM)$ and $U,V \in \Gamma(T^{\perp}M)$, where R and $R^{\perp}$ are the Riemannian curvature tensors of M and $\Gamma(T^{\perp}M)$, respectively.

Theorem 3.2. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. Then, M is a Chaki pseudoparallel with respect to a general connection ∇^G if and only if M is either totally geodesic submanifold or $\alpha(\xi) = \lambda_1 - 1$.*

Proof. Let us suppose that M is Chaki pseudoparallel. Then, from (3.1), there exists a 1-form α such that

$$(\nabla^G_X\sigma)(Y,Z) = 2\alpha(X)\sigma(Y,Z) + \alpha(Y)\sigma(X,Z) + \alpha(Z)\sigma(X,Y),$$

for all $X,Y,Z \in \Gamma(TM)$. Here, setting $Z = \xi$ and making use of (2.22) and (2.11), we have

$$-\sigma(Y,\nabla_X\xi) = \alpha(\xi)\sigma(X,Y)$$

which follows

$$(\alpha(\xi) + 1 - \lambda_1)\sigma(X,Y) = 0.$$

This proves our assertion. □

We have the following corollary because every totally geodesic submanifold is a Chaki pseudoparallel.

Corollary 3.1. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection. Then, M is a parallel submanifold if and only if M is a totally geodesic unless the general connection ∇^G is not a Schouten-van Kampen connection.*

Definition 3.1. Let M be an invariant submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If there exist forms ψ and θ such that

$$(\widetilde{\nabla}_X\widetilde{\nabla}_Y\sigma)(Z,U)=\psi(X,Y)\sigma(Z,U)+\theta(X)(\widetilde{\nabla}_Y\sigma)(Z,U), \tag{3.4}$$

then M is said to be generalized 2-recurrent submanifold (Atçeken *et al*. [1], Mert and Atçeken [7], Mert *et al*. [8]).

Theorem 3.3. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. M is generalized 2-recurrent submanifold with respect to a general connection if and only if M is a totally geodesic submanifold unless the general connection ∇^G is not a Schouten-van Kampen connection.*

Proof. Let us suppose that M be an invariant generalized 2-recurrent submanifold with respect to general connection ∇^G. Then, there exist forms ψ and θ such that

$$(\nabla^G_X\nabla^G_Y\sigma)(Z,U)=\psi(X,Y)\sigma(Z,U)+\theta(X)(\nabla^G_Y\sigma)(Z,U), \tag{3.5}$$

for all $X,Y,Z,U\in\Gamma(TM)$. Here, putting $Z=U=\xi$ in (3.2) and making use of (2.11) and (2.22), it yields to

$$\begin{aligned}\nabla^{\perp}(\nabla^G_X\sigma)(\xi,\xi)-(\nabla^G_Y\sigma)(\nabla_X\xi,\xi)-(\nabla^G_X\sigma)(\xi,\nabla_Y\xi)-(\nabla^G_{\nabla_XY}\sigma)(\xi,\xi)\\=\psi(X,Y)\sigma(\xi,\xi)+\theta(X)(\nabla^G_Y\sigma)(\xi,\xi).\end{aligned} \tag{3.6}$$

Taking into account Theorem 3.1, after the necessary adjustments, (3.6) takes the form

$$\sigma(\nabla_X\xi,\nabla_Y\xi)=(\lambda_1-1)^2\sigma(X,Y)=0.$$

This completes the proof. □

A submanifold M of a Riemannian manifold $(\widetilde{M},g)$ is called Deszcz-pseudoparallel if $\widetilde{R}\cdot\sigma$ and $Q(g,\sigma)$ are linearly dependent, that is,

$$\widetilde{R}\cdot\sigma=\ell_\sigma Q(g,\sigma), \tag{3.7}$$

where ℓ_σ is a function on $\widetilde{M}$. In particular, if $\ell_\sigma=0$, then M is said to be semiparallel (Mert and Atçeken [7]).

Theorem 3.4. *Let M be an invariant Deszcz-pseudoparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, at least one of the following is provided:*

(i) *M is totally geodesic submanifold,*

(ii) *∇^G is reduced Schouten-van Kampen connection,*

(iii) $\ell_\sigma+(\lambda_1-1)^2=0$.

