Journal of Informatics and Mathematical Sciences
Vol. 7, No. 2, pp. 99–108, 2015
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Approximate Analytical Solution of Liner Boundary Value Problems by Laplace-Differential Transform Method

Research Article

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Abstract. In this paper, we study the approximate analytical solutions of homogeneous and non-homogeneous linear PDEs with boundary conditions by using the Laplace Differential Transform method (LDTM). For this purpose, we consider three illustrations with one Dirichlet and two Neumann boundary conditions and obtain the corresponding approximate analytical solutions. This method is capable of greatly reducing the size of computational domain and a few numbers of iterations are required to reach the closed form solutions as series expansions of some known functions.

Keywords. LDTM; Linear PDEs; Boundary conditions

MSC. 41A58; 44A10; 35G15

Received: September 20, 2015 **Accepted:** November 26, 2015

1. Introduction

Many important phenomena and dynamic processes in scientific and engineering applications are governed by partial differential equations. The study of partial differential equations plays an important role in solid state physics, mathematics, fluid dynamics and chemistry etc. There are two types of these equations, the homogeneous equations and the non-homogeneous equations. For the treatment of these types of equations, various methods have been developed and the DTM is one of them. The DTM is the semi-analytical numerical method for solving partial differential equations. In 1986, J. K. Zhou was the first one to use the DTM in engineering applications with linear and nonlinear initial value problems in electric circuit analysis. Hassan

(2002) has given the different applications of differential transformation in differential equations. Ayaz (2004) has investigated applications of DTM to solve differential-algebraic equations. Bildik and Konuralp (2006) have used the Variational iteration method, DTM and Adomian decomposition method for solving different types of non-linear PDEs. Arikoglu and Ozkol (2007) have performed DTM to solve the fractional differential equations. Erturk (2007) has solved linear sixth-order boundary value problems by using DTM. Hassan and Erturk (2009) have given the higher order boundary value problems and discussed the higher order series solution for linear and non-linear differential equations.

Keskin and Oturanc (2010) performed RDTM for solving gas dynamics equation and showed to effectiveness and accuracy of the proposed method. Khan *et al.* (2010) have studied Laplace decomposition method to obtain the approximate solution of non-linear coupled PDEs and found that the Laplace decomposition method and Adomian decomposition method both can be used alternatively for the solution of higher order initial value problems. Gupta (2011) has introduced the Reduced differential transform method (RDTM) and homotopy perturbation method to find the approximate analytical solutions of fractional Benney-Lin equation. Madani *et al.* (2011) have developed a new method Laplace homotopy perturbation method which is combination of Laplace transform method and the Homotopy perturbation method being applied to solve one-dimensional non-homogeneous PDEs with variable coefficients. Alquran *et al.* (2012) have proposed the coupling of the Laplace transform method and the differential transform method for solving linear non-homogeneous partial differential equations with variable co-efficient. Mishra and Nagar (2012) have applied a combination of LTM with Homotopy perturbation method which is called He-Laplace method for solving linear and non-linear PDEs and found that the technique is capable to reduce the volume of computational work as compared to Adomian polynomials.

In this paper, we apply Laplace-Differential Transform Method for solving linear partial differential equations with boundary conditions on some examples and the results obtained by it are compared with exact solution. We make comparison between the LDTM and exact solutions and find that the proposed method shows its reasonability, reliability, validity and potential for the solution of homogeneous, non-homogeneous linear PDEs in sciences and engineering applications.

2. Differential Transformation Method

The one variable differential transform [6] of a function $u(x,t)$, is defined as:

