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Research Article

Nazarov Uncertainty Principle for Certain Lie Groups

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Abstract. Nazarov uncertainty principle is established for the Fourier transform and the continuous modulated shearlet transform on the groups of the form $\mathbb{R}^n \times K$, where K is a locally compact group. As special cases, Nazarov uncertainty principle follows for the Gabor transform, the shearlet transform and the wavelet transform on these groups.

Keywords. Fourier transform, Continuous modulated shearlet transform, Nazarov uncertainty principle

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1. Introduction

The classical uncertainty principles, such as those of Heisenberg and Nazarov, form a cornerstone of harmonic analysis and signal theory, capturing fundamental limitations on the simultaneous localization of a function and its Fourier transform. While extensively studied in Euclidean and abelian settings, their extension to non-abelian Lie groups remains a rich area of research. Lie groups naturally arise in various branches of mathematics and physics, including differential geometry, representation theory and quantum mechanics, where understanding the behavior of functions under group symmetries is crucial. The motivation behind this work is to explore how the Nazarov uncertainty principle —known for its sharp quantitative bounds— adapts to the structure of certain Lie groups. By doing so, we aim to deepen the theoretical understanding of uncertainty phenomena in non-commutative contexts and to lay the groundwork for potential applications in areas such as time-frequency analysis on manifolds, non-commutative harmonic analysis and the study of partial differential equations on Lie groups.

According to the uncertainty principle, first introduced in 1927, it is impossible to accurately measure a particle's position and momentum simultaneously. Mathematically, it states that a non-zero function and its Fourier transform cannot both be localized simultaneously with great accuracy. The classical Heisenberg uncertainty inequality quantifies the localization in terms of the dispersions of the corresponding functions. Nazarov [4] examined an alternate localization criterion, namely the support's smallness. He addressed the case of a non-zero function and its Fourier transform being small outside a compact set. Nazarov uncertainty inequality (Jaming [3]) for the Fourier transform on $\mathbb{R}^n$ can be stated as follows:

Theorem 1.1. *There exists a constant $C = C(n)$ such that, for every $E_1 \subset \mathbb{R}^n$ and $E_2 \subset \widehat{\mathbb{R}}^n$ of finite Lebesgue measures $|E_1|$ and $|E_2|$ respectively, and for every $f \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f(x)|^2\, dx \le J \int_{\mathbb{R}^n \setminus E_1} |f(x)|^2\, dx + J \int_{\widehat{\mathbb{R}}^n \setminus E_2} |\widehat{f}(\xi)|^2\, d\xi,$$

where $J = Ce^{C \min\{|E_1||E_2|, |E_1|^{1/n} w(E_2), w(E_1)|E_2|^{1/n}\}}$ and $w(E_1), w(E_2)$ are mean widths of E_1, E_2, respectively.

Nazarov uncertainty principle was proved for Shearlet transform on $\mathbb{R}^n$ (Bahri *et al*. [1]), for quadratic-phase Fourier transform (Shah *et al*. [5]), for linear canonical transform (Zhang [7]) and for non-separable linear canonical wavelet transform (Srivastava *et al*. [6]).

In Section 2, we briefly discuss some basic notations and results related to the Fourier transform and also the *continuous modulated shearlet transform* (CMST) that was introduced by Bansal *et al*. [2]. Section 3 deals with the proof of Nazarov uncertainty principle for the Fourier transform and the CMST on the class of groups $\mathbb{R}^n \times K$, K being a locally compact group of type I which is both separable and unimodular. The conclusion of this study is presented in Section 4.

2. Preliminaries and Notations

In this section, we highlight some notations and results that will be used throughout the paper. Let $E \subset \mathbb{R}^n$ be a subset of finite Lebesgue measure $|E|$ and mean width $\omega(E)$ (see, Jaming [3, p. 36]).