Proof. Since M is a Deszcz pseudoparallel, form (3.7), there exists a function such that

$$(R^G(X,Y)\cdot\sigma)(U,V)=\ell_\sigma Q(g,\sigma)(U,V;X,Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. This implies that

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$

$$= -\ell_\sigma\{\sigma((X\wedge_g Y)U,V)+\sigma(U,(X\wedge_g Y)V)\}. \tag{3.8}$$

Taking $V=\xi$ in (3.8), taking into account of (2.14) and Theorem 3.1, we obtain

$$\sigma(U,R(X,Y)\xi)=\ell_\sigma\sigma(U,(X\wedge_g Y)\xi),$$

that is,

$$(\lambda_1-1)^2\sigma(U,\eta(X)Y-\eta(Y)X)+\lambda_2(\lambda_1-1)\sigma(\eta(Y)\phi X-\eta(X)\phi Y,U)=\ell_\sigma\{U,\eta(Y)X-\eta(X)Y\},$$

or

$$(\lambda_1-1)^2+\ell_\sigma]\sigma(U,\eta(X)Y-\eta(Y)X)+\lambda_2(\lambda_1-1)\phi\sigma(U,\eta(Y)X-\eta(X)Y)=0.$$

This proves our assertion. □

We have the following corollary from Theorem 3.4.

Corollary 3.2. *Let M be an invariant semiparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, M is either totally geodesic or the ∇^G is reduced Schouten-van Kampen connection.*

Definition 3.2. Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If tensors $\widetilde{R}\cdot\sigma$ and $Q(\widetilde{S},\sigma)$ are linearly dependent, then M is said to be generalized Ricci-pseudoparallel submanifold.

In particular, $\widetilde{R}\cdot\sigma=0$, it is called generalized Ricci-symmetric.

Theorem 3.5. *Let M be an invariant generalized Ricci-pseudoparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, at least one of the following holds:*

(i) *M is totally geodesic submanifold,*
(ii) *the general connection ∇^G is reduced to at least one of the connections Schouten-van Kampen connection, Tanaka-Webster or Zamkovoy,*
(iii) *$2n\ell_{S^G}=1$ and ∇^G is a Schouten-van Kampen connection.*

Proof. If M is an invariant generalized Ricci-pseudoparallel submanifold with respect to a general conncetion ∇^G, then there exists a function on ℓ_{S^G} such that

$$(R^G(X,Y)\cdot\sigma)(U,V)=\ell_{S^G}Q(S^G,\sigma)(U,V,X;Y),$$

for all $X,Y,U,V\in\Gamma(TM)$. This means that

$$R^\perp(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$= -\ell_{S^G}\{\sigma((X\wedge_{S^G} Y)U,V)+\sigma(U,(X\wedge_{S^G} Y)V)\}. \tag{3.9}$$

Taking $V=\xi$ in (3.9), consider Theorem 3.1, we have

$$\sigma(U,R(X,Y)\xi)=\ell_{S^G}\sigma(U,(X\wedge_{S^G} Y)\xi).$$

By means of (2.8), (2.14) and (2.17), we infer

$$(\lambda_1-1)\{(\lambda_1-1)(1-2n\ell_{S^G})\sigma(U,\eta(X)Y-\eta(Y)X)-\lambda_2\phi\sigma(U,\eta(X)Y-\eta(Y)X)\}=0$$

which is equivalent to

$$(\lambda_1-1)[(\lambda_1-1)^2(1-2n\ell_G)^2+\lambda_2^2]\sigma(U,\eta(X)Y-\eta(Y)X)=0. \tag{3.10}$$

This completes the proof. □

From Theorem 3.5, we can give the following corollary.

Corollary 3.3. *Let M be an invariant generalized Ricci-symmetric submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, at least one of the following is true:*

(i) *M is a totally geodesic submanifold,*

(ii) *the general connection ∇^G is induced Schouten-van Kampen.*

Definition 3.3. Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If tensors $\widetilde{R}\cdot\widetilde{\nabla}\sigma$ and $Q(g,\widetilde{\nabla}\sigma)$ are linearly dependent, then M is said to be generalized 2-pseudoparallel submanifold.

In particular, $\widetilde{R}\cdot\widetilde{\nabla}\sigma=0$, it is called generalized 2-pseudosymmetric submanifold.

