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Research Article

Reconstruction of a Discontinuous Refractive Index Using Modified Transmission Eigenvalues

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Abstract. We consider the inverse problem of reconstructing a spherically symmetric and discontinuous refractive index using modified interior transmission eigenvalues. We first investigate the asymptotic behavior of the characteristic function. Then we establish the uniqueness of a discontinuous refractive index from modified transmission eigenvalues without assuming that the contrast has a fixed sign. Finally, numerical examples are presented to verify the uniqueness results.

Keywords. Modified transmission eigenvalues, Discontinuous refractive index, Characteristic function, Inverse transmission eigenvalue problem

Mathematics Subject Classification (2020). 35P25, 47A40, 34L25

1. Introduction

The transmission eigenvalue problem, first discussed by Colton and Monk [14] in 1988, is a new type of boundary value problem for elliptic equations and arises in inverse scattering theory for inhomogeneous media. In the case of acoustic wave, it is formulated as

$$\left.\begin{aligned} &\Delta v(x)+k^2 n(x)v(x)=0, && x\in D,\\ &\Delta w(x)+k^2 w(x)=0, && x\in D,\\ &v(x)-w(x)=0, && x\in\partial D,\\ &\frac{\partial(v(x)-w(x))}{\partial\nu}=0, && x\in\partial D, \end{aligned}\right\} \tag{1.1}$$

where $w(x)$, $v(x)$ denote acoustic wave field, d is a bounded simply connected domain with C^2 boundary ∂D and $n(x)$ denotes the refractive index which is a positive real valued function. The complex values of $k > 0$ for which the boundary value problem (1.1) has a nontrivial solution are called transmission eigenvalues. On the other hand, the problem of reconstructing the material properties, i.e., the refractive index, from the knowledge of transmission eigenvalues is called inverse transmission eigenvalue problem.

The transmission eigenvalue problem mainly arises in acoustic and electromagnetic wave scattering in inhomogeneous media. Indeed, The acoustic wave scattering problem in inhomogeneous media can be formulated as

$$\left.\begin{aligned} &\Delta u(x)+k^2n(x)u(x)=0, \quad x\in R^3,\\ &u=e^{ikx\cdot d}+u^s,\\ &\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r}-iku^s\right)=0, \end{aligned}\right\} \tag{1.2}$$

where $r = |x|$, d is a unit vector denoting the direction of propagation and u^s denotes the scattering field. From (1.2) it can easily be seen that for fixed k, the scattering field u^s has the following asymptotic behavior as $r\to\infty$,

$$u^s(x)=\frac{e^{ikr}}{r}\left\{u_\infty(\widehat{x},d)+O\left(\frac{1}{r}\right)\right\}, \tag{1.3}$$

where $u_\infty(\widehat{x},d)$ is called the far field pattern corresponding to the scattering field u^s. By (1.3), the far field operator $F: L^2(S^2)\to L^2(S^2)$ is defined by

$$(Fg)(\widehat{x}):=\int_{S^2} u_\infty(\widehat{x},d)g(d)ds(d).$$

The far field operator F plays an important role in inverse scattering theory and it is well known that this operator is injective with dense range provided that k is not a transmission eigenvalue (Colton and Kress [11]).

For a long time, research on the transmission eigenvalue problem mainly focused on proving that transmission eigenvalues form at most a discrete set. From a practical point of view, the discreteness of transmission eigenvalues is of great importance since the sampling methods for reconstructing the support of an inhomogeneous medium fail if the interrogating frequency corresponds to a transmission eigenvalue (Colton and Kress [11]).

On the other hand, due to the non-selfadjointness of (1.1), the existence of transmission eigenvalues for general media remained open for more than twenty years until Päivärinta and Sylvester [26] showed the existence of at least one transmission eigenvalue provided that the contrast m in the medium is large enough. The study on the existence of transmission eigenvalues was completed by Cakoni *et al*. [6] where the existence of an infinite set of transmission eigenvalues was proven only under the assumption that the contrast m in the medium does not change sign and is bounded away from zero. The first transmission eigenvalue was also estimated here.

It was shown by Cakoni *et al*. [4,5], and Harris [19,21] that transmission eigenvalues can be determined by the scattering data and since they provide information about the material properties of the scattering object, can play an important role in a variety of problems in target identification. In [17, 23, 24, 28] various numerical algorithms were studied to compute transmission eigenvalues. For example, Gintides and Pallikarakis [17] transformed the transmission eigenvalue problem (1.1) into a variational form, i.e., finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{\eta^2 - n(x)}(\Delta u + k^2\eta^2 u)(\Delta\bar{\phi} + k^2 n(x)\bar{\phi}) = 0, \quad \text{for all } \phi \in H_0^2(D)$$

and computed the transmission eigenvalues by Galerkin method. The eigenfunctions of the eigenvalue problem

$$L\phi_i(x) = \mu_i \phi_i(x), \qquad x \in D,$$

$$\phi_i(x) = 0, \ \frac{\partial \phi_i(x)}{\partial \nu} = 0, \quad x \in \partial D$$

were chosen as a Hilbert basis in $H_0^2(D)$, where $L = \Delta\Delta$.

Recently, transmission eigenvalue problems with conductive boundary conditions have attracted great attention (Diao *et al*. [15], Harris [20], Harris and Kleefeld [22]).

The study on inverse transmission eigenvalue problems have mainly focused on the case of spherically symmetric refractive index. In this case, transmission eigenvalue problem becomes a boundary value problem of nonstandard ordinary differential equations.

Suppose that the inhomogeneous medium domain is a ball of radius 1 and $n(x)$ is spherically symmetric, i.e., $n(x) = n(r)$. The Laplacian in r^3 is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial \varphi^2}$$

in spherical coordinates (r,θ,φ), therefore, the boundary value problem (1.1) is transformed into

$$v'' + \frac{2}{r}v' + k^2 n(r)v = 0, \quad 0 < r < 1,$$

$$w'' + \frac{2}{r}w' + k^2 w = 0, \quad 0 < r < 1,$$

$$v(1) - w(1) = 0,$$

$$v'(1) - w'(1) = 0.$$

Now applying the transformation $u := rv$, $V := rw$, the above boundary value problem reads

$$u'' + k^2 n(r)u = 0, \quad 0 < r < 1,$$

$$V'' + k^2 V = 0, \quad 0 < r < 1,$$

$$u(0) = V(0) = 0, \ \ u(1) = V(1), \ \ u'(1) = V'(1).$$

From the second equation, we can see that the solution $V(r)$ satisfying $V(0) = 0$ must be a constant multiple of $\frac{\sin(kr)}{k}$. Thus, the transmission eigenvalue problem (1.1) becomes a boundary value problem for an ordinary differential equation

$$\left.\begin{aligned} & u'' + \lambda n(r)u = 0, \quad r \in (0,1), \\ & u(0,\lambda) = 0, \ \ u(1,\lambda)\cos\sqrt{\lambda} - u'(1,\lambda)\tfrac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0, \end{aligned}\right\} \tag{1.4}$$

where $\lambda = k^2$ is a spectral parameter and the refractive index $n(r)$ is a positive real function.

The function

$$\Delta(\lambda) := u(1,\lambda)\cos\sqrt{\lambda} - u'(1,\lambda)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} \tag{1.5}$$

is called the characteristic function of the transmission eigenvalue problem (1.4). It is well known that this is an entire function of order less than $\frac{1}{2}$ and all eigenvalues λ_n of the transmission eigenvalue problem (1.4) coincide with the zeros of the characteristic function $\Delta(\lambda)$([1]), where $u(r,\lambda)$ is the solution to the initial value problem

$$\left.\begin{aligned} &u'' + \lambda n(r)u = 0, \quad r \in (0,1),\\ &u(0,\lambda) = 0,\ u'(0,\lambda) = 1. \end{aligned}\right\} \tag{1.6}$$

In addition, there exists a real constant γ such that

$$\Delta(\lambda) = \gamma\lambda^d \prod_{n=d+1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right). \tag{1.7}$$

We define the quantity $A := \int_0^1 = \sqrt{n(r)}dr$.

It was shown by Aktosun *et al*. [1] that $n(r)$ can be uniquely determined from all transmission eigenvalues in case $A < 1$ and from all transmission eigenvalues and constant γ in case $A = 1$. Similar problems were discussed by Bondarenko and Buterin [2], Cakoni *et al*. [3], Cakoni and Haddar [7], Chen [8] and in general, there must be additional information (for example information about the refractive index in subintervals) (Gesztesy and Simon [16], Wang and Shieh [30], Wei and Wei [31]).

Modified transmission eigenvalues were introduced in order to reconstruct the refractive index without above constrains and additional information and to avoid the dependence of transmission eigenvalues on the frequency (Cogar *et al*. [9, 10], Stratouras [27]).

If there exist nontrivial solutions $\widetilde{w}$, $\widetilde{v}$ satisfying

$$\left.\begin{aligned} &\Delta\widetilde{w} + k^2 n(r)\widetilde{w} = 0, && x \in D,\\ &\Delta\widetilde{v} + k^2\eta^2\widetilde{v} = 0, && x \in D,\\ &\widetilde{v} = \widetilde{w},\ \tfrac{\partial\widetilde{v}}{\partial v} = \tfrac{\partial\widetilde{w}}{\partial v}, && x \in \partial D, \end{aligned}\right\} \tag{1.8}$$

then k was called a modified transmission eigenvalue, where $\eta > 0$ is a given sufficiently large constant. It was proved that the refractive index $n(r)$ can be uniquely determined from the modified transmission eigenvalues (Cogar *et al*. [9]).