2.1 Fourier Transform

Consider G to be a locally compact group of type I which is both separable and unimodular and is equipped with the left Haar measure ν_G. The unitary dual of G, denoted by $\widehat{G}$, is the set of equivalence classes of irreducible unitary representations of G endowed with the Mackey-Borel structure. The Plancherel measure $\nu_{\widehat{G}}$ on $\widehat{G}$ is uniquely determined by the fixed Haar measure on G. For each $\pi \in \widehat{G}$, let $\mathrm{HS}(\mathcal{H}_\pi)$ be the space of all Hilbert-Schmidt operators on the Hilbert space $\mathcal{H}_\pi$, the representation space of π. It forms a Hilbert space with respect to the inner product $\langle T,S\rangle = \mathrm{tr}(S^*T)$. The family of Hilbert spaces indexed by $\widehat{G}$, denoted as $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi\in\widehat{G}}$, is a field of Hilbert spaces over $\widehat{G}$. $\mathscr{H}^2(\widehat{G})$, the direct integral of the family $\{\mathrm{HS}(\mathcal{H}_\pi)\}_{\pi\in\widehat{G}}$ with respect to $\nu_{\widehat{G}}$, is the space of all measurable vector fields F on $\widehat{G}$ such that

$$\|F\|^2_{\mathscr{H}^2(\widehat{G})} = \int_{\widehat{G}} \|F(\pi)\|^2_\pi \, d\nu_{\widehat{G}}(\pi) < \infty,$$

where $\|\cdot\|_\pi$ is the norm on the Hilbert space $\mathrm{HS}(\mathcal{H}_\pi)$. Also, $\mathscr{H}^2(\widehat{G})$ forms a Hilbert space with the inner product given by

$$\langle F,K\rangle_{\mathscr{H}^2(\widehat{G})} = \int_{\widehat{G}} \mathrm{tr}[F(\pi)^* K(\pi)] \, d\nu_{\widehat{G}}(\pi).$$

For $f \in L^1(G)$, the Fourier transform of f is defined by

$$\mathcal{F}f(\pi) = \pi(f) = \int_G f(x)\pi(x)^* \, d\nu_G(x).$$

The Plancherel formula states that for all $f \in L^1(G)\cap L^2(G)$,

$$\|\mathcal{F}f\|_{\mathscr{H}^2(\widehat{G})} = \|f\|_2.$$

2.2 Continuous Modulated Shearlet Transform

Generalizing the Gabor transform, the shearlet transform and the wavelet transform, the CMST has been defined and studied in [2]. We briefly describe the CMST for the convenience of the reader. Let $\mathfrak{L}$ be a locally compact group equipped with left Haar measure $d\nu_{\mathfrak{L}}(l)$ and $\mathrm{Aut}(\mathfrak{H})$ denote the automorphism group of $\mathfrak{H}$, where $\mathfrak{H}$ is a locally compact abelian group which is second countable and has Haar measure $d\nu_{\mathfrak{H}}(h)$.

Consider the homomorphism $\alpha : \mathfrak{L} \to \mathrm{Aut}(\mathfrak{H})$ by $l \mapsto \alpha_l$, ensuring the continuity of the map $(l,h) \mapsto \alpha_l(h)$ from the product space $\mathfrak{L}\times\mathfrak{H}$ onto $\mathfrak{H}$. The set $\mathfrak{L}\times\mathfrak{H}$ endowed with the product topology and the operations

$$(l,h)(l',h') = (l\,l', h\,\alpha_l(h')),$$
$$(l,h)^{-1} = (l^{-1}, \alpha_{l^{-1}}(h^{-1}))$$

is a locally compact group called the semi-direct product of $\mathfrak{L}$ and $\mathfrak{H}$ and is denoted by $\mathfrak{L}\times_\alpha\mathfrak{H}$. The set $\mathcal{S} = (\mathfrak{L}\times_\alpha\mathfrak{H})\times G$ forms a locally compact group. Let $1_{\mathfrak{L}}$, $1_{\mathfrak{H}}$ and 1_G denote the identity elements of $\mathfrak{L}$, $\mathfrak{H}$ and G respectively. For $u = (l,h,x,\pi) \in \mathcal{S}\times\widehat{G}$, assume that

$$\mathscr{H}_u = \pi(x)\mathrm{HS}(\mathscr{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathscr{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathscr{H}_\pi)\}$. Then, $\mathscr{H}_u$ is a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S\rangle_{\mathscr{H}_u} = \mathrm{tr}(S^*T) = \langle T,S\rangle_{\mathrm{HS}(\mathscr{H}_\pi)}.$$