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0}; \quad k \geq 0 \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x-x_0)^k, \tag{2.2}$$

where x_0 is the initial point for the given initial condition. Then the function $u(x,t)$ can be written as

$$u(x,t)=\sum_{k=0}^{\infty}U_k(t)x^k. \tag{2.3}$$

3. Algorithm of LDTM

To illustrate the basic idea of Laplace differential transform method [1], we consider the general form of inhomogeneous PDEs with variable or constant coefficients

$$\mathscr{L}[u(x,t)]+\mathscr{R}[u(x,t)]=f(x,t);\quad x\in \mathbb{R},\ \mathrm{t}\in\mathbb{R}^+, \tag{3.1}$$

subject to the initial conditions

$$u(x,0)=g_1(x),\ \ u_t(x,0)=g_2(x), \tag{3.2}$$

and the Dirichlet boundary conditions

$$u(0,t)=h_1(t),\ \ u(1,t)=h_2(t), \tag{3.3}$$

or the Neumann boundary conditions

$$u(0,t)=h_1(t),\ \ u_x(1,t)=h_3(t), \tag{3.4}$$

where $\mathscr{L}[\cdot]$ is linear operator with respect to 't', $\mathscr{R}[\cdot]$ is remaining operator and f is a known analytical function.

First, we take the Laplace transform on both sides of equation (3.1), with respect to 't', and we get

$$L[\mathscr{L}[u(x,t)]]+L[\mathscr{R}[u(x,t)]]=L[f(x,t)]. \tag{3.5}$$

By using initial conditions from equation (3.2), we get

$$\bar{u}(x,s)+L[\mathscr{R}[u(x,t)]]=\left[\bar{f}(x,s)\right], \tag{3.6}$$

where $\bar{u}(x,s)$ and $\bar{f}(x,s)$ are the Laplace transform on $u(x,t)$ and $f(x,t)$ respectively.

Afterwards, we apply differential transform method on the equation (3.6) with respect to 'x', and we get

$$\bar{U_k}(s)+L[\mathscr{R}[U_k(t)]]=\left[\bar{F_k}(s)\right], \tag{3.7}$$

where $\bar{U_k}(s)$ and $\bar{F_k}(s)$ are the differential transform of $\bar{u}(x,s)$ and $\bar{f}(x,s)$ respectively.

In the next step, we apply inverse Laplace transform on both sides of the equation (3.7) with respect to 's', and then we get

$$L^{-1}\left[\bar{U_k}(s)\right]+L^{-1}L[\mathscr{R}[U_k(t)]]=L^{-1}\left[\bar{F_k}(s)\right],$$

or

$$U_k(t)+[\mathscr{R}[U_k(t)]]=[F_k(t)]. \tag{3.8}$$

Now, apply the differential transform method on the given Dirihlet and Neumann boundary conditions (3.3) and (3.4), we get

$$U_0(t)=h_1(t). \tag{3.9}$$

Let us assume

$$U_1(t)=aq(t). \tag{3.10}$$

By the definition of DTM, we take

$$u(1,t)=\sum_{i=0}^{\infty}U_i(t), \quad u_x(1,t)=\sum_{i=0}^{\infty}iU_i(t). \tag{3.11}$$

By equation (3.11), we calculate the value of a. Now by the equation (3.9) and the equation (3.10)in (3.8), the closed form series solution can be written as

$$u(x,t)=\sum_{k=0}^{\infty}U(k,t)x^k. \tag{3.12}$$

4. Illustrative Examples

To illustrate the applicability of LDTM, we have applied it to linear or non-linear PDEs which are homogeneous as well as non-homogeneous.

Example 4.1. The homogeneous linear PDE is:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+u=0, \tag{4.1}$$