On the other hand, applying Liouville transformation

$$\xi = \int_0^r \sqrt{n(s)}ds,\ \ z(\xi) := (n(r))^{\frac{1}{4}}y(r), \quad r = r(\xi)$$

to the transmission eigenvalue problem, we get Sturm-Liouville problem

$$-z'' + q(\xi)z = \lambda z, \quad \xi \in (0,a),$$

$$z(0,\lambda) = 0,\ \ z(a,\lambda)\cos\sqrt{\lambda} - z'(a,\lambda)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} = 0.$$

The existence of real eigenvalues and asymptotic behavior of the Sturm-Liouville problem and related inverse problems have also received great attention (Mclaughlin and Polyakov [25], Wei and Xu [32], Xu *et al*. [34]). The classic inverse Sturm-Liouville problem for mixed spectrum data has also generated much interest (Wang [29], Wei and Xu [33]).

In most studies on the inverse transmission eigenvalue problem, the refractive index was assumed to be continuous, especially, twice continuously differentiable and the problem of reconstructing it from the knowledge of transmission eigenvalues has been discussed in case the refractive index is of relatively small value.

In practice, the refractive index of the media can be discontinuous and may have arbitrary values. The inverse transmission eigenvalue problem with discontinuous refractive index was considered by Gintides and Pallikarakis [18], where the refractive index was required to satisfy $n(1)=1$ and either $n(r)>1$ or $0<n(r)<1$. Accordingly, the aim of this paper is to eliminate the above constraints on the refractive index by using the modified transmission eigenvalues.

This paper is organized as follows. In Section 2, we formulate the modified transmission eigenvalue problem in discontinuous media and discuss the asymptotic behavior of the characteristic function. In Section 3, we establish the uniqueness of the inverse modified transmission eigenvalue problem in discontinuous media and present some numerical examples to validate our theoretical results.

2. The Modified Transmission Eigenvalue Problem in Discontinuous Media and the Characteristic Function

2.1 Formulation of the Modified Transmission Eigenvalue Problem in Discontinuous Media

We assume that d is the unit sphere in r^3 and the refractive index is spherically symmetric, i.e., $n(x)=n(r)$. For the modified transmission eigenvalue problem

$$\left.\begin{array}{ll} \Delta\widetilde{w}+k^2n(r)\widetilde{w}=0, & x\in D, \\ \Delta\widetilde{v}+k^2\eta^2\widetilde{v}=0, & x\in D, \\ \widetilde{w}-\widetilde{v}=0, & x\in\partial D, \\ \frac{\partial(\widetilde{w}-\widetilde{v})}{\partial\nu}=0, & x\in\partial D. \end{array}\right\} \tag{2.1}$$

We introduce the spherical coordinates (r,θ,ϕ). We seek spherically symmetric eigenfunctions in the form

$$\widetilde{w}(r,\theta)=b_l\frac{y_l(r)}{r}P_l(\cos\theta),\quad \widetilde{v}(r,\theta)=a_lj_l(k\eta r)P_l(\cos\theta), \tag{2.2}$$

where j_l is a spherical Bessel function, P_l Legendre polynomial, a_l, b_l constants and $y_l(r)$ satisfies the following initial value problem:

$$\left.\begin{array}{l} y_l''(r)+\left(k^2n(r)-\frac{l(l+1)}{r^2}\right)y_l(r)=0,\quad 0<r<1, \\ \lim\limits_{r\to0}\left(\frac{y_l(r)}{r}-j_l(k\eta r)\right)=0. \end{array}\right\} \tag{2.3}$$

It is well known (Colton and Kress [11]) that k is a modified transmission eigenvalue if and only if

$$\widetilde{d}_l(k)=\det\begin{pmatrix} y_l(1) & -j_l(k\eta) \\ \frac{d}{dr}\left(\frac{y_l(r)}{r}\right)_{r=1} & -kj_l'(k\eta) \end{pmatrix}=0. \tag{2.4}$$

In this paper, we consider the case where the refractive index $n(r)$ is discontinuous in $d\in(0,1)$. We assume that $n(r)$ is twice continuously differentiable in $[0,d)$, $(d,1]$ and has finite one-sided limits $\lim\limits_{r\to d-0} n''(r)$, $\lim\limits_{r\to d+0} n''(r)$, and $n'(1)=0$. We consider the following jump conditions (Gintides and Pallikarakis [18]),

$$n(d^+)=an(d^-), \tag{2.5}$$

$$n'(d^+)=a^{-1}n'(d^-)+bn(d^-), \tag{2.6}$$

$$a>0, \quad |a-1|+|b|>0. \tag{2.7}$$

Using Liouville transformation

$$\xi(r)=\int_0^r \sqrt{n(\rho)}d\rho, \quad z(\xi)=n(r)^{\frac{1}{4}}y_l(r),$$

the differential equation of (2.3) is transformed into the following form:

$$\frac{d^2z(\xi)}{d\xi^2}+\left(k^2-\frac{l(l+1)}{\xi^2}-p(\xi)\right)z(\xi)=0, \quad 0<\xi<A, \tag{2.8}$$

where

$$A=\int_0^1 \sqrt{n(\rho)}d\rho, \ \ p(\xi)=\frac{l(l+1)}{r^2n(r)}-\frac{l(l+1)}{\xi^2}+\frac{n''(r)}{4n(r)^2}-\frac{5}{16}\frac{n'(r)^3}{n(r)^3}.$$

The following result holds for the solution z to (2.8).

Lemma 2.1 ([18])**.** *The solution $z(\xi)$ to (2.8) is discontinuous at $\xi=\widetilde{d}$ and satisfies the jump conditions:*

$$z(\widetilde{d}^+)=\widetilde{a}z(\widetilde{d}^-), \tag{2.9}$$

$$\frac{dz(\widetilde{d}^+)}{d\xi}=\widetilde{a}^{-1}\frac{dz(\widetilde{d}^-)}{d\xi}+\widetilde{b}z(\widetilde{d}^-), \tag{2.10}$$

$$|\widetilde{a}-1|+|\widetilde{b}|>0, \tag{2.11}$$

where

$$\widetilde{d}=\int_0^d \sqrt{n(\rho)}d\rho, \ \ \widetilde{a}=a^{\frac{1}{4}}, \ \ \widetilde{b}=\frac{1}{4}\left[\frac{n'(d^+)}{n(d^+)^{3/2}}\widetilde{a}-\frac{n'(d^-)}{n(d^-)^{5/4}n(d^+)^{\frac{1}{4}}}\right].$$

2.2 The Asymptotic Behavior of the Characteristic Function

First, we give the expression of the characteristic function and asymptotic behavior in case $L=0$. In this case, $y_0(r)$ satisfies

$$y_0''(r)+k^2n(r)y_0(r)=0, \quad 0<r<1, \ y_0(0)=0, \ y_0'(0)=1$$

and by Liouville transformation, it is transformed into

$$\frac{d^2z}{d\xi^2}+(k^2-p(\xi))z=0, \quad 0<\xi<A, \ z(0)=0, \ \frac{dz(0)}{d\xi}=n(0)^{-1/4}, \tag{2.12}$$

where

$$p(\xi)=\frac{n''(r)}{4n(r)^2}-\frac{5}{16}\frac{n'(r)^3}{n(r)^3}.$$

The characteristic function can be rewritten as

$$\widetilde{d}_0(k)=\det\begin{pmatrix} y_0(1) & \frac{\sin k\eta}{k\eta} \\ y_0'(1) & \cos k\eta \end{pmatrix}.$$

Lemma 2.2 ([18]). *The solution $z(\xi)$ to (2.12) satisfies the following Volterra integral equation,*

$$z(\xi)=\frac{\sin k\xi}{kn(0)^{\frac{1}{4}}}+\int_0^{\xi}p(t)\frac{\sin(\xi-t)}{k}z(t)dt,\quad 0\le\xi\le\tilde{d} \tag{2.13}$$

$$\begin{aligned}&=\frac{1}{kn(0)^{\frac{1}{4}}}\left[\tilde{a}\sin k\tilde{d}\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k\tilde{d}\sin k(\xi-\tilde{d})+\frac{\tilde{b}}{k}\sin k\tilde{d}\sin k(\xi-\tilde{d})\right]\\&+\frac{1}{k}\int_0^{\tilde{d}}[\tilde{a}\sin k(\tilde{d}-t)\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k(\tilde{d}-t)\sin k(\xi-\tilde{d})\\&+\frac{\tilde{b}}{k}\sin k(\tilde{d}-t)\sin k(\xi-\tilde{d})]p(t)z(t)dt+\int_{\tilde{d}}^{\xi}p(t)\frac{\sin k(\xi-t)}{k}z(t)dt,\quad \tilde{d}<\xi\le A.\end{aligned} \tag{2.14}$$

Furthermore, the solution z to (2.13), (2.14) is an entire function of order $\frac{1}{2}$ with respect to k^2. The characteristic function corresponding to the modified transmission eigenvalue problem is more complicated than that of the classical transmission eigenvalue problem due to the absence of the assumption $n(1)=1$.

The following lemma shows the asymptotic behavior of the characteristic function $\tilde{d}_0(k)$.