Suppose that the inner product on $\mathscr{H}_u$ induces the norm $\|\cdot\|_u$. It may be easily verified that $\mathscr{H}_u = \mathrm{HS}(\mathscr{H}_\pi)$, for all $u \in \mathcal{S}\times\widehat{G}$. The family $\{\mathscr{H}_u : u \in \mathcal{S}\times\widehat{G}\}$ is a field of Hilbert spaces over $\mathcal{S}\times\widehat{G}$. The direct integral of $\{\mathscr{H}_u : u \in \mathcal{S}\times\widehat{G}\}$, denoted by $\mathscr{H}^2(\mathcal{S}\times\widehat{G})$, is the space of all vector fields F on $\mathcal{S}\times\widehat{G}$ which are measurable and satisfy

$$\|F\|^2_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})} = \int_{\mathcal{S}\times\widehat{G}} \|F(u)\|^2_u \, d\sigma(u) < \infty.$$

The space $\mathscr{H}^2(\mathcal{S}\times\widehat{G})$ is a Hilbert space having the inner product

$$\langle F,K\rangle_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}=\int_{\mathcal{S}\times\widehat{G}}\mathrm{tr}[K(u)^*\,F(u)]\,d\sigma(u)$$

and is equipped with the product measure

$$d\sigma(u)=d\nu_{\mathcal{S}}(l,h,x)\,d\nu_{\widehat{G}}(\pi).$$

For each $(l,h,x)\in\mathcal{S}$ and $f,\psi\in L^2(\mathfrak{H}\times G)$, define $\mathcal{T}^{\psi}_{(l,h,x)}:\mathfrak{H}\times G\to\mathbb{C}$ by

$$\mathcal{T}^{\psi}_{(l,h,x)}(k,y)=\delta_{\alpha}^{1/2}(l)\,\psi(\alpha_{l^{-1}}(h^{-1}k),x^{-1}y)$$

and for all $(k,y)\in\mathfrak{H}\times G$, define $\mathcal{J}^{\psi}_{(l,h,x)}f:\mathfrak{H}\times G\to\mathbb{C}$ by

$$\mathcal{J}^{\psi}_{(l,h,x)}f(k,y)=f(k,y)\,\overline{\mathcal{T}^{\psi}_{(l,h,x)}(k,y)}.$$

We call a function $\psi\in L^2(\mathfrak{H}\times G)$ *admissible* if

$$C_{\psi}:=\int_{\mathfrak{L}\times G}|\mathcal{F}_{\mathfrak{H}}\widetilde{\psi}(\eta\circ\lambda_l,x)|^2\,d\nu_{\mathfrak{L}}(l)\,d\nu_G(x)<\infty$$

which is independent of a.e. $\eta\in\widehat{\mathfrak{H}}$.

For $f\in C_{00}(\mathfrak{H}\times G)$, the set of all continuous complex-valued functions on $\mathfrak{H}\times G$ with compact supports and admissible function $\psi\in L^2(\mathfrak{H}\times G)$, the CMST of f with respect to ψ is a measurable field of operators on $\mathcal{S}\times\widehat{G}$ defined by

$$\mathcal{MS}_{\psi}f(l,h,x,\pi)=\int_{\mathfrak{H}}\int_G f(k,y)\,\overline{\mathcal{T}^{\psi}_{(l,h,x)}(k,y)}\,\pi(y)^*\,d\nu_{\mathfrak{H}}(k)\,d\nu_G(y).$$

$\mathcal{MS}_{\psi}f(l,h,x,\pi)$ is a bounded linear operator on $\mathscr{H}_{\pi}$ such that

$$\|\mathcal{MS}_{\psi}f(l,h,x,\pi)\|\le\|f\|_{L^2(\mathfrak{H}\times G)}\,\|\psi\|_{L^2(\mathfrak{H}\times G)}.$$

The operator $\mathcal{MS}_{\psi}:C_{00}(\mathfrak{H}\times G)\to\mathscr{H}^2(\mathcal{S}\times\widehat{G})$ defined by $f\mapsto\mathcal{MS}_{\psi}f$ satisfies

$$\|\mathcal{MS}_{\psi}f\|_{\mathscr{H}^2(\mathcal{S}\times\widehat{G})}=C_{\psi}^{1/2}\,\|f\|_{L^2(\mathfrak{H}\times G)}\tag{2.1}$$

and thus can be extended uniquely to a bounded linear operator from $L^2(\mathfrak{H}\times G)$ into $\mathscr{H}^2(\mathcal{S}\times\widehat{G})$. This extension, which we still denote by $\mathcal{MS}_{\psi}$, satisfies (2.1) for each $f\in L^2(\mathfrak{H}\times G)$.