subject to the initial conditions

$$u(x,0)=e^{-x}+x, \quad u_t(x,0)=0, \tag{4.2}$$

and the Dirichlet boundary conditions

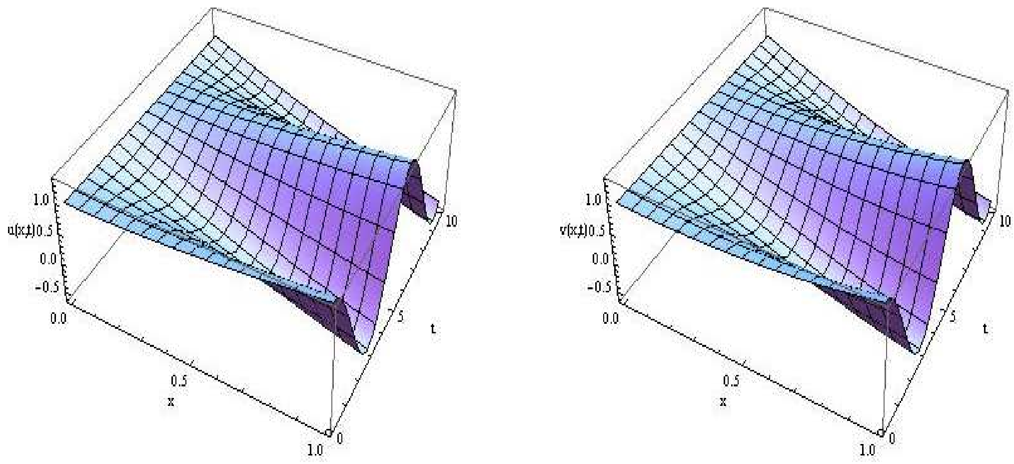
$$u(0,t)=1, \quad u(1,t)=\frac{1}{e}+\cos(t). \tag{4.3}$$

The exact solution can be expressed as:

$$u(x,0)=e^{-x}+x\cos(t).$$

In this technique, first we apply the Laplace transformation on equation (4.1) with respect to 't', therefore, we get

$$s^2L[u(x,t)]-su(x,0)-u_t(x,0)=L\left[\frac{\partial^2 u}{\partial x^2}-u\right].$$



(a) (b)

Figure 1. The behavior of the (a) Exact solution, (b) LDTM solution, with respect to 'x' and 't', for Example 4.1.

By using initial conditions from equation (4.2), we get

$$L[u(x,t)] = \frac{e^{-x}+x}{s} + \frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} - u\right].$$

Now we applying the Inverse Laplace transformation with respect to 's' on both sides:

$$u(x,t) = e^{-x} + x + L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} - u\right]\right]. \tag{4.4}$$

The next step is applying the Differential transformation method on equations (4.3) and (4.4) with respect to space variable 'x', we get

$$U_k(t) = \frac{(-1)^k}{k!} + \delta(k-1,t) + L^{-1}\left[\frac{1}{s^2}L\left[(k+2)(k+1)U_{k+2}(t) - U_k(t)\right]\right] \tag{4.5}$$

or

$$U_0(t) = 1. \tag{4.6}$$

Let us assume

$$U_1(t) = a\cos(t). \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) and by straightforward iterative steps, we obtain

$$U_2(t) = \frac{1}{2},\ U_3(t) = 0,\ U_4(t) = \frac{1}{24},\ \dots\ . \tag{4.8}$$

Now, by equation (3.11) we get

$$a = \frac{24 - 37e + 24e\cos(t)}{24e\cos(t)}.$$

Now putting the value of a in equation (4.7), and we $U_k(t)$, $k \geq 0$ get the component of the DTM can be obtained. When we substitute all values of $U_k(t)$ from equations (4.6), (4.7) and (4.8) into equation (3.12), then the series solution can be formed as

$$u(x,t) = 1 + \frac{x(24 - 37e + 24e\cos(t))}{24e} + \frac{x^2}{2} + \frac{x^4}{24} + \dots .$$

Example 4.2. The homogeneous linear PDE is:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{4.9}$$

subject to the initial conditions

$$u(x,0) = e^x - x, \tag{4.10}$$

and the Neumann boundary conditions

$$u(0,t) = 1 + t, \quad u_x(1,t) = e - 1. \tag{4.11}$$

The exact solution can be expressed as:

$$u(x,0) = e^x - x + t.$$

In this technique, first we apply the Laplace transformation on equation (4.9) with respect to 't', therefore, we get

$$sL[u(x,t)] - u(x,0) = L\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right].$$

By using initial conditions from equation (4.10), we get

$$L[u(x,t)] = \frac{e^x - x}{s} + \frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right].$$

Now we applying the Inverse Laplace transformation with respect to 's' on both sides:

$$u(x,t) = e^x - x + L^{-1}\left[\frac{1}{s}L\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right]\right]. \tag{4.12}$$