Lemma 2.3. *Assume that the refractive index is piecewise C^2 and* (2.5)-(2.7) *hold. Then, the characteristic function $\tilde{d}_0(k)$ satisfies the following asymptotic formula,*

$$\begin{aligned}\tilde{d}_0(k)=\frac{1}{k[n(0)n(1)]^{\frac{1}{4}}}&\left[\frac{1}{4}(\tilde{a}+\tilde{a}^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta)\right.\\&+\frac{1}{4}(\tilde{a}+\tilde{a}^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A-\eta)+\frac{1}{4}(\tilde{a}-\tilde{a}^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta-A+2\tilde{d})\\&\left.-\frac{1}{4}(\tilde{a}-\tilde{a}^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta+A-2\tilde{d})\right]+O\left(\frac{1}{k^2}\right).\end{aligned} \tag{2.15}$$

Proof. By applying the Liouville transformation, we obtain

$$y_0(1)=\frac{1}{n(1)^{\frac{1}{4}}}z(A),\quad y_0'(1)=\frac{dz(\xi)}{d\xi}n(r)^{\frac{1}{2}}n(r)^{-1/4}-\frac{1}{4}z(\xi)n(r)^{-5/4}n'(r)\Big|_{\xi=A,r=1}$$

and from the asymptotic formula for $z(\xi)$ and $\frac{dz(\xi)}{d\xi}$ ([18, Proposition 8]),

$$z(\xi)=\frac{1}{kn(0)^{\frac{1}{4}}}[\tilde{a}\sin k\tilde{d}\cos k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k\tilde{d}\sin k(\xi-\tilde{d})]+O\left(\frac{1}{k^2}\right),$$

$$\frac{dz(\xi)}{d\xi}=\frac{1}{n(0)^{\frac{1}{4}}}[-\tilde{a}\sin k\tilde{d}\sin k(\xi-\tilde{d})+\tilde{a}^{-1}\cos k\tilde{d}\cos k(\xi-\tilde{d})]+O\left(\frac{1}{k}\right).$$

We get the following asymptotic formula for $\tilde{d}_0(k)$:

$$\begin{aligned}\tilde{d}_0(k)&=\cos k\eta\cdot y_0(1)-\frac{\sin k\eta}{k\eta}\cdot y_0'(1)\\&=\frac{1}{k[n(0)n(1)]^{\frac{1}{4}}}\{\tilde{a}\sin k\tilde{d}\cos k(A-\tilde{d})\cos k\eta+\tilde{a}^{-1}\cos k\tilde{d}\sin k(A-\tilde{d})\cos k\eta\\&\cdot\frac{\sqrt{n(1)}}{\eta}[-\tilde{a}\sin k\tilde{d}\sin k(A-\tilde{d})\sin k\eta+\tilde{a}^{-1}\cos k\tilde{d}\cos k(A-\tilde{d})\sin k\eta]\}+O\left(\frac{1}{k^2}\right).\end{aligned}$$

Simplifying the above expression using trigonometric formulas, we obtain (2.15).

Next, we consider the expression of the characteristic function and its asymptotic behavior for the case $L \geq 1$.

Lemma 2.4 ([18])**.** *In case r>d, the solution to* (2.3) *satisfies the asymptotic behavior*

$$y_l(r) = \frac{1}{kn(r)^{\frac{1}{4}}n(0)^{\frac{l}{2}+\frac{1}{4}}}\left[\frac{\tilde{a}+1}{2\tilde{a}}\sin\left(k\xi - \frac{l\pi}{2}\right) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin\left(k\xi + \frac{l\pi}{2} - 2k\tilde{d}\right)\right] + O\left(\frac{\ln k}{k^2}\right).$$

From Lemma 2.4*,* $y_l(1)$ *and* $y_l'(1)$ *can be rewritten as*

$$y_l(1) = \frac{1}{kn(1)^{\frac{1}{4}}n(0)^{\frac{l}{2}+\frac{1}{4}}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin\left(Ak - \frac{l\pi}{2}\right) + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin\left(Ak + \frac{l\pi}{2} - 2k\tilde{d}\right)\right] + O\left(\frac{\ln k}{k^2}\right),$$

$$y_l'(1) = \frac{n(1)^{\frac{1}{2}}}{n(1)^{\frac{1}{4}}n(0)^{\frac{l}{2}+\frac{1}{4}}}\left[\frac{\tilde{a}^2+1}{2\tilde{a}}\cos\left(Ak - \frac{l\pi}{2}\right) + \frac{1-\tilde{a}^2}{2\tilde{a}}\cos\left(Ak + \frac{l\pi}{2} - 2k\tilde{d}\right)\right] + O\left(\frac{\ln k}{k}\right).$$

Substituting this result together with the asymptotic formula of $j_l(k\eta)$ *and* $j_l'(k\eta)$*,*

$$j_l(k\eta) = \frac{1}{k\eta}\cos\left(k\eta - \frac{l\pi}{2} - \frac{\pi}{2}\right)\left(1 + O\left(\frac{1}{k\eta}\right)\right),$$

$$j_l'(k\eta) = \frac{1}{k\eta}\cos\left(k\eta - \frac{l\pi}{2}\right)\left(1 + O\left(\frac{1}{k\eta}\right)\right)$$

to (2.4)*, we have*

$$\begin{aligned}
\tilde{d}_l(k) &= \frac{1}{kn(1)^{\frac{1}{4}}n(0)^{\frac{l}{2}+\frac{1}{4}}}\left\{\left[\frac{n(1)^{\frac{1}{2}}}{\eta}\frac{\tilde{a}^2+1}{2\tilde{a}}\cos\left(Ak - \frac{l\pi}{2}\right)\sin\left(k\eta - \frac{l\pi}{2}\right) + \frac{n(1)^{\frac{1}{2}}}{\eta}\right.\right. \\
&\quad \left. \cdot \frac{1-\tilde{a}^2}{2\tilde{a}}\cos\left(Ak + \frac{l\pi}{2} - 2k\tilde{d}\right)\sin\left(k\eta - \frac{l\pi}{2}\right)\right] + \left[\frac{\tilde{a}^2+1}{2\tilde{a}}\sin\left(Ak - \frac{l\pi}{2}\right)\cos\left(k\eta - \frac{l\pi}{2}\right)\right. \\
&\quad \left.\left. + \frac{1-\tilde{a}^2}{2\tilde{a}}\sin\left(Ak + \frac{l\pi}{2} - 2k\tilde{d}\right)\cos\left(k\eta - \frac{l\pi}{2}\right)\right]\right\} + O\left(\frac{\ln k}{k^2}\right) \\
&= \frac{1}{kn(1)^{\frac{1}{4}}n(0)^{\frac{l}{2}+\frac{1}{4}}}\left[\frac{1}{4}\left(\tilde{a} + \tilde{a}^{-1}\right)\left(1 - \frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA + k\eta - l\pi)\right. \\
&\quad + \frac{1}{4}(\tilde{a} + \tilde{a}^{-1})\left(1 + \frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA - k\eta) + \frac{1-\tilde{a}^2}{4\tilde{a}}\left(1 - \frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA + k\eta - 2k\tilde{d}) \\
&\quad \left. + \frac{1-\tilde{a}^2}{4\tilde{a}}\left(1 + \frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA - k\eta - 2k\tilde{d} + l\pi)\right] + O\left(\frac{\ln k}{k^2}\right).
\end{aligned} \tag{2.16}$$

3. Uniqueness of the Modified Inverse Transmission Eigenvalue Problem and Numerical Examples

3.1 Uniqueness of the Modified Transmission Eigenvalue Problem

We use only the transmission eigenvalues with spherically symmetric eigenfunctions as the knowledge of the modified transmission eigenvalues, i.e., we restrict our attention to the case when $L = 0$.

As seen in the asymptotic formula (2.15), the characteristic function includes A, $n(1)$ and jump constants $\widetilde{\alpha}$, $\widetilde{d}$. So our first goal is to show that these values are uniquely determined by the knowledge of the modified transmission eigenvalues.

Firstly, we prove that A is uniquely determined by the knowledge of the modified transmission eigenvalues.

Let us introduce the notation $\widetilde{D}_0(k) = k[n(0)n(1)]^{\frac{1}{4}}\widetilde{d}_0(k)$. Then we have

$$\begin{aligned}\widetilde{D}_0(k) = &\frac{1}{4}(\widetilde{\alpha}+\widetilde{\alpha}^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta)+\frac{1}{4}(\widetilde{\alpha}+\widetilde{\alpha}^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A-\eta)\\ &+\frac{1}{4}(\widetilde{\alpha}-\widetilde{\alpha}^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta-A+2\widetilde{d})\\ &-\frac{1}{4}(\widetilde{\alpha}-\widetilde{\alpha}^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta+A-2\widetilde{d})+O\left(\frac{1}{k}\right),\end{aligned} \tag{3.1}$$

where $A+\eta > |\eta - A|$, $|\eta - A + 2\widetilde{d}|$, $|\eta + A - 2\widetilde{d}|$. Therefore, by Theorem 2.5 and corollaries in [13], it is straightforward to verify that if $n(1) \neq \eta^2$ and $A \neq \eta$, then the density of all zeros on the right half plane of $\widetilde{D}_0(k)$ is $\frac{A+\eta}{\pi}$. Therefore, A can be uniquely determined by the modified transmission eigenvalues.

Secondly, we prove that $n(1)$ is uniquely determined by the knowledge of the modified transmission eigenvalues.

Since $y(1)$ and $y(1)'$ in the expression of the characteristic function $\widetilde{d}_0(k)$ are entire functions of order $\frac{1}{2}$ with respect to k^2 ([18, Corollary 7]), it implies that the characteristic function $\widetilde{d}_0(k)$ is also an entire function of order $\frac{1}{2}$ with respect to k^2. Hence, we can apply Hadamard's factorization theorem and conclude that

$$\widetilde{d}_0(k) = \widetilde{c}_0 k^2 \prod_{j=1}^{\infty}\left(1-\frac{k^2}{k_{j0}^2}\right). \tag{3.2}$$

Lemma 3.1 ([9])**.** *If $\varphi(x)$ is an entire function which is almost-periodic and bounded on the real line, then each of the limits*

$$\lim_{T\to\infty}\frac{1}{T}\int_a^T \varphi(k)\sin(\alpha k)dk\,,$$
$$\lim_{T\to\infty}\frac{1}{T}\int_a^T \varphi(k)\cos(\alpha k)dk$$

exists for any real α and a fixed constant a.