3. $\mathbb{R}^n\times K$, K a Locally Compact Group

In this section, we shall prove Nazarov uncertainty principle for the Fourier transform and the CMST on $\mathbb{R}^n\times K$.

Let $G=\mathbb{R}^n\times K$, where K is locally compact group of type I which is both separable and unimodular and is equipped with Haar measure $dg=dx\,dk$, dx being Lebesgue measure on $\mathbb{R}^n$ and dk the left Haar measure on K. $\widehat{G}=\widehat{\mathbb{R}}^n\times\widehat{K}$ is the dual of G, where $\widehat{K}$ is the dual space of K. The following gives the Nazarov inequality in the case of Fourier transform:

Theorem 3.1. *For every $f\in L^2(\mathbb{R}^n\times K)$ and for every sets $E_1,E_2\subset\mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C=C(n)$ such that*

$$\|f\|_2^2\le J\int_{\mathbb{R}^n\setminus E_1}\int_K|f(x,k)|^2\,dx\,dk+J\int_{\widehat{\mathbb{R}}^n\setminus E_2}\int_{\widehat{K}}\|\widehat{f}(\xi,\delta)\|_{HS}^2\,d\xi\,d\delta,$$

where $J=Ce^{C\min\{|E_1||E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$.

Proof. Let $f \in L^2(\mathbb{R}^n \times K)$ and $E_1, E_2 \subset \mathbb{R}^n$ are of finite Lebesgue measures.
There exists a zero measure set $A \subseteq K$ such that

$$\int_{\mathbb{R}^n} |f(x,k)|^2\, dx < \infty,$$

for $k \in K \setminus A = A'$ (say). For $k \in K$, consider $f_k(x) = f(x,k)$, for all $x \in \mathbb{R}^n$.
Then, $f_k \in L^2(\mathbb{R}^n)$ for all $k \in A'$ and for all $\xi \in \widehat{\mathbb{R}}^n$,

$$\widehat{f}_k(\xi) = \int_{\mathbb{R}^n} f(x,k)\, e^{-2\pi i \langle \xi, x\rangle}\, dx = \mathcal{F}_1 f(\xi,k).$$

Using Theorem 1.1, there exists a constant $C = C(n)$ such that

$$\int_{\mathbb{R}^n} |f(x,k)|^2\, dx \le J \int_{\mathbb{R}^n \setminus E_1} |f_k(x)|^2\, dx + J \int_{\widehat{\mathbb{R}}^n \setminus E_2} |\widehat{f}_k(\xi)|^2\, d\xi,$$

where $J = Ce^{C \min\{|E_1||E_2|, |E_1|^{1/n} w(E_2), w(E_1)|E_2|^{1/n}\}}$. On integrating both the sides of above inequality with respect to dk, we obtain

$$\int_{A' \times \mathbb{R}^n} |f(x,k)|^2\, dx\, dk \le J \int_{A'} \int_{\mathbb{R}^n \setminus E_1} |f_k(x)|^2\, dk\, dx + J \int_{A'} \int_{\widehat{\mathbb{R}}^n \setminus E_2} |\widehat{f}_k(\xi)|^2\, dk\, d\xi.$$

The integral on the L.H.S. is equal to $\|f\|_2^2$. Using Fubini's theorem we obtain

$$\|f\|_2^2 \le J \int_{\mathbb{R}^n \setminus E_1} \int_K |f(x,k)|^2\, dx\, dk + J \int_{\widehat{\mathbb{R}}^n \setminus E_2} \int_{A'} |\widehat{f}_k(\xi)|^2\, d\xi\, dk. \tag{3.1}$$

Also, we can write

$$\int_{\widehat{\mathbb{R}}^n \times A'} |\mathcal{F}_1 f(\xi,k)|^2\, d\xi\, dk = \int_{\mathbb{R}^n \times A'} |f(x,k)|^2\, dx\, dk = \|f\|_2^2 < \infty.$$

It implies that $\mathcal{F}_1 f \in L^2(\widehat{\mathbb{R}}^n \times A')$ and $\mathcal{F}_2 \mathcal{F}_1 f$ is well defined a.e.
The function $f \in L^2(\mathbb{R}^n \times A')$ can be approximated by the functions in the space $L^1 \cap L^2(\mathbb{R}^n \times A')$ and we have $\mathcal{F}_2 \mathcal{F}_1 f = \widehat{f}$, for all $f \in L^2(\mathbb{R}^n \times A')$.
Using Plancherel formula for K, we have

$$\int_{A'} |\widehat{f}_k(\xi)|^2\, dk = \int_K |\mathcal{F}_1 f(\xi,k)|^2\, dk = \int_{\widehat{K}} \|\widehat{f}(\xi,\delta)\|_{\mathrm{HS}}^2\, d\delta.$$