Theorem 3.6. *Let M be an invariant generalized 2-pseudoparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g$ admitting a general connection ∇^G. Then, at least one of the following is true:*

(i) *M is totally geodesic submanifold,*

(ii) *the general connection ∇^G is reduced to at least one of Schouten-van Kampen connection, Tanaka-Webster connection or Zamkovoy connection,*

(iii) *$\ell_{\nabla^G\sigma}+(\lambda_1-1)^2=0$ and ∇^G is a quarter symmetric connection.*

Proof. Since M be an invariant generalized 2-pseudoparallel submanifold, we have the form

$$(R^G(X,Y)\cdot\nabla^G\sigma)(U,V,Z)=\ell_{\nabla^G\sigma}Q(g,\nabla^G\cdot\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z\in\Gamma(TM)$, where $\ell_{\nabla^G\sigma}$ is a function $\widetilde{M}$. This leads to

$$R^{\perp}(X,Y)(\nabla^G_U\sigma)(V,Z)-(\nabla^G_{R(X,Y)U}\sigma)(V,Z)-(\nabla^G_U\sigma)(R(X,Y)V,Z)-(\nabla^G_U\sigma)(V,R(X,Y)Z)$$
$$=-\ell_{\nabla^G\sigma}\{(\nabla^G_{(X\wedge_g Y)U}\sigma)(V,Z)+(\nabla^G_U\sigma)((X\wedge_g Y)V,Z)+(\nabla^G_U\sigma)(V,(X\wedge_g Y)Z)\}. \tag{3.11}$$

Here if $X=Z=\xi$ is taken in (3.11), we have

$$R^{\perp}(\xi,Y)(\nabla^G_U\sigma)(V,\xi)-(\nabla^G_{R(\xi,Y)U}\sigma)(V,\xi)-(\nabla^G_U\sigma)(R(\xi,Y)V,\xi)-(\nabla^G_U\sigma)(V,R(\xi,Y)\xi)$$
$$=-\ell_{\nabla^G\sigma}\{(\nabla^G_{(\xi\wedge_g Y)U}\sigma)(V,\xi)+(\nabla^G_U\sigma)((\xi\wedge_g Y)V,\xi)+(\nabla^G_U\sigma)(V,(\xi\wedge_g Y)\xi)\}. \tag{3.12}$$

Next, we will calculate these statements. Making use of (2.11) and (2.22), non-zero component of the first term takes the form

$$\begin{aligned}R^{\perp}(\xi,Y)(\nabla^G_U\sigma)(V,\xi)&=R(\xi,Y)^{\perp}\{\nabla^{\perp}_U\sigma(V,\xi)-\sigma(\nabla_UV,\xi)-\sigma(\nabla_U\xi,V)\}\\&=-R^{\perp}(\xi,Y)(\lambda_1-1)\phi^2\sigma(U,V)\\&=(\lambda_1-1)R^{\perp}(\xi,Y)\sigma(U,V).\end{aligned} \tag{3.13}$$

As the second term, one can easily to see

$$\begin{aligned}(\nabla^G_{R(\xi,Y)U}\sigma)(V,\xi)&=-\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\&=-(\lambda_1-1)\phi^2\sigma(R(\xi,Y)U,V)\\&=(\lambda_1-1)\sigma((\lambda_1-1)^2\eta(U)Y+(\lambda_1-1)g(Y,U)\xi+\lambda_1(1-\lambda_1)\eta(U)\eta(Y)\xi\\&\quad-\lambda_2(\lambda_1-1)\sigma([g(Y,\phi U)\xi+\eta(U)\phi Y],V)\\&=(\lambda_1-1)^2\eta(U)\{(\lambda_1-1)\sigma(V,Y)-\lambda_2\phi\sigma(V,Y)\}.\end{aligned} \tag{3.14}$$

In the same way, the non-zero component of the third term is

$$(\nabla^G_U\sigma)(R(\xi,Y)V,\xi) = -\sigma(\nabla_U\xi, R(\xi,Y)V)$$
$$= -(\lambda_1-1)\phi^2\sigma(U,R(\xi,Y)V)$$
$$= (\lambda_1-1)\sigma(U,R(\xi,Y)V)$$
$$= (\lambda_1-1)^2\eta(V)\{(\lambda_1-1)\sigma(U,Y)-\lambda_2\phi\sigma(U,Y)\}. \tag{3.15}$$

On the other hand, the non-zero components of the last term of the left side of the equality are

$$(\nabla^G_U\sigma)(V,R(\xi,Y)\xi) = (\nabla^G_U\sigma)(V,-(\lambda_1-1)^2\phi^2Y-\lambda_2(\lambda_1-1)\phi Y)$$
$$= (\lambda_1-1)^2(\nabla^G_U\sigma)(V,Y-\eta(Y)\xi)-\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(V,\phi Y)$$
$$= (\lambda_1-1)^2(\nabla^G_U\sigma)(V,Y)-(\lambda_1-1)^2(\nabla^G_U\sigma)(V,\eta(Y)\xi)-\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(V,\phi Y)$$
$$= (\lambda_1-1)^2(\nabla^G_U\sigma)(V,Y)-(\lambda_1-1)^3\eta(Y)\sigma(U,V)-\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(V,\phi Y). \tag{3.16}$$