Lemma 3.2. *If $n(r) < \eta^2$ $(0 \le r \le 1)$, then the modified transmission eigenvalues corresponding to η uniquely determine $n(1)$.*

Proof. We introduce the following notations:

$$\psi(k) = k^3\prod_{j=1}^{\infty}\left(1-\frac{k^2}{k_{j0}^2}\right),$$
$$\gamma_0 = \frac{1}{\widetilde{c}_0[n(0)n(1)]^{\frac{1}{4}}}.$$

Then $\psi(k) = \gamma_0 \widetilde{D}_0(k)$ and from (3.1), we have

$$\psi(k) = \gamma_0[M \sin k(A+\eta) + N \sin k(A-\eta) + P \sin k(\eta - A + 2\widetilde{d}) + Q \sin k(\eta + A - 2\widetilde{d})] + O\left(\frac{1}{k}\right), \tag{3.3}$$

where

$$M = \frac{1}{4}(\widetilde{\alpha} + \widetilde{\alpha}^{-1})\left(1 - \frac{\sqrt{n(1)}}{\eta}\right), \quad N = \frac{1}{4}(\widetilde{\alpha} + \widetilde{\alpha}^{-1})\left(1 + \frac{\sqrt{n(1)}}{\eta}\right),$$

$$P = \frac{1}{4}(\widetilde{\alpha} - \widetilde{\alpha}^{-1})\left(1 + \frac{\sqrt{n(1)}}{\eta}\right), \quad Q = -\frac{1}{4}(\widetilde{\alpha} - \widetilde{\alpha}^{-1})\left(1 - \frac{\sqrt{n(1)}}{\eta}\right).$$

Since A is known by the modified transmission eigenvalues, applying Lemma 3.1 with $a > 0$ sufficiently large, we see that the limits

$$L_1 = \lim_{T\to\infty} \int_a^T \psi(k) \sin k(A+\eta)dk,$$

$$L_2 = \lim_{T\to\infty} \int_a^T \psi(k) \sin k(A-\eta)dk$$

exist and are known. Computing these limits by (3.3), we have

$$L_1 = \frac{\gamma M}{2}, \quad L_2 = \frac{\gamma N}{2}$$

and therefore we have that

$$\frac{L_1}{L_2} = \frac{1 - \sqrt{n(1)}/\eta}{1 + \sqrt{n(1)}/\eta}.$$

Hence, $n(1)$ is determined.

Next, we show that jump constants $\widetilde{\alpha}$, $\widetilde{d}$ of the refractive index can be uniquely determined by the modified transmission eigenvalues.

The characteristic function can be rewritten as

$$\frac{\widetilde{d}_0(k)}{\widetilde{c}_0} = \frac{\gamma_0}{k}\left[\frac{1}{4}(\widetilde{\alpha} + \widetilde{\alpha}^{-1})\left(1 - \frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta) + \frac{1}{4}(\widetilde{\alpha} + \widetilde{\alpha}^{-1})\left(1 + \frac{\sqrt{n(1)}}{\eta}\right)\sin k(A-\eta) + \frac{1}{4}(\widetilde{\alpha} - \widetilde{\alpha}^{-1})\left(1 + \frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta - A + 2\widetilde{d}) - \frac{1}{4}(\widetilde{\alpha} - \widetilde{\alpha}^{-1})\left(1 - \frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta + A - 2\widetilde{d})\right] + O\left(\frac{1}{k^2}\right), \tag{3.4}$$

where A and $n(1)$ are uniquely determined by the transmission eigenvalues. □

Lemma 3.3 ([18]). *Let z be the solution of* (2.12) *and* (2.13) *and $\nu = |\mathrm{Im}\, k|$. Then, for k sufficiently large, there exist some positive constants C and d such that:*

$$\left| z(\xi) - \frac{\sin k\xi}{kn(0)^{\frac{1}{4}}} \right| \le \frac{1}{|k|^2} Ce^{\nu\xi}, \quad 0 \le \xi \le \widetilde{d},$$

$$\left| \frac{dz(\xi)}{d\xi} - \frac{\cos k\xi}{n(0)^{\frac{1}{4}}} \right| \le \frac{1}{|k|} Ce^{\nu\xi}, \quad 0 \le \xi \le \widetilde{d},$$

$$\left|z(\xi)-\frac{1}{kn(0)^{\frac{1}{4}}}[\widetilde{\alpha}\sin k\widetilde{d}\cos k(\xi-\widetilde{d})+\widetilde{\alpha}^{-1}\cos k\widetilde{d}\sin k(\xi-\widetilde{d})]\right|\le\frac{1}{|k|^2}De^{\nu\xi},\qquad \widetilde{d}\le\xi\le A,$$

$$\left|\frac{dz(\xi)}{d\xi}-\frac{1}{n(0)^{\frac{1}{4}}}[-\widetilde{\alpha}\sin k\widetilde{d}\sin k(\xi-\widetilde{d})+\widetilde{\alpha}^{-1}\cos k\widetilde{d}\cos k(\xi-\widetilde{d})]\right|\le\frac{1}{|k|}De^{\nu\xi},\quad \widetilde{d}\le\xi\le A.$$

Lemma 3.4 ([35])**.** *Let $f(z)$ be an entire function such that*

$$|f(z)|\le Ce^{A|z|}$$

for positive constants A and C and all values of z, and

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx<\infty.$$

Then there exists a function $\phi\in L^2[-A,A]$ such that

$$f(z)=\int_{-A}^{A}\phi(t)e^{izt}dt.$$

Lemma 3.5. *Jump constants $\widetilde{\alpha}$, $\widetilde{d}$ can be uniquely determined from the modified transmission eigenvalues.*

Proof. Assume that two transmission eigenvalue problems corresponding to different discontinuous refractive indices $n_1(r)$ and $n_2(r)$ have the same transmission eigenvalues. We denote the characteristic quantities for each problem by $\widetilde{d}_i(k)$, $\widetilde{c}_i$, γ_i, $\widetilde{d}_i$, $\widetilde{\alpha}_i$, M_i, N_i, P_i, Q_i for $i=1,2$, where $\widetilde{d}_i(k)$, $\widetilde{c}_i$, γ_i denote $\widetilde{d}_{0i}(k)$, $\widetilde{c}_{0i}$, γ_{0i}. Since two problems have the same eigenvalues, (3.2) implies that for any $k\in C$,

$$\widetilde{d}_1(k)/\widetilde{c}_1=\widetilde{d}_2(k)/\widetilde{c}_2$$

and from (3.4), we have that

$$\begin{aligned}&\frac{\gamma_1}{k}[M_1\sin k(A+\eta)+N_1\sin k(A-\eta)+P_1\sin k(\eta-A+2\widetilde{d}_1)+Q_1\sin k(\eta+A-2\widetilde{d}_1)]\\&\quad=\frac{\gamma_2}{k}[M_2\sin k(A+\eta)+N_2\sin k(A-\eta)+P_2\sin k(\eta-A+2\widetilde{d}_2)+Q_2\sin k(\eta+A-2\widetilde{d}_2)]\end{aligned}$$

for all sufficiently large $k>0$. Thus, we get

$$\begin{aligned}&\big((\gamma_1M_1-\gamma_2M_2)\sin k(A+\eta)+(\gamma_1N_1-\gamma_2N_2)\sin k(A-\eta)+\gamma_1P_1\sin k(\eta-A+2\widetilde{d}_1)\\&\quad-\gamma_2P_2\sin k(\eta-A+2\widetilde{d}_2)+\gamma_1Q_1\sin k(\eta+A-2\widetilde{d}_1)-\gamma_2Q_2\sin k(\eta+A-2\widetilde{d}_2)\big)=0.\end{aligned}$$

In the case $\widetilde{d}_1=\widetilde{d}_2=\widetilde{d}$,

$$\begin{aligned}&\big((\gamma_1M_1-\gamma_2M_2)\sin k(A+\eta)+(\gamma_1N_1-\gamma_2N_2)\sin k(A-\eta)\\&\quad+(\gamma_1P_1-\gamma_2P_2)\sin k(\eta-A+2\widetilde{d})+(\gamma_1Q_1-\gamma_2Q_2)\sin k(\eta+A-2\widetilde{d})\big)=0.\end{aligned}$$

If $\widetilde{d}\ne\frac{A}{2}$, then $\eta-A+2\widetilde{d}\ne\eta+A-2\widetilde{d}$ and therefore, $\sin k(\eta-A+2\widetilde{d})$ and $\sin k(\eta+A-2\widetilde{d})$ are linearly independent. Since all trigonometric terms are independent, it implies that each coefficient equals zero respectively in order that the equality holds, i.e.,

$$\begin{cases}\gamma_1M_1=\gamma_2M_2,\\\gamma_1N_1=\gamma_2N_2,\\\gamma_1P_1=\gamma_2P_2,\\\gamma_1Q_1=\gamma_2Q_2.\end{cases}\tag{3.5}$$

Eq. (3.5) implies that $\frac{\tilde{a}_1^2+1}{\tilde{a}_1^2-1}=\frac{\tilde{a}_2^2+1}{\tilde{a}_2^2-1}$ and $1+\frac{2}{\tilde{a}_1^2-1}=1+\frac{2}{\tilde{a}_2^2-1}$. Therefore, $\tilde{a}_1=\tilde{a}_2$ and accordingly $\gamma_1=\gamma_2$.