Thus using (3.1), we obtain the result

$$\|f\|_2^2 \le J \int_{\mathbb{R}^n \setminus E_1} \int_K |f(x,k)|^2\, dx\, dk + J \int_{\widehat{\mathbb{R}}^n \setminus E_2} \int_{\widehat{K}} \|\widehat{f}(\xi,\delta)\|_{\mathrm{HS}}^2\, d\xi\, d\delta. \qquad \square$$

Corollary 3.2. *For every $f \in L^2(\mathbb{R}^n \times K)$, $E_1, E_2 \subset \mathbb{R}^n$ of finite Lebesgue measures, $N_1 \subset K$ of finite Haar measure and $N_2 \subset \widehat{K}$ of finite Plancherel measure, there exists a constant $C = C(n)$ such that*

$$\|f\|_2^2 \le J \int_{(\mathbb{R}^n \times K) \setminus M_1} |f(x,k)|^2\, dx\, dk + J \int_{(\widehat{\mathbb{R}}^n \times \widehat{K}) \setminus M_2} \|\widehat{f}(\xi,\delta)\|_{HS}^2\, d\xi\, d\delta,$$

where $J = Ce^{C \min\{|E_1||E_2|, |E_1|^{1/n} w(E_2), w(E_1)|E_2|^{1/n}\}}$, $M_1 = E_1 \times N_1 \subset \mathbb{R}^n \times K$ and $M_2 = E_2 \times N_2 \subset \widehat{\mathbb{R}}^n \times \widehat{K}$.

Proof. The proof follows from the fact that $(\mathbb{R}^n \setminus E_1) \times K \subset (\mathbb{R}^n \times K) \setminus M_1$ and $(\widehat{\mathbb{R}}^n \setminus E_2) \times \widehat{K} \subset (\widehat{\mathbb{R}}^n \times \widehat{K}) \setminus M_2$. $\square$

The next theorem provides the Nazarov uncertainty inequality for the continuous modulated shearlet transform when $G = \mathbb{R}^n \times K$.

Theorem 3.3. *Let $G=\mathbb{R}^n\times K$. For every $f,\psi\in L^2(\mathfrak{H}\times G)$ such that ψ is an admissible function and $E_1,E_2\subset\mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C=C(n)$ such that*

$$\begin{aligned}C_\psi\|f\|^2_{L^2(\mathfrak{H}\times G)}\le JC_\psi\int_{\mathfrak{H}}\int_{(\mathbb{R}^n\setminus E_1)\times K}|f(t,y,u)|^2\,d\nu_{\mathfrak{H}}(t)\,d\nu_G(y,u)\\ +J\int_{\mathcal{S}\times(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\mathcal{MS}_\psi f(l,h,x,k,\xi,\delta)\|^2_{HS}\,d\sigma(l,h,x,k,\xi,\delta),\end{aligned}$$

where $J=Ce^{C\min\{|E_1||E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$.

Proof. Let $f,\psi\in L^2(\mathfrak{H}\times G)$ such that ψ is an admissible function and $E_1,E_2\subset\mathbb{R}^n$ be of finite Lebesgue measures.
By [2, eq. (6.1)], we have $\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,\cdot)\in L^2(\mathbb{R}^n\times K)$, for almost every $(l,h,x,k)\in\mathcal{S}$.
By Theorem 3.1, we obtain for almost all $(l,h,x,k)\in\mathcal{S}$,

$$\begin{aligned}\int_G|\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,y,u)|^2\,d\nu_G(y,u)&\le J\int_{(\mathbb{R}^n\setminus E_1)\times K}|\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,y,u)|^2\,d\nu_G(y,u)\\&\quad+J\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\mathcal{F}_G\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,\xi,\delta)\|^2_{\mathrm{HS}}\,d\nu_{\widehat{G}}(\xi,\delta)\\&=J\int_{(\mathbb{R}^n\setminus E_1)\times K}|\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,y,u)|^2\,d\nu_G(y,u)\\&\quad+J\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\mathcal{MS}_\psi f(l,h,x,k,\xi,\delta)\|^2_{\mathrm{HS}}\,d\nu_{\widehat{G}}(\xi,\delta).\end{aligned}$$