The next step is applying the Differential transformation method on equations (4.11) and (4.12) with respect to space variable 'x', we get

$$U_k(t) = \frac{1}{k!} - \delta(k-1,t) + L^{-1}\left[\frac{1}{s}L[(k+2)(k+1)U_{k+2}(t) - (k-1)U_{k-1}(t)]\right] \tag{4.13}$$

or

$$U_0(t) = 1 + t. \tag{4.14}$$

Let us assume

$$U_1(t) = at^4. \tag{4.15}$$

Substituting (4.14) and (4.15) into (4.13) and by straightforward iterative steps, we obtain

$$U_2(t) = \frac{1+at^4}{2},\ U_3(t) = \frac{1+4at^3+at^4}{3!},\ U_4(t) = \frac{1+12at^2+12at^3+at^4}{5!}, \dots . \tag{4.16}$$

Now, by equation (3.11) we get

$$a = \frac{-65+24e}{t^2(12+92t+65t^2)}.$$

Now putting the value of a in equation (4.15), and we $U_k(t)$, $k \geq 0$ get the component of the DTM can be obtained. When we substitute all values of $U_k(t)$ from equations (4.14), (4.15) and (4.16) into equation (3.12), then the series solution can be formed as

$$\begin{aligned} u(x,t) = 1 + t + \frac{(-65+24e)t^2 x}{12+92t+65t^2} + \frac{1}{2}\left(1 + \frac{(-65+24e)t^2}{12+92t+65t^2}\right)x^2 \\ + \frac{1}{6}\left(1 + \frac{4(-65+24e)t}{12+92t+65t^2} + \frac{(-65+24e)t^2}{12+92t+65t^2}\right)x^3 \\ + \frac{1}{12}\left(\frac{2(-65+24e)t}{12+92t+65t^2} + \frac{1}{2}\left(1 + \frac{4(-65+24e)t}{12+92t+65t^2} + \frac{(-65+24e)t^2}{12+92t+65t^2}\right)\right)x^4 + \dots . \end{aligned}$$

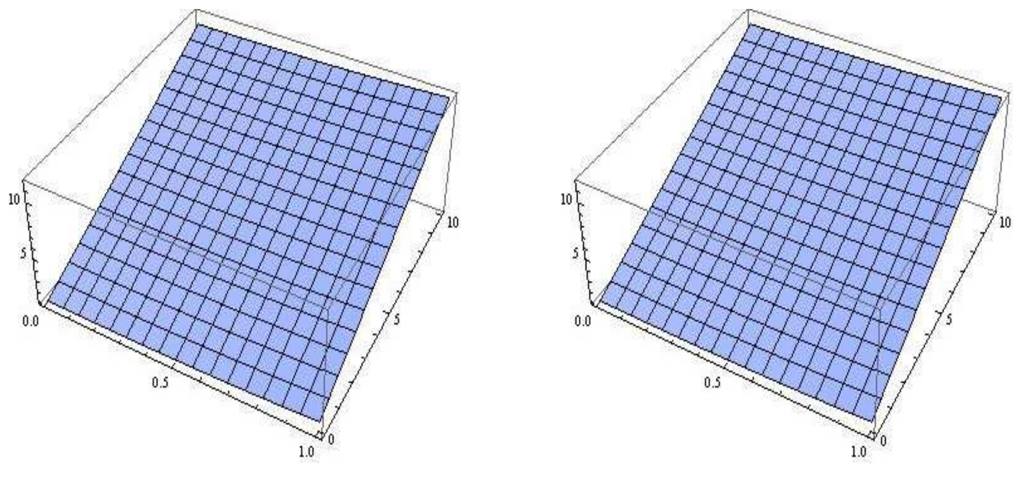


Figure 2. The behavior of the (a) Exact solution, (b) LDTM solution, with respect to 'x' and 't', for Example 4.2.