Likewise, if $\tilde{d}=\frac{A}{2}$, then by the linear independency of trigonometric functions

$$\begin{cases}\gamma_1 M_1=\gamma_2 M_2,\\ \gamma_1 N_1=\gamma_2 N_2,\\ (\gamma_1 P_1-\gamma_2 P_2)+(\gamma_1 Q_1-\gamma_2 Q_2)=0.\end{cases} \tag{3.6}$$

From the third of (3.6), we have that

$$\begin{aligned}&\gamma_1(P_1+Q_1)=\gamma_2(P_2+Q_2),\\ &\gamma_1\left[\frac{1}{4}(\tilde{a}_1-\tilde{a}_1^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)-\frac{1}{4}(\tilde{a}_1-\tilde{a}_1^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\right]\\ &\quad=\gamma_2\left[\frac{1}{4}(\tilde{a}_2-\tilde{a}_2^{-1})\left(1+\frac{\sqrt{n(1)}}{\eta}\right)-\frac{1}{4}(\tilde{a}_2-\tilde{a}_2^{-1})\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\right].\end{aligned}$$

By straightforward calculations, we have $\gamma_1(\tilde{a}_1-\tilde{a}_1^{-1})=\gamma_2(\tilde{a}_2-\tilde{a}_2^{-1})$ and combining this with the first of (3.6), we have $\tilde{a}_1=\tilde{a}_2$ and $\gamma_1=\gamma_2$.

In the case $\tilde{d}_1\neq\tilde{d}_2$,

$$\begin{aligned}&\big((\gamma_1 M_1-\gamma_2 M_2)\sin k(A+\eta)+(\gamma_1 N_1-\gamma_2 N_2)\sin k(A-\eta)+\gamma_1 P_1\sin k(\eta-A+2\tilde{d}_1)\\ &-\gamma_2 P_2\sin k(\eta-A+2\tilde{d}_2)+\gamma_1 Q_1\sin k(\eta+A-2\tilde{d}_1)-\gamma_2 Q_2\sin k(\eta+A-2\tilde{d}_2)\big)=0.\end{aligned}$$

. If $A=\tilde{d}_1+\tilde{d}_2$, then

$$\begin{cases}\gamma_1 M_1=\gamma_2 M_2,\\ \gamma_1 N_1=\gamma_2 N_2,\\ \gamma_1 P_1=\gamma_2 Q_2,\\ \gamma_1 Q_1=\gamma_2 P_2.\end{cases}$$

Similar arguments show that $\tilde{a}_1=\tilde{a}_2$ and $\gamma_1=\gamma_2$.

Likewise, in case $A=2\tilde{d}_1$ or $A=2\tilde{d}_2$ or else, we have $\tilde{a}_1=\tilde{a}_2$ and $\gamma_1=\gamma_2$ from

$$\begin{cases}\gamma_1 P_2=0,\\ \gamma_1 Q_2=0,\end{cases}\quad\text{or}\quad\begin{cases}\gamma_1 P_1=0,\\ \gamma_1 Q_1=0,\end{cases}\quad\begin{cases}\gamma_1 P_1=0,\\ \gamma_1 P_2=0.\end{cases}$$

Eq. (2.11) implies that we should obtain $\tilde{b}_1=\tilde{b}_2=0$ in order to get a contradiction.

Since $\tilde{a}_1=\tilde{a}_2=1$, using Liouville transformation formula together with (2.14) and the assumption $n'(1)=0$, we have the following characteristic function:

$$\begin{aligned}\tilde{d}_0(k)=&\frac{1}{k[n(0)n(1)]^{\frac{1}{4}}}\left[-\frac{1}{2}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta)+\frac{1}{2}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta-A)\right]\\ &+\frac{\tilde{b}}{k^2[n(0)n(1)]^{\frac{1}{4}}}\left[\frac{\sqrt{n(1)}}{\eta}\sin k\tilde{d}\cos k(A-\tilde{d})\sin k\eta-\sin k\tilde{d}\sin k(A-\tilde{d})\cos k\eta\right]\\ &+\frac{1}{kn(1)^{\frac{1}{4}}}\int_0^{\tilde{d}}\Big[-\frac{\sqrt{n(1)}}{\eta}\sin k(\tilde{d}-t)\sin k(A-\tilde{d})\sin k\eta+\frac{\sqrt{n(1)}}{\eta}\cos k(\tilde{d}-t)\cos k(A-\tilde{d})\sin k\eta\\ &-\sin k(\tilde{d}-t)\cos k(A-\tilde{d})\cos k\eta-\cos k(\tilde{d}-t)\sin k(A-\tilde{d})\cos k\eta\Big]p(t)z(t)dt\end{aligned}$$

$$+\frac{\widetilde{b}}{k^2 n(1)^{\frac{1}{4}}}\int_0^{\widetilde{d}}[\frac{\sqrt{n(1)}}{\eta}\sin k(\widetilde{d}-t)\cos k(A-\widetilde{d})\sin k\eta-\sin k(\widetilde{d}-t)\sin k(A-\widetilde{d})\cos k\eta]$$

$$\cdot p(t)z(t)dt+\frac{1}{kn(1)^{\frac{1}{4}}}\int_{\widetilde{d}}^{A}[\frac{\sqrt{n(1)}}{\eta}\cos k(A-t)\sin k\eta-\sin k(A-t)\cos k\eta]p(t)z(t)dt. \tag{3.7}$$

Simplifying the second term yields

$$\frac{\widetilde{b}}{k^2[n(0)n(1)]^{\frac{1}{4}}}\left[\frac{\sqrt{n(1)}}{\eta}\sin k\widetilde{d}\cos k(A-\widetilde{d})\sin k\eta-\sin k\widetilde{d}\sin k(A-\widetilde{d})\cos k\eta\right]$$

$$=\frac{\widetilde{b}}{k^2[n(0)n(1)]^{\frac{1}{4}}}\left[\frac{1}{4}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A-\eta)+\frac{1}{4}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A+\eta)\right.$$

$$\left.-\frac{1}{4}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A+2\widetilde{d})-\frac{1}{4}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A-2\widetilde{d})\right].$$

Hence, we can rewrite (3.7) as

$$\widetilde{d}_0(k)=\frac{1}{k[n(0)n(1)]^{\frac{1}{4}}}\left[-\frac{1}{2}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta)+\frac{1}{2}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(\eta-A)\right]$$

$$+\frac{\widetilde{b}}{k^2[n(0)n(1)]^{\frac{1}{4}}}\left[\frac{1}{4}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A-\eta)+\frac{1}{4}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A+\eta)\right.$$

$$\left.-\frac{1}{4}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A+2\widetilde{d})-\frac{1}{4}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A-2\widetilde{d})\right]$$

$$+\frac{\sin k\eta}{k^2[n(0)n(1)]^{\frac{1}{4}}}\int_0^A\frac{\sqrt{n(1)}}{\eta}\cos k(A-t)\sin kt\cdot p(t)dt$$

$$-\frac{\cos k\eta}{k^2[n(0)n(1)]^{\frac{1}{4}}}\int_0^A\sin k(A-t)\sin kt\cdot p(t)dt+\frac{E(k)}{k^2 n(1)^{\frac{1}{4}}}, \tag{3.8}$$

where

$$E(k)=k\sin k\eta\int_0^{\widetilde{d}}\frac{\sqrt{n(1)}}{\eta}\cos k(A-t)p(t)\left[z(t)-\frac{\sin kt}{kn(0)^{\frac{1}{4}}}\right]dt$$

$$+k\sin k\eta\int_{\widetilde{d}}^{A}\frac{\sqrt{n(1)}}{\eta}\cos k(A-t)p(t)\left[z(t)-\frac{\sin kt}{kn(0)^{\frac{1}{4}}}\right]dt$$

$$+\widetilde{b}\sin k\eta\int_0^{\widetilde{d}}\frac{\sqrt{n(1)}}{\eta}\cos k(A-\widetilde{d})\sin k(\widetilde{d}-t)p(t)z(t)dt$$

$$-k\cos k\eta\int_0^{\widetilde{d}}\sin k(A-t)p(t)\left[z(t)-\frac{\sin kt}{kn(0)^{\frac{1}{4}}}\right]dt$$

$$-k\cos k\eta\int_{\widetilde{d}}^{A}\sin k(A-t)p(t)\left[z(t)-\frac{\sin kt}{kn(0)^{\frac{1}{4}}}\right]dt$$

$$-\widetilde{b}\cos k\eta\int_0^{\widetilde{d}}\sin k(A-\widetilde{d})\sin k(\widetilde{d}-t)p(t)z(t)dt. \tag{3.9}$$

Note that $E(k)$ is an even function of k and if k is a real number, then $E(k)$ is also a real valued function.

