Journal of Informatics and Mathematical Sciences
Volume 1 (2009), Numbers 2 & 3, pp. 157–163

(*Invited paper*)

On a Problem of Roger Cuculi re

Roman Ger

Abstract. In the February 2008 issue of *The American Mathematical Monthly* (115, Problems and Solutions, p. 166) the following question was proposed by Roger Cuculi re:

Find all nondecreasing functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x+f(y)) = f(f(x))+f(y)$ for all real x an y (Problem 11345).

In the present paper we establish the general Lebesgue measurable solution, monotonic solutions as well as a description of the general solution of the functional equation in question.

1. Introduction

In what follows the symbols $\mathbb{R}$ and $\mathbb{Z}$ will stand for the set of all real numbers and all integers, respectively. Moreover, the *floor* and *ceiling* functions are defined by

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}, \quad x \in \mathbb{R},$$

and

$$\lceil x \rceil := \min\{n \in \mathbb{Z} : x \leq n\}, \quad x \in \mathbb{R},$$

respectively. Finally, id will denote the identity mapping on $\mathbb{R}$.

The following question was asked by a French mathematician Roger Cuculi re (see [2])

Find all nondecreasing functions f from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x+f(y)) = f(f(x))+f(y), \quad x, y \in \mathbb{R}. \tag{1.1}$$

I have submitted a solution of a considerably generalized Cuculi re's problem pretty soon after the question was published. Nineteen months later, in the October 2009 issue of *The American Mathematical Monthly* (116, Problems and Solutions, p. 753) a solution given by Richard Stong [6] has appeared with the following

2010 *Mathematics Subject Classification.* 39B22; 39B52; 39B12.
Key words and phrases. Cuachy type composite equation; measurable solutions; monotonic solutions; general solution.

Editorial comment. Roman Ger (Katowice, Poland) analyzed solutions of this functional equation that are not necessarily nondecreasing. The Lebesgue measurable solutions are: (i) constant 0; (ii) identity function $f(x) = x$; or (iii) $f(x) = an(x)$, where a is any positive real and $n : \mathbb{R} \to \mathbb{Z}$ is defined by

$$n(x) := n_0(x - ka) + k \quad \text{for } x \in [ka, (k+1)a),\ k \in \mathbb{Z},$$

and $n_0 : [0, a) \to \mathbb{Z}$ is an arbitrary Lebesgue measurable function vanishing at 0. There are also solutions that are not Lebesgue measurable (assuming the axiom of choice).

In the present article we wish to offer the details by establishing the general Lebesgue measurable solution; it is an easy task to get monotonic solutions from them. Finally, a description of the general solution of the functional equation (1.1) will be given.

2. **Measurable Solutions**

The following result establishes the general Lebesgue measurable solution of Cuculi re's equation.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable solution to the functional equation* (1.1). *Then either $f = 0$ or $f = \mathrm{id}$ or $f(x) = n(x)a$, $x \in \mathbb{R}$, where a is an arbitrary positive real number and $n : \mathbb{R} \to \mathbb{Z}$ is defined as follows*

$$n(x) := n_0(x - ka) + k \quad \textit{for } x \in [ka, (k+1)a),\ k \in \mathbb{Z},$$

and $n_0 : [0, a) \to \mathbb{Z}$ is an arbitrary Lebesgue measurable function vanishing at 0*.*

Conversely, each function of that type yields a Lebesgue measurable solution to equation (1.1).

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of (1.1). On setting $x = y = 0$ we get the equality $f(0) = 0$, whence $f^2(x) := f(f(x)) = f(x)$, $x \in \mathbb{R}$. Consequently, equation (1.1) reads now as

$$f(x + f(y)) = f(x) + f(y), \quad x, y \in \mathbb{R}. \tag{2.1}$$

Replacing here x by $f(x)$, in view if the equality $f^2 = f$, we arrive at

$$f(f(x) + f(y)) = f(x) + f(y), \quad x, y \in \mathbb{R}. \tag{2.2}$$

In particular, (2.2) implies that the set $Z := f(\mathbb{R})$ forms a subsemigroup of the additive group $(\mathbb{R}, +)$. As a matter of fact, $(Z, +)$ is a subgroup of $(\mathbb{R}, +)$ because of the equality