Integrating both sides with respect to $d\nu_{\mathcal{S}}(l,h,x,k)$, we have

$$\begin{aligned}&\int_{\mathcal{S}}\int_G|\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,y,u)|^2\,d\nu_{\mathcal{S}}(l,h,x,k)d\nu_G(y,u)\\&\le J\int_{\mathcal{S}}\int_{(\mathbb{R}^n\setminus E_1)\times K}|\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x,k)}f)(I,y,u)|^2\,d\nu_{\mathcal{S}}(l,h,x,k)d\nu_G(y,u)\\&\quad+J\int_{\mathcal{S}\times(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\mathcal{MS}_\psi f(l,h,x,k,\xi,\delta)\|^2_{\mathrm{HS}}\,d\sigma(l,h,x,k,\xi,\delta).\end{aligned}$$

Hence, using again [2, eq. (6.1)], we get

$$\begin{aligned}C_\psi\|f\|^2_{L^2(\mathfrak{H}\times G)}\le JC_\psi\int_{\mathfrak{H}}\int_{(\mathbb{R}^n\setminus E_1)\times K}|f(t,y,u)|^2\,d\nu_{\mathfrak{H}}(t)\,d\nu_G(y,u)\\ +J\int_{\mathcal{S}\times(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\mathcal{MS}_\psi f(l,h,x,k,\xi,\delta)\|^2_{\mathrm{HS}}\,d\sigma(l,h,x,k,\xi,\delta).\end{aligned}$$

□

Corollary 3.4. *Let $G=\mathbb{R}^n\times K$. For every $f,\psi\in L^2(G)$ and $E_1,E_2\subset\mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C=C(n)$ such that*

$$\begin{aligned}\|\psi\|^2_{L^2(G)}\,\|f\|^2_{L^2(G)}\le J\|\psi\|^2_{L^2(G)}\int_{(\mathbb{R}^n\setminus E_1)\times K}|f(y,u)|^2\,d\nu_G(y,u)\\ +J\int_G\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|G_\psi f(x,k,\xi,\delta)\|^2_{HS}\,d\nu_G(x,k)\,d\nu_{\widehat{G}}(\xi,\delta),\end{aligned}$$

where $J=Ce^{C\min\{|E_1||E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$ and $G_\psi f$ denotes the Gabor transform of f with respect to ψ.

The following Nazarov uncertainty principle for Gabor transform on $\mathbb{R}^n$ can be deduced from the above corollary:

Corollary 3.5. *For every $f,\psi \in L^2(\mathbb{R}^n)$ and $E_1,E_2 \subset \mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C = C(n)$ such that*

$$\|\psi\|_2^2\,\|f\|_2^2 \le J\|\psi\|_2^2 \int_{\mathbb{R}^n\setminus E_1} |f(y)|^2\,dy + J\int_{\mathbb{R}^n}\int_{\widehat{\mathbb{R}}^n\setminus E_2} |G_\psi f(x,\xi)|^2\,dx\,d\xi,$$

where $J = Ce^{C\min\{|E_1|\,|E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$.

The next theorem provides the Nazarov uncertainty inequality for the continuous modulated shearlet transform when $\mathfrak{H} = \mathbb{R}^n \times K$.

Theorem 3.6. *Let $\mathfrak{H} = \mathbb{R}^n \times K$, where K is a second countable, locally compact abelian group. For every $f,\psi \in L^2(\mathfrak{H}\times G)$ such that ψ is an admissible function and $E_1,E_2 \subset \mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C = C(n)$ such that*

$$\begin{aligned} C_\psi\|f\|^2_{L^2(\mathfrak{H}\times G)} \le J\int_{\mathfrak{L}\times(\mathbb{R}^n\setminus E_1)\times K\times G\times\widehat{G}} \|\mathcal{MS}_\psi f(l,h,x,\pi)\|^2_{HS}\,d\sigma(l,h,x,\pi) \\ + JC_\psi\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\int_G |\mathcal{F}_{\mathfrak{H}} f(\eta,y)|^2\,d\nu_{\widehat{\mathfrak{H}}}(\eta)\,d\nu_G(y), \end{aligned}$$

where $J = Ce^{C\min\{|E_1|\,|E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$.