Moreover, by lemma 3.3 we get

$$\int_{-\infty}^{\infty} |E(x)|^2 < \infty,\ |E(x)| \le Ce^{(A+\eta)|\mathrm{Im}\,k|}, \quad C>0.$$

Thus, by Lemma 3.4, there exists $V \in L^2(0,A+\eta)$ such that

$$E(k) = \int_0^{A+\eta} V(t)\cos kt\,dt.$$

On the other hand,

$$\begin{aligned}
&\frac{\sin k\eta}{k^2[n(0)n(1)]^{\frac14}}\int_0^A \frac{\sqrt{n(1)}}{\eta}\cos k(A-t)\sin kt\,p(t)dt - \frac{\cos k\eta}{k^2[n(0)n(1)]^{\frac14}}\int_0^A \sin k(A-t)\sin kt\,p(t)dt\\
&= \frac{1}{4k^2[n(0)n(1)]^{\frac14}}\left[\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A) + \left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A)\right]\int_0^A p(t)dt\\
&\quad - \frac{\cos k\eta}{2k^2[n(0)n(1)]^{\frac14}}\int_0^A \cos k(A-2t)p(t)dt - \frac{\sin k\eta}{2k^2[n(0)n(1)]^{\frac14}}\frac{\sqrt{n(1)}}{\eta}\int_0^A \sin k(A-2t)p(t)dt.
\end{aligned}$$

Now, applying variable transformation to the last two terms of the above equation, we can rewrite the integrals as

$$\int_0^A \sin k(A-2t)p(t)dt = \int_0^A W(t)\sin kt\,dt,$$

$$\int_0^A \cos k(A-2t)p(t)dt = \int_0^A U(t)\cos kt\,dt,$$

where

$$U(t) := \frac12\left[p\left(\frac{A+t}{2}\right) - p\left(\frac{A-t}{2}\right)\right],\quad W(t) := -\frac12\left[p\left(\frac{A+t}{2}\right) - p\left(\frac{A-t}{2}\right)\right].$$

Thus,

$$\begin{aligned}
\widetilde{d}_0(k) &= \frac{1}{k[n(0)n(1)]^{\frac14}}\left[-\frac12\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A+\eta) + \frac12\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\sin k(A-\eta)\right]\\
&\quad + \frac{\widetilde{b}}{k^2[n(0)n(1)]^{\frac14}}\left[\frac14\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A-\eta) + \frac14\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A+\eta)\right.\\
&\quad \left. -\frac14\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A+2\widetilde{d}) - \frac14\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A-2\widetilde{d})\right]\\
&\quad + \frac{1}{4k^2[n(0)n(1)]^{\frac14}}\left[\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A) + \left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A)\right]\int_0^A p(t)dt\\
&\quad - \frac{\cos k\eta}{2k^2[n(0)n(1)]^{\frac14}}\int_0^A U(t)\cos kt\,dt - \frac{\sin k\eta}{2k^2[n(0)n(1)]^{\frac14}}\frac{\sqrt{n(1)}}{\eta}\int_0^A W(t)\sin kt\,dt\\
&\quad + \frac{1}{k^2 n(1)^{\frac14}}\int_0^{A+\eta} V(t)\cos kt\,dt. \qquad (3.10)
\end{aligned}$$

Since $\gamma_1 = \gamma_2$ and $\widetilde{d}_1(k)/\widetilde{c}_1 = \widetilde{d}_2(k)/\widetilde{c}_2$, it can be seen that

$$\widetilde{d}_1(k) = \frac{n_2(0)^{\frac14}}{n_1(0)^{\frac14}}\widetilde{d}_2(k).$$

Therefore, with the aid of (3.10), we have

$$\begin{aligned}
&\frac{(\widetilde{b}_1-\widetilde{b}_2)}{4k^2[n_1(0)n(1)]^{\frac{1}{4}}}\left[\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A-\eta)+\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(A+\eta)\right]\\
&\quad-\frac{1}{4k^2[n_1(0)n(1)]^{\frac{1}{4}}}\left(1+\frac{\sqrt{n(1)}}{\eta}\right)[\widetilde{b}_1\cos k(\eta-A+2\widetilde{d}_1)-\widetilde{b}_2\cos k(\eta-A+2\widetilde{d}_2)]\\
&\quad-\frac{1}{4k^2[n_1(0)n(1)]^{\frac{1}{4}}}\left(1-\frac{\sqrt{n(1)}}{\eta}\right)[\widetilde{b}_1\cos k(\eta+A-2\widetilde{d}_1)-\widetilde{b}_2\cos k(\eta+A-2\widetilde{d}_2)]\\
&\quad+\frac{1}{4k^2[n_1(0)n(1)]^{\frac{1}{4}}}\left[\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta-A)+\left(1-\frac{\sqrt{n(1)}}{\eta}\right)\cos k(\eta+A)\right]\\
&\quad\cdot\int_0^A(p_1(t)-p_2(t))dt\\
&=\frac{\cos k\eta}{2k^2[n_1(0)n(1)]^{\frac{1}{4}}}\int_0^A(U_1(t)-U_2(t))\cos kt\,dt\\
&\quad+\frac{\sin k\eta}{2k^2[n_1(0)n(1)]^{\frac{1}{4}}}\frac{\sqrt{n(1)}}{\eta}\int_0^A(W_1(t)-W_2(t))\sin kt\,dt\\
&\quad+\frac{1}{k^2n(1)^{\frac{1}{4}}}\int_0^{A+\eta}\left(\frac{n_2(0)^{\frac{1}{4}}}{n_1(0)^{\frac{1}{4}}}V_2(t)-V_1(t)\right)\cos kt\,dt.
\end{aligned}\tag{3.11}$$

Multiplying both sides of (3.11) by $2k^2[n_1(0)n(1)]^{\frac{1}{4}}\cos k(\eta-A+2\widetilde{d}_1)T^{-1}$ and integrating with respect to k from τ to T yields

$$\begin{aligned}
&(\widetilde{b}_1-\widetilde{b}_2)O\left(\frac{1}{T}\right)+\left[\int_0^A(p_1(t)-p_2(t))dt\right]O\left(\frac{1}{T}\right)-\widetilde{b}_1\left(1+\frac{\sqrt{n(1)}}{\eta}\right)\left(\frac{1}{4}+O\left(\frac{1}{T}\right)\right)+\widetilde{b}_2O\left(\frac{1}{T}\right)\\
&=\frac{\sqrt{n(1)}}{\eta}\int_0^A\left[(W_1(t)-W_2(t))\int_\tau^T\frac{1}{T}\sin k\eta\cos k(\eta-A+2\widetilde{d}_1)\sin kt\,dk\right]dt\\
&\quad+\int_0^A\left[(U_1(t)-U_2(t))\int_\tau^T\frac{1}{T}\cos k\eta\cos k(\eta-A+2\widetilde{d}_1)\cos kt\,dk\right]dt\\
&\quad+2\int_0^{A+\eta}\left[(n_2(0)^{\frac{1}{4}}V_2(t)-n_1(0)^{\frac{1}{4}}V_1(t))\int_\tau^T\frac{1}{T}\cos k(\eta-A+2\widetilde{d}_1)\cos kt\right]dk.
\end{aligned}\tag{3.12}$$

We can rewrite the right hand side of (3.12) as

$$\int_0^A(W_1(t)-W_2(t))f_T\,dt+\int_0^A(U_1(t)-U_2(t))h_T\,dt+2\int_0^{A+\eta}(n_2(0)^{\frac{1}{4}}V_2(t)-n_1(0)^{\frac{1}{4}}V_1(t))g_T\,dt,$$

where $|f_T|,|h_T|,|g_T|\le 1$, f_T,h_T,g_T tends to zero as $T\to\infty$ almost everywhere. Thus, we have $\widetilde{b}_1=0$.

Similarly, multiplying both sides of (3.12) by $2k^2[n_1(0)n(1)]^{\frac{1}{4}}\cos k(\eta-A+2\widetilde{d}_2)T^{-1}$ and integrating with respect to k from τ to T yields $\widetilde{b}_2=0$.

This is a contradiction to (2.11). Hence it can be concluded that $\widetilde{d}_1=\widetilde{d}_2$.

Finally, we prove that the refractive index can be uniquely reconstructed from the modified transmission eigenvalues.

Let us find the solution to the first differential equation of the modified transmission eigenvalue problem (2.1) in the form $\widetilde{w}(r,\theta) = b_l y_l(r) P_l(\cos\theta)$ in case $n(r)$ is twice continuously differentiable. Then, $y_l(r)$ satisfies

$$y_l''(r) + \frac{2}{r} y_l'(r) + (k^2 n(r) - \frac{l(l+1)}{r}) y_l(r) = 0 \tag{3.13}$$

and can be written in the form ([10, 20]),

$$y_l(r) = j_l(k\eta r) + \int_0^r G(r,s,k) j_l(k\eta s) ds, \tag{3.14}$$

where $j_l(k\eta r)$ is a Bessel function of order L and is the solution to $y_l''(r) + \frac{2}{r} y_l'(r) + k^2\eta^2 y_l(r) = 0$. Furthermore, recalling the results from [12, 18], a similar argument shows that in case $n(r)$ is discontinuous, the integral kernel $G(r,S,k)$ also satisfies

$$r^2 \left[\frac{\partial^2 G}{\partial r^2} + \frac{2}{r} \frac{\partial G}{\partial r} + k^2 n(r) G \right] = s^2 \left[\frac{\partial^2 G}{\partial s^2} + \frac{2}{s} \frac{\partial G}{\partial s} + k^2 \eta^2 G \right], \tag{3.15}$$

$$G(r,r,k) = \frac{k^2}{2r} \int_0^r t m(t) dt, \tag{3.16}$$

$$G(r,s,k) = O((rs)^{\frac{1}{2}}), \quad 0 < s \le r < 1 \tag{3.17}$$

and integral equation

$$G(r,s,k) = \frac{1}{2} \frac{k^2}{\sqrt{rs}} \int_0^{\sqrt{rs}} t m(t) dt - \frac{k^2}{\sqrt{rs}} \int_1^{\sqrt{\frac{r}{s}}} \int_0^{\sqrt{rs}} t^2 \tau \left[n(t\tau) - \frac{\eta^2}{\tau^4} \right] G(t\tau, t/\tau, k) dt d\tau, \tag{3.18}$$

where $m(r) = \eta^2 - n(r)$.