Furthermore, non-zero the first components of the right side of the equality are also

$$(\nabla^G_{(\xi\wedge_g Y)U}\sigma)(V,\xi) = -\sigma(\nabla_{(\xi\wedge_g Y)U}\xi, V)$$
$$= -(\lambda_1-1)\phi^2\sigma(g(Y,U)\xi-\eta(U)Y,V)$$
$$= -(\lambda_1-1)\eta(U)\sigma(Y,V). \tag{3.17}$$

The second term also gives us

$$(\nabla^G_U\sigma)((\xi\wedge_g Y)V,\xi) = -\sigma(\nabla_U\xi,(\xi\wedge_g Y)V)$$
$$= -(\lambda_1-1)\phi^2\sigma(U,g(Y,V)\xi-\eta(V)Y)$$
$$= -(\lambda_1-1)\eta(V)\sigma(U,Y). \tag{3.18}$$

Finally, for the last term of the left of equality, we have

$$(\nabla^G_U\sigma)(V,(\xi\wedge_g Y)\xi) = (\nabla^G_U\sigma)(V,\eta(Y)\xi-Y)$$
$$= (\nabla^G_U\sigma)(V,\eta(Y)\xi)-(\nabla^G_U\sigma)(V,Y)$$
$$= -\sigma(\nabla_U\eta(Y)\xi,V)-(\nabla^G_U\sigma)(V,Y)$$
$$= (\lambda_1-1)\eta(Y)\sigma(U,V)-(\nabla^G_U\sigma)(V,Y). \tag{3.19}$$

Finally, the statements (3.13)-(3.19) are put in (3.12), we reach at

$$(\lambda_1-1)R^\perp(\xi,Y)\sigma(U,V)-(\lambda_1-1)^2\eta(U)\{(\lambda_1-1)\sigma(Y,V)-\lambda_2\phi\sigma(V,Y)\}$$
$$-(\lambda_1-1)^2\eta(V)[(\lambda_1-1)\sigma(U,Y)-\lambda_2\phi\sigma(U,Y)]-(\lambda_1-1)^2(\nabla^G_U\sigma)(V,Y)$$
$$+(\lambda_1-1)^3\eta(Y)\sigma(V,Y)+\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(V,\phi Y)$$
$$= -\ell_{\nabla^G\sigma}\{-(\lambda_1-1)\eta(U)\sigma(Y,V)-(\lambda_1-1)\eta(V)\sigma(U,Y)+(\lambda_1-1)\eta(Y)\sigma(U,V)-(\nabla^G_U\sigma)(V,Y)\}.$$

In the last equality, taking $V=\xi$ and after the necessary revisions are made, we conclude that

$$(\lambda_1-1)\{(\ell_{\nabla^G\sigma}+(\lambda_1-1)^2)\sigma(U,Y)-\lambda_2(\lambda_1-1)\phi\sigma(U,Y)\}=0,$$

which is equivalent to

$$(\lambda_1-1)[[\ell_{\nabla^G\sigma}+(\lambda_1-1)]^2+\lambda_2^2[\lambda_1-1]^2]\sigma(U,Y)=0 \tag{3.20}$$

which proves our assertions. □

We can give the following corollary from Theorem 3.6.

Corollary 3.4. *Let M be an invariant generalized 2-pseudosymmetric submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, at least one of the following is true:*

(i) *M is a totally geodesic submanifold,*

(ii) *the general connection ∇^G is reduced to at least one of Schouten-van Kampen connection, Zamkovoy connection, Tanaka-Webster connection.*

Definition 3.4. Let M be a submanifold of a semi-Riemannian manifold $(\widetilde{M},g)$. If tensors $\widetilde{R}\cdot\widetilde{\nabla}\sigma$ and $Q(\widetilde{S},\widetilde{\nabla}\sigma)$ are linearly dependent, then M is said to be generalized 2-Ricci pseudoparallel submanifold.