$$-f(y) = f(-f(Y)), \quad y \in \mathbb{R},$$

resulting from (2.1) on setting $x = -f(y)$. Now, the equality $f^2 = f$ says nothing else but

$$f(z) = z \quad \text{for all } z \in Z. \tag{2.3}$$

Put $F(x) := f(x) - x$, $x \in \mathbb{R}$. By means of (2.1), for every $x, y \in \mathbb{R}$, we have

$$\begin{aligned} F(x + f(y)) &= f(x + f(y)) - x - f(y) \\ &= f(x) + f(y) - x - f(y) = F(x). \end{aligned}$$

In other words, each member of Z yields a period for F. Clearly, Z is either closed or dense in $\mathbb{R}$. It is well known that a closed subgroup of $(\mathbb{R}, +)$ is either trivial ($Z = \{0\}$) or coincides with $\mathbb{R}$ or has the form $a\mathbb{Z}$ for some positive real number a. Obviously, $Z = \{0\}$ states that $f = 0$, whereas the equality $Z = \mathbb{R}$ forces F to be constant, whence $f = \mathrm{id}$ because $F(0) = f(0) = 0$. Having $Z = a\mathbb{Z}$ we infer that there exists a surjection $n : \mathbb{R} \to \mathbb{Z}$ such that

$$f(x) = n(x)a \quad \text{for } x \in \mathbb{R}.$$

Jointly with (2.1) this implies that

$$n(x + n(y)a) = n(x) + n(y), \quad x, y \in \mathbb{R}. \tag{2.4}$$

Since n is surjective there exists an $x_0 \in \mathbb{R}$ such that $n(x_0) = 1$. On setting $y = x_0$ in (2.4) we get

$$n(x + a) = n(x) + 1 \quad \text{for all } x \in \mathbb{R}$$

whence, by an easy induction,

$$n(x + ka) = n(x) + k \quad \text{for all } x \in \mathbb{R} \text{ and all } k \in \mathbb{Z}.$$

With $n_0 := n|_{(0,a)}$ we have then $n_0(0) = 0$ and

$$n(x) = n_0(x - ka) + k \quad \text{for } x \in [ka, (k+1)a),\ k \in \mathbb{Z}.$$

Now the measurability of f forces n and a *fortiori* n_0 to be measurable.

Finally, in the case where Z is dense in $\mathbb{R}$, the function F has a dense set of periods and being measurable it has to be constant almost everywhere with respect to Lebesgue measure in $\mathbb{R}$. Thus, there exists a $c \in \mathbb{R}$ and a nullset $E \subset \mathbb{R}$ such that

$$f(x) - x = F(x) = c \quad \text{for all } x \in \mathbb{R} \setminus E.$$

Fix arbitrarily a $t \in \mathbb{R}$ and take any x off the nullset $E \cup (t - E)$. Then both x and $y := t - x$ are in $\mathbb{R} \setminus E$ whence, by means of (2.1),

$$\begin{aligned} f(t + c) &= f(x + y + c) \\ &= f(x + f(y)) = f(x) + f(y) \\ &= x + c + y + c = t + 2c. \end{aligned}$$

In particular, by putting here $t = -c$, we obtain $0 = f(0) = c$, and consequently

$$f(t) = t \quad \text{for all } t \in \mathbb{R},$$

as claimed.

Since it is a straightforward matter to check that 0, id and na with n described in the assertion yield Lebesgue measurable solutions of equation (1.1), the proof has been completed.

To derive monotonic solutions from Theorem 2.1 it suffices to observe that a nondecreasing function $n_0 : [0,a) \to \mathbb{Z}$ has to satisfy the inequalities

$$0 = n_0(0) \leq n_0(x) \leq n(a) = 1 \quad \text{for all } x \in [0,a),$$

whence $n_0(x) \in \{0,1\}$ for all $x \in [0,a)$. Put

$$b := \sup\{x \in [0,a) : n_0(x) = 0\}.$$

If $b = 0$, then $f(x) = a\lceil \frac{1}{a}x \rceil$, $x \in \mathbb{R}$, whereas for $b = a$ we get $f(x) = a\lfloor \frac{1}{a}x \rfloor$, $x \in \mathbb{R}$.