Furthermore, G is an entire function of k, of exponential type and satisfies

$$G(r,s,k) = \frac{1}{2} \frac{k^2}{\sqrt{rs}} \int_0^{\sqrt{rs}} t m(t) dt (1 + O(k^2)). \tag{3.19}$$

Substituting (3.19) to (3,14) and again substituting (3.14) to the characteristic function $\widetilde{d}_l(k)$ and using the asymptotic behavior of spherical Bessel function j_l, similar arguments with [3] show that the coefficient $\widetilde{c}_{2l+2}$ of k^{2l+2} in the Taylor expansion is determined as

$$\widetilde{c}_{2l+2} = \frac{\pi}{(2^{l+1}\Gamma(l+3/2))^2} \int_0^1 t^{2l+2} m(t) dt. \tag{3.20}$$

On the other hand, from the asymptotic formula (2.16) of the characteristic function $\widetilde{d}_l(k)$, it can be seen that $\widetilde{d}_l(k)$ is an entire function of order l with respect to k and therefore, by Hadamard's factorization theorem it can be rewritten as

$$\widetilde{d}_l(k) = k^{2l+2} \widetilde{c}_{2l+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{nl}^2} \right).$$

Now, we define a constant

$$\gamma_l = \frac{1}{\widetilde{c}_{2l+2} n(0)^{\frac{l}{2}+\frac{1}{4}}} \tag{3.21}$$

and show that it can be uniquely determined by the knowledge of the modified transmission eigenvalues. □

Lemma 3.6. *The constant γ_l is uniquely determined by the modified transmission eigenvalues.*

Proof. If k is sufficiently large, from the asymptotic formula (2.16) of $\widetilde{d}_l(k)$, we have that

$$\frac{\widetilde{d}_l(k)}{\widetilde{c}_{2l+2}} = \frac{\gamma_l}{kn(1)^{\frac{1}{4}}}\left[\frac{1}{4}(\widetilde{\alpha}+\widetilde{\alpha}^{-1})\left(1-\frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA+k\eta-l\pi)+\frac{1}{4}(\widetilde{\alpha}+\widetilde{\alpha}^{-1})\left(1+\frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA-k\eta)\right.$$
$$\left.+\frac{1-\widetilde{\alpha}^2}{4\widetilde{\alpha}}\left(1-\frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA+k\eta-2k\widetilde{d})+\frac{1-\widetilde{\alpha}^2}{4\widetilde{\alpha}}\left(1+\frac{n(1)^{\frac{1}{2}}}{\eta}\right)\sin(kA-k\eta-2k\widetilde{d}+l\pi)\right].$$

We follow similar arguments with the proof of Lemma 3.5 and assume that two transmission eigenvalue problems corresponding to the discontinuous refractive indices $n_1(r)$ and $n_2(r)$ respectively, have the same transmission eigenvalues. Let us denote the coefficients of trigonometric functions $\sin(kA+k\eta-l\pi)$, $\sin(kA-k\eta)$, $\sin(kA+k\eta-2k\widetilde{d})$ and $\sin(kA-k\eta-2k\widetilde{d}+l\pi)$ by M_i, N_i, P_i, Q_i $(i=1,2)$. Then, we have

$$\frac{\gamma_{l1}}{k}[M_1\sin(kA+k\eta-l\pi)+N_1\sin(kA-k\eta)+P_1\sin(kA+k\eta-2k\widetilde{d})+Q_1\sin(kA-k\eta-2k\widetilde{d}+l\pi)]$$
$$=\frac{\gamma_{l2}}{k}[M_2\sin(kA+k\eta-l\pi)+N_2\sin(kA-k\eta)$$
$$+P_2\sin(kA+k\eta-2k\widetilde{d})+Q_2\sin(kA-k\eta-2k\widetilde{d}+l\pi)].$$

Here, we took into account the fact that $n(1)$, A, $\widetilde{\alpha}$ and $\widetilde{d}$ are uniquely determined.

Thus, we have

$$(\gamma_{l1}M_1-\gamma_{l2}M_2)\sin(kA+k\eta-l\pi)+(\gamma_{l1}N_1-\gamma_{l2}N_2)\sin(kA-k\eta)$$
$$+(\gamma_{l1}P_1-\gamma_{l2}P_2)\sin(kA+k\eta-2k\widetilde{d})+(\gamma_{l1}Q_1-\gamma_{l2}Q_2)\sin(kA-k\eta-2k\widetilde{d}+l\pi)]$$
$$=(-1)^l(\gamma_{l1}M_1-\gamma_{l2}M_2)\sin(kA+k\eta)+(\gamma_{l1}N_1-\gamma_{l2}N_2)\sin(kA-k\eta)$$
$$+(\gamma_{l1}P_1-\gamma_{l2}P_2)\sin(kA+k\eta-2k\widetilde{d})+(-1)^l(\gamma_{l1}Q_1-\gamma_{l2}Q_2)\sin(kA-k\eta-2k\widetilde{d})=0.$$

Since $\sin(kA+k\eta)$, $\sin(kA-k\eta)$, $\sin(kA+k\eta-2k\widetilde{d})$ and $\sin(kA-k\eta-2k\widetilde{d})$ are linearly independent for sufficiently large $\eta>0$, we have that

$$\begin{cases}(-1)^l(\gamma_{l1}M_1-\gamma_{l2}M_2)=0,\\ \gamma_{l1}N_1-\gamma_{l2}N_2=0,\\ \gamma_{l1}P_1-\gamma_{l2}P_2=0,\\ (-1)^l(\gamma_{l1}Q_1-\gamma_{l2}Q_2)=0.\end{cases}$$

Thus, we can conclude that $\gamma_{l1}=\gamma_{l2}$. □

Theorem 3.1. *Suppose that the refractive index $n(r)$ is a piecewise C^2 function satisfying* (2.5)-(2.7) *and $n'(1)=0$. If $n(0)$ is known, then $n(r)$ is uniquely determined by the knowledge of all modified transmission eigenvalues counting multiplicity.*

Proof. Assume that two refractive indices $n_1(r)$ and $n_2(r)$ satisfying $n_1(0)=n_2(0)=n(0)$ have the same corresponding modified transmission eigenvalues. From (3.20), the following equality holds:

$$\int_0^1 t^{2l+2}m_i(t)dt=\frac{\left(2^{l+1}\Gamma\left(l+\frac{3}{2}\right)\right)^2}{n(0)^{\frac{l}{2}+\frac{1}{4}}\gamma_{l_i}\pi},\quad i=1,2. \tag{3.22}$$

Clearly, $m_i(r) = \eta^2 - n_i(r) > 0$ and by Lemmas 3.2 and 3.6, $n(1)$ and

$$\gamma_l = \frac{1}{c_{2l+2} n(0)^{\frac{l}{2}+\frac{1}{4}}}$$

are uniquely determined by the knowledge of the modified transmission eigenvalues. Furthermore, as $n(0)$ is already known, the right hand side of (3.22) is uniquely determined. Hence, taking into account the proof of Theorem 16 in [18], we can conclude that $n_1(r) = n_2(r)$. □

3.2 Numerical Examples

Using similar arguments with [17], it can easily be seen that the modified transmission eigenvalue problem (2.1) is equivalent to the boundary value problem

$$(\Delta + k^2 n(x)) \frac{1}{\eta^2 - n(x)} (\Delta + k^2 \eta^2) u = 0, \quad u = w - v \in H_0^2(D)$$

and the corresponding variational problem is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{\eta^2 - n(x)} (\Delta u + k^2 \eta^2 u)(\Delta \bar{\phi} + k^2 n(x) \bar{\phi}) = 0, \quad \forall\ \phi \in H_0^2(D). \tag{3.23}$$

For the numerical experiment, we use the Galerkin method as in [17] to seek the weak solution of the modified transmission eigenvalue problem.

Let $\{\phi_i\}_i^\infty$ be a set of eigenfunctions of the problem

$$\left.\begin{aligned} &L\phi_i(x) = \mu_i \phi_i(x), && x \in D, \\ &\phi_i(x) = 0,\ \tfrac{\partial \phi_i(x)}{\partial \nu} = 0, && x \in \partial D, \end{aligned}\right\} \tag{3.24}$$

where $L = \Delta\Delta$.

Eigenfunctions can be easily computed and form a Hilbert basis in $H_0^2(D)$.

Then, the weak solution of the modified transmission eigenvalue problem, i.e., the solution u to the variational form (3.23) can be approximated as

$$u_k^{(N)} = \sum_{i=1}^{N} c_i \phi_i. \tag{3.25}$$

We substitute $u_k^{(N)}$ into (3.23) and use the eigenfunctions ϕ_i, $i = 1, \dots, n$, as test functions.

Then, the approximate nonlinear eigenvalue problem is written in the following matrix form

$$[A^{(N)} - (k^{(N)})^2 B^{(N)} + (k^{(N)})^4 C^{(N)}]c = 0, \tag{3.26}$$

where

$$A^{(N)} = \int_D \frac{1}{\eta^2 - n(x)} \Delta\phi_i \Delta\bar{\phi}_j dx, \tag{3.27}$$

$$B^{(N)} = -\left(\int_D \frac{n(x)}{\eta^2 - n(x)} \Delta\phi_i \bar{\phi}_j dx + \int_D \frac{\eta^2}{\eta^2 - n(x)} \phi_i \Delta\bar{\phi}_j dx \right), \tag{3.28}$$

$$C^{(N)} = \int_D \frac{\eta^2 n(x)}{\eta^2 - n(x)} \phi_i \bar{\phi}_j dx \tag{3.29}$$

are $N \times N$ matrices and $c = (c_1, c_2, \dots, c_N)^T$, $i, j = 1, \dots, n$.

In our numerical experiments we show that the refractive index can be uniquely determined by the modified transmission eigenvalues without assuming that the contrast does not change sign.

We consider the case where the refractive index is constant and piecewise constant.

First, let d be a circular domain of radius R with constant refractive index $n(x) = n$. It is well known that n is uniquely determined by the knowledge of only the lowest positive transmission eigenvalue provided that it is known a priori that either $n > 1$ or $0 < n < 1$ [11].

Accordingly, we show through numerical experiment that in a circular domain with constant refractive index, we can recover the refractive index from the lowest positive modified transmission eigenvalue without its prior knowledge.