Theorem 3.7. *Let M be an invariant generalized 2-Ricci pseudoparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G. Then, at least one of the following is true:*

(i) *M is totally geodesic submanifold,*

(ii) *$2n\ell_{S^G}=\lambda_1-1$ and $\lambda_2=0$,*

(iii) *the general connection ∇^G is reduced to at least one of Schouten-van Kampen connection, Zamkovoy connection, Tanaka-Webster connection.*

Proof. If M is an invariant generalized 2-Ricci pseudoparallel submanifold of a Kenmotsu manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ admitting a general connection ∇^G, then there is a function such on $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ that

$$(R^G(X,Y)\cdot\nabla^G\sigma)(U,V,Z)=\ell_{S^G}Q(S^G,\nabla^G\sigma)(U,V,Z;X,Y),$$

for all $X,Y,Z,U,V\in\Gamma(TM)$. Explanation of this expression is

$$\begin{aligned}&R^{\perp}(X,Y)(\nabla^G_U\sigma)(V,Z)-(\nabla^G_{R(X,Y)U}\sigma)(V,Z)-(\nabla^G_U\sigma)(R(X,Y)V,Z)-(\nabla^G_U\sigma)(V,R(X,Y)Z)\\&\quad=-\ell_{S^G}\{(\nabla^G_{(X\wedge_{S^G}Y)U}\sigma)(V,Z)+(\nabla^G_U\sigma)((X\wedge_{S^G}Y)V,Z)+(\nabla^G_U\sigma)(V,(X\wedge_{S^G}Y)Z)\}.\end{aligned}$$

If this expression is re-written for $X=V=\xi$, we obtain

$$\begin{aligned}&R^{\perp}(\xi,Y)(\nabla^G_U\sigma)(\xi,Z)-(\nabla^G_{R(\xi,Y)U}\sigma)(\xi,Z)-(\nabla^G_U\sigma)(R(\xi,Y)\xi,Z)-(\nabla^G_U\sigma)(V,R(\xi,Y)Z)\\&\quad=-\ell_{S^G}\{(\nabla^G_{(\xi\wedge_{S^G}Y)U}\sigma)(\xi,Z)+(\nabla^G_U\sigma)((\xi\wedge_{S^G}Y)\xi,Z)+(\nabla^G_U\sigma)(\xi,(\xi\wedge_{S^G}Y)Z)\}.\end{aligned}\tag{3.21}$$

Now we will calculate each of these expressions separately. For this reason, as non-zero components of the first term are

$$\begin{aligned}R^{\perp}(\xi,Y)(\nabla^G_U\sigma)(\xi,Z)&=-R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,Z)\\&=-(\lambda_1-1)R^{\perp}(\xi,Y)\phi^2\sigma(U,Z)\\&=(\lambda_1-1)R^{\perp}(\xi,Y)\sigma(U,Z).\end{aligned}\tag{3.22}$$

Also, non-zero components of the second term give us

$$\begin{aligned}(\nabla^G_{R(\xi,Y)U}\sigma)(\xi,Z)&=-\sigma(\nabla_{R(\xi,Y)U}\xi,Z)\\&=-(\lambda_1-1)\phi^2\sigma(R(\xi,Y)U,Z)\\&=(\lambda_1-1)\sigma(R(\xi,Y)U,Z)\\&=\eta(U)(\lambda_1-1)^2\{(\lambda_1-1)\sigma(Y,Z)-\lambda_2\phi\sigma(Y,Z)\}.\end{aligned}\tag{3.23}$$

If we calculate non-zero components in the third term from the right of the equality, it takes the form

$$\begin{aligned}(\nabla^G_U\sigma)(R(\xi,Y)\xi,Z) &= (\nabla^G_U\sigma)(-(\lambda_1-1)^2\phi^2Y-\lambda_2(\lambda_1-1)\phi Y,Z)\\ &= (\lambda_1-1)^2(\nabla^G_U\sigma)(Y-\eta(Y)\xi,Z)-\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(\phi Y,Z)\\ &= (\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)-(\lambda_1-1)\eta(Y)\sigma(U,Z)\}-\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(\phi Y,Z). \quad (3.24)\end{aligned}$$

For the non-zero components of the last term of the right, one can easily to see

$$\begin{aligned}(\nabla^G_U\sigma)(\xi,R(\xi,Y)Z) &= -\sigma(\nabla_U\xi,R(\xi,Y)Z)\\ &= -(\lambda_1-1)\phi^2\sigma(U,R(\xi,Y)Z)\\ &= \eta(Z)(\lambda_1-1)^2\{(\lambda_1-1)\sigma(U,Y)-\lambda_2\phi\sigma(U,Y)\}. \quad (3.25)\end{aligned}$$