Proof. Let $f,\psi \in L^2(\mathfrak{H}\times G)$ such that ψ is an admissible function and $E_1,E_2 \subset \mathbb{R}^n$ be of finite Lebesgue measures.

By [2, eq. (6.3)], we have $(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(\cdot,y) \in L^2(\mathfrak{H})$ for a.e. $l \in \mathfrak{L}$ and $x,y \in G$. Using Theorem 3.1, there exists a constant $C = C(n)$ such that

$$\begin{aligned} \int_{\mathfrak{H}} |(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(h,y)|^2\,d\nu_{\mathfrak{H}}(h) \le J\int_{(\mathbb{R}^n\setminus E_1)\times K} |(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(h,y)|^2\,d\nu_{\mathfrak{H}}(h) \\ + J\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}} |\mathcal{F}_{\mathfrak{H}}(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(\eta,y)|^2\,d\nu_{\widehat{\mathfrak{H}}}(\eta), \end{aligned}$$

for a.e. $l \in \mathfrak{L}$ and $x,y \in G$, where $J = Ce^{C\min\{|E_1|\,|E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$.

Integrating both sides with respect to $\delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_G(x)\,d\nu_G(y)$, we get

$$\begin{aligned} &\int_{\mathfrak{L}}\int_{\mathfrak{H}}\int_G\int_G |(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(h,y)|^2\,\delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\mathfrak{H}}(h)\,d\nu_G(x)\,d\nu_G(y) \\ &\le J\int_{\mathfrak{L}}\int_{(\mathbb{R}^n\setminus E_1)\times K}\int_G\int_G |(f *_{\mathfrak{H}} \mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(h,y)|^2\,\delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\mathfrak{H}}(h)\,d\nu_G(x)\,d\nu_G(y) \\ &\quad + J\int_{\mathfrak{L}}\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\int_G\int_G |\mathcal{F}_{\mathfrak{H}} f(\eta,y)|^2 |\mathcal{F}_{\mathfrak{H}}(\mathcal{T}^{\widetilde{\psi}}_{(l,1_{\mathfrak{H}},yx^{-1}y)})(\eta,y)|^2 \\ &\quad \times \delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\widehat{\mathfrak{H}}}(\eta)\,d\nu_G(x)\,d\nu_G(y). \end{aligned}$$

Using [2, Lemma 2.7, Lemma 2.8 and eq. (6.3)], we obtain

$$\begin{aligned} C_\psi\|f\|^2_{L^2(\mathfrak{H}\times G)} &\le J\int_{\mathfrak{L}}\int_{(\mathbb{R}^n\setminus E_1)\times K}\int_G\int_G |\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x)} f)(I,y)|^2 \delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\mathfrak{H}}(h)\,d\nu_G(x)\,d\nu_G(y) \\ &\quad + J\int_{\mathfrak{L}}\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\int_G\int_G |\mathcal{F}_{\mathfrak{H}} f(\eta,y)|^2 |\mathcal{F}_{\mathfrak{H}}\widetilde{\psi}(\eta\circ\lambda_l, y^{-1}x)|^2 d\nu_{\mathfrak{L}}(l)\,d\nu_{\widehat{\mathfrak{H}}}(\eta)\,d\nu_G(x)\,d\nu_G(y) \\ &= J\int_{\mathfrak{L}}\int_{(\mathbb{R}^n\setminus E_1)\times K}\int_G\int_{\widehat{G}} |\mathcal{F}_G\mathcal{F}_{\mathfrak{H}}(\mathcal{J}^{\psi}_{(l,h,x)} f)(I,\pi)|^2 \delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\mathfrak{H}}(h)\,d\nu_G(x)\,d\nu_{\widehat{G}}(\pi) \end{aligned}$$

$$+J\int_{\mathfrak{L}}\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\int_G\int_G |\mathcal{F}_{\mathfrak{H}}f(\eta,y)|^2|\mathcal{F}_{\mathfrak{H}}\widetilde{\psi}(\eta\circ\lambda_l,x)|^2 d\nu_{\mathfrak{L}}(l)\,d\nu_{\widehat{\mathfrak{H}}}(\eta)\,d\nu_G(x)\,d\nu_G(y)$$

$$=J\int_{\mathfrak{L}\times(\mathbb{R}^n\setminus E_1)\times K\times G\times\widehat{G}} \|\mathcal{MS}_\psi f(l,h,x,\pi)\|^2_{\mathrm{HS}}\, d\sigma(l,h,x,\pi)$$

$$+JC_\psi\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\int_G |\mathcal{F}_{\mathfrak{H}}f(\eta,y)|^2\, d\nu_{\widehat{\mathfrak{H}}}(\eta)\, d\nu_G(y).$$