The lowest positive modified transmission eigenvalue can be computed analytically. By similar arguments with [17], we see that the lowest positive modified transmission eigenvalue is the lowest positive solution to

$$\det\begin{pmatrix} J_m(k\eta R) & J_m(k\sqrt{n}R) \\ J'_m(k\eta R) & J'_m(k\sqrt{n}R) \end{pmatrix} = 0, \quad m = 0, 1, \dots, \tag{3.30}$$

where J_m are Bessel functions of the first kind. This relation can be derived easily from separation of variables for Helmholtz equation.

We construct a basis $\{\phi_i\}_{i=1}^N$ with the eigenfunctions of (3.24) in order to approximate the lowest positive modified transmission eigenvalue by (3.26). In polar coordinates, the eigenfunctions for one eigenvalue μ are linear combinations of

$$J_i(\mu r)\cos i\theta,\ J_i(\mu r)\sin i\theta,\ I_i(\mu r)\cos i\theta,\ I_i(\mu r)\sin i\theta.$$

The eigenvalues μ can be computed from the relation:

$$\det\begin{pmatrix} J_i(kR) & J'_i(kR) \\ I_i(kR) & I'_i(kR) \end{pmatrix} = 0, \quad i = 0, 1, \dots.$$

We construct a basis with 12 eigenfunctions $\{\phi_i\}_{i=1}^{12}$ and compute the 12×12 matrices $A^{(N)}, B^{(N)}, C^{(N)}$ for $r = 1$ in (3.26).

Then, we use the MATLAB function `polyeig` to solve the eigenvalue problem (3.26) for different values of n in the interval $(0, 20]$.

We estimate n by minimizing $|k_0^{(N)} - k_0|$, the absolute difference between the lowest positive solution k_0 to (3.30) and the approximation $k_0^{(N)}$ of (3.26).

The results are shown in Table 1.

Table 1 illustrates that the constant refractive index can be reconstructed from the modified transmission eigenvalues without prior knowledge of the refractive index.

Next, let d be a circular domain of radius R with piecewise constant refractive index

$$n(x) = \begin{pmatrix} n_1, & x \in D_1, \\ n_2, & x \in D_2. \end{pmatrix}$$

We use four modified transmission eigenvalues for the numerical experiment of reconstruction.

Table 1. Reconstructions for constant refractive index

Original n	Parameter η	Eigenvalue k_0	Approximation $k_0^{(N)}$	Estimated n
0.4	3	1.3095	1.3097	0.39
0.7	3	1.3405	1.3407	0.69
3	5	0.8372	0.8371	3
6	10	0.3969	0.3969	6
10	10	0.4099	0.4099	10
12	10	0.4186	0.4186	12
20	15	0.2706	0.2706	20

These transmission eigenvalues can be computed analytically from the equation:

$$\det\begin{pmatrix} J_m(k\eta R) & 0 & J_m(k\sqrt{n_2}R) & N_m(k\sqrt{n_2}R) \\ J'_m(k\eta r)|_{r=R} & 0 & J'_m(k\sqrt{n_2}r)|_{r=R} & N'_m(k\sqrt{n_2}r)|_{r=R} \\ 0 & J_m(k\sqrt{n_1}r_1) & J_m(k\sqrt{n_2}r_1) & N_m(k\sqrt{n_2}r_1) \\ 0 & J'_m(k\sqrt{n_1}r)|_{r=r_1} & J'_m(k\sqrt{n_2}r)|_{r=r_1} & N'_m(k\sqrt{n_2}r)|_{r=r_1} \end{pmatrix} = 0. \quad (3.31)$$

This relation is analogous with (3.30). Analogously to the case of constant refractive index, we estimate n_1, n_2 and r_1 by minimizing the sum of absolute differences between analytically computed modified transmission eigenvalue and approximated transmission eigenvalues when varying n_1 in $(0,1)$, n_2 in $[2,20]$ and r_1 in $(0,1)$, respectively. The results are shown in Table 2.

Table 2. Reconstructions of piecewise constant refractive index

Original n_1, n_2, r_1	Parameter η	Eigenvalues k_0, k_1, k_2, k_3	Approximations $\tilde{k}_0$, $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$	Estimated $\tilde{n}_1, \tilde{n}_2, \tilde{r}_1$
13, 5, 0.5	10	0.4017, 0.5216, 0.6439, 0.7640	0.4027, 0.5235, 0.6460, 0.7662	13, 4.5, 0.5
10, 8, 0.4	10	0.4043, 0.5257 0.6477, 0.7673	0.4043, 0.5265 0.6489, 0.7689	10,7.5, 0.5
0.5, 3, 0.5	10	0.3880,0.7111, 0.5174,0.6413	0.3881, 0.5183, 0.6425, 0.7135	0.4, 2.5, 0.6
0.6, 5, 0.6	10	0.3901,0.5196, 0.6435, 0.7156	0.3892, 0.5199 0.6444, 0.7156	0.8, 4.8, 0.7

Table 2 illustrates that the piecewise constant refractive index can be reconstructed from the modified transmission eigenvalues without prior knowledge of the refractive index.

Finally, we assume that d is a spherically stratified domain with k-layers such that $D = \bigcup_{i=1}^k D_i$ and $\{\partial D_i\}_{i=1}^k$ are concentric circles. The refractive index is given by

$$n(x) = \begin{cases} n_1, & x \in D_1, \\ \vdots \\ n_k, & x \in D_k. \end{cases}$$

The transmission eigenvalues are the zeros of the determinant of a $2k \times 2k$ matrix, analogous to (3.31). With similar discussions to [17], we approximate the transmission eigenvalues using a Newton method.

We can write the $N \times N$ matrices $A^{(N)}$, $B^{(N)}$ and $C^{(N)}$ in the following form:

$$A^{(N)} = \sum_{l=1}^{k} \frac{1}{\eta^2 - n_l} A_l , \tag{3.32}$$

$$B^{(N)} = \sum_{l=1}^{k} \frac{n_l}{\eta^2 - n_l} B_l^{(1)} + \sum_{l=1}^{k} \frac{\eta^2}{\eta^2 - n_l} B_l^{(2)} , \tag{3.33}$$

$$C^{(N)} = \sum_{l=1}^{k} \frac{n_l \eta^2}{\eta^2 - n_l} C_l , \tag{3.34}$$

where $A_l = \int_{D_l} \Delta\phi_i \Delta\bar{\phi}_j dx$, $B_l^{(1)} = \int_{D_l} \Delta\phi_i \bar{\phi}_j dx$, $B_l^{(2)} = \int_{D_l} \phi_i \Delta\bar{\phi}_j dx$ and $C_l = \int_{D_l} \phi_i \bar{\phi}_j dx$ for $i,j = 1,\dots n$, $L = 1,\dots,k$.

If we set $a_l := 1/(\eta^2 - n_l)$, then (3.32)-(3.34) can be rewritten as

$$A^{(N)} = \sum_{l=1}^{k} a_l A_l , \tag{3.35}$$

$$B^{(N)} = -\sum_{l=1}^{k} B_l^{(1)} + \sum_{l=1}^{k} \eta^2 a_l (B_l^{(1)} + B_l^{(2)}) , \tag{3.36}$$

$$C^{(N)} = \sum_{l=1}^{k} (\eta^4 a_l - \eta^2) C_l . \tag{3.37}$$

Now the inverse transmission eigenvalue problem has the following form: given a set of modified transmission eigenvalues $S = \{\mu_i\}_{i=1}^k$, find scalars $\{a_l\}_{l=1}^k$ such that the pencil $P(\lambda) = \lambda^4 C^{(N)} + \lambda^2 B^{(N)} + A^{(N)}$ has spectrum $\sigma(A^{(N)}, B^{(N)}, C^{(N)}) = S$.

We denote the set of unknown coefficients by $a = (a_1, a_2, \dots, a_k)$.

We solve the nonlinear system $f(a) := (f_1(a), \dots, f_k(a))^T = (0, \dots, 0)^T$ using the Newton method where

$$f_i(a) = \det \left[\mu_i^4 \sum_{l=1}^{k} (\eta^4 a_l - \eta^2) C_l + \mu_i^2 \left(-\sum_{l=1}^{k} B_l^{(1)} + \sum_{l=1}^{k} \eta^2 a_l (B_l^{(1)} + B_l^{(2)}) \right) + \sum_{l=1}^{k} a_l A_l \right].$$

The numerical algorithm of inverse transmission eigenvalue problem based on the Newton method is constructed analogously to [17].

In this paper, we have tested the algorithm for the simple case of spherically stratified domain with two layers.

In case of unit disc with $n_1 = 5$, $n_2 = 8$ and inner radius $r_1 = 0.6$, given as initial estimate for the indices, the mean value 6.5 was reconstructed as $n_1 = 5.000$, $n_2 = 8.000$ after 7 iterations with tolerance 10^{-12}.

In case of unit disc with $n_1 = 12$, $n_2 = 6$ and inner radius $r_1 = 0.8$, given as initial estimate for the indices, the mean value 9.5 was reconstructed as $n_1 = 12.000$, $n_2 = 6.000$ after 9 iterations with tolerance 10^{-12}.

4. Conclusion

In this paper, we have studied the transmission eigenvalue problem and its inverse problem which are of great practical importance such as Radar, sonar, geophysical exploration, medical imaging and non-destructive test. First, we have formulated the modified transmission eigenvalue problem in case of discontinuous refractive index and estimated the asymptotic behavior of the characteristic function. Second, we have proved the uniqueness of the inverse transmission eigenvalue problem of reconstructing the refractive index and its discontinuous positions from the knowledge of the modified transmission eigenvalues. Our numerical examples indicated the validity of our theoretical results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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