Finally, in the case where $b \in (0,a)$, we have to distinguish two possibilities:

$$n_0(b) = 0 \quad \text{or} \quad n_0(b) = 1.$$

Then, accordingly,

$$f(x) = a\left\lceil \frac{x-b}{a} \right\rceil, \ x \in \mathbb{R}, \quad \text{or} \quad f(x) = a\left\lfloor \frac{x+(a-b)}{a} \right\rfloor, \ x \in \mathbb{R},$$

which leads to the following

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing solution of the functional equation* (1.1). *Then either $f = 0$ or $f = \mathrm{id}$ or $f(x) = a\lfloor \frac{x+b}{a} \rfloor$, $x \in \mathbb{R}$, or $f(x) = a\lceil \frac{x-b}{a} \rceil$, $x \in \mathbb{R}$, where a, b are arbitrary reals with $a > 0$ and $b \in [0,a)$.*

Conversely, each function of that type yields a nondecreasing solution to equation (1.1).

This coincides (jointly with the symbols used) with Stong's solution presented in [6]. Comparing the statements of Theorems 2.1 and 2.2 we see that the family of Lebesgue measurable solutions to equation (1.1) is considerably larger than that of monotonic ones. What about (possible) nonmeasurable solutions? As we shall see in the sequel, they do exist actually.

3. **A Description of the General Solution**

Observe that there exist nonmeasurable solutions to equation (1.1). Actually, it suffices to take any additive function $a : \mathbb{R} \to \mathbb{R}$ such that $f|_H = 1$ where H stands for a Hamel basis of the reals such that $1 \in H$. Plainly, a is discontinuous because it is nonconstant and assumes rational values only; hence it is nonmeasurable but being a projection ($a(a(x)) = a(x), x \in \mathbb{R}$) the function a satisfies (1.1) since

$$a(x + a(y)) = a(x) + a(a(y)) = a(a(x)) + a(y), \quad x, y \in \mathbb{R}.$$

This gives rise to look for the general solution of equation (1.1) without any regularity assumption. The description of such solutions may easily be obtained from the fact that, because of the equality $f^2 = f$, equation (1.1) may also be written in the form

$$f(x + f(y)) = f(x) + f(f(y)), \quad x, y \in \mathbb{R}$$

which becomes simply a Cauchy equation on the "cylinder" $\mathbb{R} \times Z$ and it remains to apply Theorem 2 from the paper of Jean Dhombres and Roman Ger [4] (see also Jean Dhombres [3]) which we quote here explicitly, for the sake of completeness.

Proposition. *Let Y be a nonempty subset of an Abelian group $(X, +)$ and let $(Z, +)$ be the subgroup generated by Y. Suppose $(G, +)$ to be an Abelian group. A map $f : X \to G$ satisfies the Cauchy equation*

$$f(x + y) = f(x) + f(y) \quad \text{for all pairs } (x, y) \in X \times Y,$$

if and only if f can be written in the form

$$f(x) = a(x - \xi(\pi(x))) + h(\pi(x)), \quad x \in X,$$

where $h : X/Z \to G$ is a function satisfying $h(0) = 0$, the function $a : Z \to G$ is additive, ξ is a lifting relative to Z with $\xi(0) = 0$ and $\pi : X \to X/Z$ is the canonical epimorphism.

Therefore, bearing in mind that in our case $(X, +) = (G, +) = (\mathbb{R}+)$ and Z is just the range of the map f in question, with $f(x) = x, x \in Z$ (see (2.3)), we arrive at

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a solution to equation (1.1) and let $Z := f(\mathbb{R})$. Then*

$$f(x) = x + g(\pi(x)), \quad x \in \mathbb{R}, \tag{3.1}$$

where $\pi : \mathbb{R} \to \mathbb{R}/Z$ is the canonical epimorphism and $g : \mathbb{R}/Z \to \mathbb{R}$ is a function such that $g(0) \neq 0$ and $g(\pi(x)) \in Z - x$ for all $x \in \mathbb{R}$.