Example 4.3. The non-homogeneous linear PDE is:

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} = 2t + 2x^2 + 2, \tag{4.17}$$

subject to the initial conditions

$$u(x,0) = x^2, \tag{4.18}$$

and the Neumann boundary conditions

$$u(0,t) = t^2, \quad u_x(1,t) = 2. \tag{4.19}$$

The exact solution can be expressed as:

$$u(x,0) = x^2 + t^2.$$

In this technique, first we apply the Laplace transformation on equation (4.17) with respect to 't', therefore, we get

$$sL[u(x,t)] - u(x,0) = \frac{2}{s^2} + \frac{2x^2+2}{s} - L\left[x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.18), we get

$$L[u(x,t)] = \frac{x^2}{s} + \frac{2}{s^3} + \frac{2x^2+2}{s^2} - \frac{1}{s}L\left[x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right].$$

Now we applying the Inverse Laplace transformation with respect to 's' on both sides:

$$u(x,t) = x^2 + t^2 + (2x^2+2)t - L^{-1}\left[\frac{1}{s}L\left[x\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right]\right]. \tag{4.20}$$

The next step is applying the Differential transformation method on equations (4.19) and (4.20) with respect to space variable 'x', we get

$$\begin{aligned} U_k(t) &= (1+2t)\delta(k-2,t) + (t^2+2t)\delta(k,t) \\ &\quad - L^{-1}\left[\frac{1}{s}L\left[\sum_{r=0}^{k}\delta(k-r-1,t)(r+1)U_{r+1}(t) + (k+2)(k+1)U_{k+2}(t)\right]\right]; \end{aligned} \tag{4.21}$$

or

$$U_0(t) = t^2. \tag{4.22}$$

Let us assume

$$U_1(t) = at. \tag{4.23}$$

Substituting (4.22) and (4.23) into (4.21) and by straightforward iterative steps, we obtain

$$U_2(t) = 1, \; U_3(t) = \frac{-(a+at)}{3!}, \; U_4(t) = 0, \dots . \tag{4.24}$$

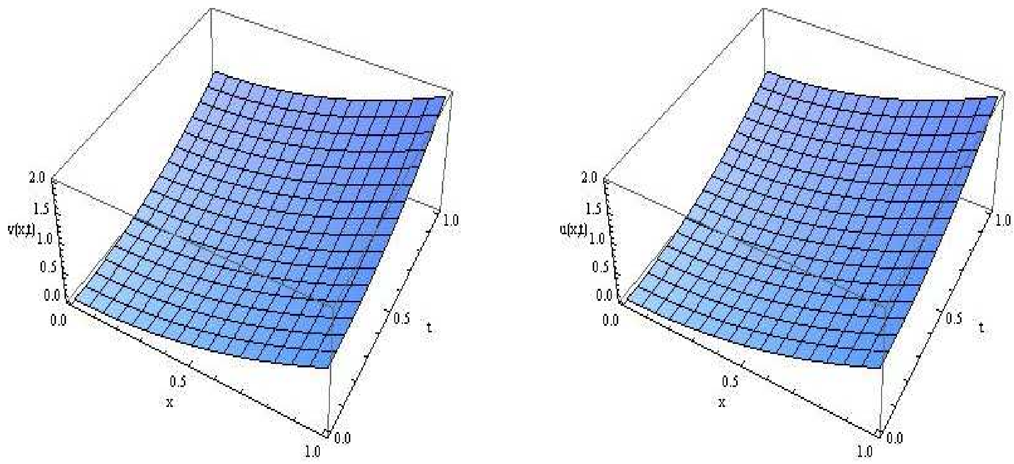
Now, by equation (3.11) we get

$$a = 0.$$

Now putting the value of a in equation (4.23), and we $U_k(t)$, $k \geq 0$ get the component of the DTM can be obtained. When we substitute all values of $U_k(t)$ from equations (4.22), (4.23) and (4.24) into equation (3.12), then the series solution can be formed as

$$u(x,t) = x^2 + t^2$$

which is the exact solution.



(a) (b)

Figure 3. The behavior of the (a) Exact solution, (b) LDTM solution, with respect to 'x' and 't', for Example 4.3.

5. Conclusion

In this paper, the LDTM has been successfully applied to find the exact solution of homogeneous and non-homogeneous linear PDEs with boundary conditions. The aim of this paper is to describe that the LDTM gives approximate analytical solution of PDEs and it is more closer to exact solution. The proposed method requires less computational work compared with the DTM. The LDTM solution can be calculated easily in short time and the graphs were performed by using Mathematica 8.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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