□

Corollary 3.7. *Let $\mathfrak{H}=\mathbb{R}^n\times K$. For every $f,\psi\in L^2(\mathfrak{H})$ such that ψ is an admissible function and $E_1,E_2\subset\mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C=C(n)$ such that*

$$C_\psi\|f\|^2_{L^2(\mathfrak{H})}\le J\int_{\mathfrak{L}\times(\mathbb{R}^n\setminus E_1)\times K}|\mathcal{W}_\psi f(l,x,u)|^2\,\delta_\alpha(l)\,d\nu_{\mathfrak{L}}(l)\,d\nu_{\mathfrak{H}}(x,u)$$

$$+JC_\psi\int_{(\widehat{\mathbb{R}}^n\setminus E_2)\times\widehat{K}}\|\widehat{f}(\xi,\delta)\|^2_{HS}\,d\nu_{\widehat{\mathfrak{H}}}(\xi,\delta),$$

where $J=Ce^{C\min\{|E_1||E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$ and $\mathcal{W}_\psi f$ denotes the wavelet transform of f with respect to ψ.

Remark 3.8. Let I_n be the $n\times n$ identity matrix and O_n be the $n\times 1$ column vector with all entries 0. Take $\mathfrak{L}=\mathbb{R}^*\times\mathbb{R}^{n-1}$ and $\mathfrak{H}=\mathbb{R}^n$, where $\mathbb{R}^*=\mathbb{R}\setminus\{0\}$. The set $\mathfrak{L}\times\mathfrak{H}$ endowed with the operation $(a,s,t)\circ(a',s',t')=(aa',s+|a|^{1-1/n}s',t+S_sA_at')$ is a locally compact group called the shearlet group, where $S_s=\begin{bmatrix}1 & s^T\\ O_{n-1} & I_{n-1}\end{bmatrix}$ and $A_a=\begin{bmatrix}a & O^T_{n-1}\\ O_{n-1} & \mathrm{sgn}(a)|a|^{1/n}I_{n-1}\end{bmatrix}$, for all $a\in\mathbb{R}^*$, $s\in\mathbb{R}^{n-1}$. The left Haar measure is given by $d\nu(a,s,t)=\frac{1}{|a|^{n+1}}\,da\,ds\,dt$. The following corollary was proved by Bahri *et al*. [1].

Corollary 3.9. *Let $\mathfrak{L}\times\mathfrak{H}=\mathbb{R}^*\times\mathbb{R}^{n-1}\times\mathbb{R}^n$ with the operation defined above. For every $f,\psi\in L^2(\mathfrak{H})$ such that ψ is an admissible function and $E_1,E_2\subset\mathbb{R}^n$ of finite Lebesgue measures, there exists a constant $C=C(n)$ such that*

$$C_\psi\|f\|_2^2\le J\int_{\mathbb{R}^*}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n\setminus E_1}|SH_\psi f(a,s,t)|^2\,\frac{1}{|a|^{n+1}}\,da\,ds\,dt+JC_\psi\int_{\widehat{\mathbb{R}}^n\setminus E_2}|\widehat{f}(\delta)|^2\,d\delta,$$

where $J=Ce^{C\min\{|E_1||E_2|,|E_1|^{1/n}w(E_2),w(E_1)|E_2|^{1/n}\}}$ and $SH_\psi f$ denotes the shearlet transform of f with respect to ψ.

4. Conclusion

In this paper, we have investigated the Nazarov uncertainty principle within the framework of certain Lie groups of the form $\mathbb{R}^n\times K$, where K is a locally compact group. By extending the classical formulation of the uncertainty principle to these more intricate algebraic structures, we have demonstrated how the interplay between group representations, harmonic analysis and geometric properties of Lie groups influences localization phenomena. The analysis in our paper reveals that, while the essence of Nazarov's principle —limiting simultaneous concentration in space and frequency— persists, the nature of this trade-off is deeply shaped by the underlying group geometry. These findings not only broaden the scope of uncertainty principles in abstract harmonic analysis but also suggest new avenues for exploring applications in quantum mechanics, signal processing on manifolds, and representation theory.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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