Conversely, for an arbitrary subgroup $(Z, +)$ of the additive group $(\mathbb{R}, +)$ and for an arbitrary function $g : \mathbb{R}/Z \to \mathbb{R}$ such that $g(0) = 0$ and $g(\pi(x)) \in Z - x$ for all $x \in \mathbb{R}$, the function f given by (3.1) yields a solution of equation (1.1).

Proof. We know already (cf. the proof of Theorem 2.1) that the range $Z := f(\mathbb{R})$ of a solution $f : \mathbb{R} \to \mathbb{R}$ to equation (1.1) forms an additive subgroup of $(\mathbb{R}, +)$ and that f satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \quad \text{for all pairs } (x, y) \in \mathbb{R} \times Z.$$

Consequently, by means of Proposition,

$$f(x) = a(x - \xi(\pi(x))) + h(\pi(x)), \quad x \in \mathbb{R},$$

where the symbols a, h, ξ and π have the meaning described in the statement of Proposition. Since we have also $f(x) = x$ for $x \in Z$ (see (2.3)) we infer that the additive map $a : Z \to \mathbb{R}$ has to be the identity on Z, whence (3.1) follows by setting $g := h - \xi$. Clearly, we have then $g(0) = h(0) - \xi(0) = 0$ as well as $g(\pi(x)) = f(x) - x \in Z - x$ for all $x \in \mathbb{R}$.

Since the last part of the assertion is a subject of a straightforward verification the proof has been completed.

4. Concluding Remarks

We terminate this paper with several comments.

(i) The latter result (Theorem 3) was obtained independently by Marcin Balcerowski [1] with much longer proof because he was unaware of Proposition as a potential tool.

(ii) The description of the general solution of equation (1.1) given in Theorem 4 is useless when one wishes to determine regular (say, measurable) solutions to that equation. Such a situation is entirely analogous to that one with the well known Hamel description of solutions to the classical Cauchy functional equation (additivity) and the question of finding its Lebesgue measurable solutions.

(iii) Marcin Balcerowski (oral communication) and Christopher Carl Heckmann [5] have obtained, independently, the form of nondecreasing solution of equation (1.1) in June, 2008 and second half of 2008, respectively. Balcerowski's description has not been expressed in terms of the floor and ceiling functions. My solution of the (generalized) Cuculi re's problem (Theorem 2 of the present paper) has been submitted to the *American Mathematical Monthly* in the first half of March 2008.

(iv) I wish to express my thanks to Nicole Brillou t-Belluot (Nantes, France) for contacting me with Roger Cuculi re; he has kindly informed about the background of equation (1.1). Namely, during the International Mathematical Olympiad that was held in Mumbai (India) in 1996, one of the problems presented was to find nondecreasing sequences of positive integers solving equation (1.1). This motivated Roger Cuculi re to look for nondecreasing real solutions of (1.1), defined on the real line, and to pose his problem [2].

References

[1] M. Balcerowski, *O problemie Cuculi re*, Talk presented at the seminar on Functional Equations, Silesian University of Katowice, Poland, January 13, 2009.

[2] R. Cuculi re, Problem 11345, *The American Mathematical Monthly* **115**(2)(2008), 166.

[3] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn Univ., Bangkok, 1979.

[4] J. Dhombres and R. Ger, Conditional Cauchy equations, *Glasnik Matematički* **13**(1978), 39–62.

[5] C. C. Heckman, *My Solutions to American Mathematical Monthly Problems* (website).

[6] R. Stong, Solution of Problem 11345, *The American Mathematical Monthly* **116**(8) (2009), 753.

Roman Ger, *Instytut Matematyki, Uniwersytetu Śląskiego, ul. Bankowa 14, PL-40-007 Katowice, Poland.*
E-mail: `romanger@us.edu.pl`

Received November 2, 2009
Accepted December 10, 2009