For the non-zero components of the first term of the left of equality, we have

$$\begin{aligned}(\nabla^G_{(\xi\wedge_{S^G}Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla_{(\xi\wedge_{S^G}Y)U}\xi,Z)\\ &= -(\lambda_1-1)\phi^2\sigma((\xi\wedge_S Y)U,Z)\\ &= (\lambda_1-1)\sigma(S^G(Y,U)\xi-S^G(\xi,U)Y,Z)\\ &= -(\lambda_1-1)S^G(U,\xi)\sigma(Y,Z)\\ &= 2n(\lambda_1-1)^3\eta(U)\sigma(Y,Z). \quad (3.26)\end{aligned}$$

Non-zero component of the second term of the left of equality is

$$\begin{aligned}(\nabla^G_U\sigma)((\xi\wedge_{S^G}Y)\xi,Z) &= (\nabla^G_U\sigma)(S^G(Y,\xi)\xi-S^G(\xi,\xi)Y,Z)\\ &= 2n(\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)-(\nabla^G_U\sigma)(\eta(Y)\xi,Z)\}\\ &= 2n(\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)+\sigma(\nabla_U\eta(Y)\xi,Z)\}\\ &= 2n(\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)+(\lambda_1-1)\eta(Y)\phi^2\sigma(U,Z)\}\\ &= 2n(\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)-(\lambda_1-1)\eta(Y)\sigma(U,Z)\}, \quad (3.27)\end{aligned}$$

and the last term is give us

$$\begin{aligned}(\nabla^G_U\sigma)(\xi,(\xi\wedge_{S^G}Y)Z) &= -\sigma(\nabla_U\xi,S^G(Y,Z)\xi-S^G(\xi,Z)Y)\\ &= S^G(\xi,Z)(\lambda_1-1)\phi^2\sigma(U,Y)\\ &= 2n(\lambda_1-1)^2\eta(Z)\sigma(U,Y). \quad (3.28)\end{aligned}$$

Consequently, if (3.22)-(3.28) are put (3.21), we have

$$\begin{aligned}&(\lambda_1-1)R^\perp(\xi,Y)\sigma(U,Z)-\eta(U)(\lambda_1-1)^2[(\lambda_1-1)\sigma(Y,Z)-\lambda_2\phi\sigma(Y,Z)]\\ &-(\lambda_1-1)^2[(\nabla^G_U\sigma)(Y,Z)-(\lambda_1-1)\eta(Y)\sigma(U,Z)]+\lambda_2(\lambda_1-1)(\nabla^G_U\sigma)(\phi Y,Z)\\ &-\eta(Z)(\lambda_1-1)^2[(\lambda_1-1)\sigma(U,Y)-\lambda_2\phi\sigma(U,Y)]\\ &\quad = -\ell_{S^G}[2n(\lambda_1-1)^3\eta(U)\sigma(Y,Z)+2n(\lambda_1-1)^2\{(\nabla^G_U\sigma)(Y,Z)-(\lambda_1-1)\eta(Y)\sigma(U,Z)\}\\ &\qquad +2n(\lambda_1-1)^2\eta(Z)\sigma(U,Y)]. \quad (3.29)\end{aligned}$$

Putting $Z=\xi$ in (3.29) and after making the necessary abbreviations, we can infer

$$(\lambda_1-1)^2[2n\ell_{S^G}-(\lambda_1-1)]\sigma(U,Y)+\lambda_2(\lambda_1-1)\phi\sigma(U,Y)=0$$

which proves our assertions. □

4. Conclusion

In this study, the structure of invariant submanifolds of Kenmotsu manifolds with respect to the generalized connection has been examined in detail. Initially, the geometric properties of these submanifolds have been analyzed using the Tachibana operator, and the behavior curvature of the manifolds have systematically investigated.

In the subsequent analysis, the invariant submanifolds have been studied within the framework of various pseudoparallel concepts. In particular, the properties of pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and Ricci generalized 2-pseudoparallel submanifolds topic has been studied. These investigations clarified the constraints imposed on the submanifolds under different pseudoparallel conditions and how their geometric structures are affected.

The obtained results provide a new and comprehensive perspective on the geometric structure of invariant submanifolds of Kenmotsu manifolds under the generalized connection. Moreover, this study is significant in demonstrating the applicability of pseudoparallel concepts in manifold theory and contributing to the characterization of invariant submanifolds. In future research, these findings may be extended to more general classes of manifolds and different connections, representing a potential direction for further exploration.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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