Journal of Informatics and Mathematical Sciences
Vol. 13, No. 1, pp. 9–24, 2021
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
DOI: 10.26713/jims.v13i1.1533

http://www.rgnpublications.com

Research Article

The Numerical Solutions of FIEs of the Second Kind of Degenerated Type Using Bownd's Methods

Atefa J. Saleh

Department of Mathematics, College of Basic Education, Mustansiriyah University, Baghdad, Iraq
atifa_jalil.edbs@uomustansiriyah.edu.iq

Received: March 7, 2021 **Accepted:** March 19, 2021 **Published:** March 31, 2021

Abstract. In this work, we consider the numerical solutions of some types of Fredholm integro-differential equations (FIDEs). Namely, Bownd's, successive, Adomian's and the improved methods are used to solve linear Fredholm integro-differential equations (FIDEs). In addition, some numerical examples are given to show the efficiency and accuracy of the proposed methods. The numerical results show that using the improved methods is the best technique to solve such types of problems.

Keywords. Runge-Kutta method; Fredholm equations; Single-step equation; Adomian's decomposition method; Degenerated type equation

Mathematics Subject Classification (2020). 45B05; 65R20

1. Introduction

In several articles, Bownd and Wood [4] have developed a new method for solving FIE based on reducing the (FIE) to an initial-value (ODEs), which can be solved using some numerical methods.

Clearly, there is a strong similarity between Fredholm Equations and initial value problems for ordinary differential equations. Indeed any such problem can be reformulated as Fredholm equations.

Consequently, the methods for solving initial value problemsare of considerable help in suggesting methods for dealing with Fredholm equations.

Adomian's decomposition method is a mathematical method, which can be applied to solve linear differential equations, deterministic and stochastic operator equations, and many algebraic equations. In this method, the solve is found as an infinite series which converges rapidly to an accurate solution.

Moreover this method can be considered as an extension to the successive approximation methods, while being much more powerful [1].

In this work, Bownd's, successive, Adomian's and the improved methods are used to solve linear Fredholm integro-differential equations (FIDEs). Moreover, some numerical examples are given to show the efficiency and accuracy of the proposed methods.

2. Solving System of (DEs)

In this search contains an introduction to the numerical solution of n-th-order system of first-order initial-value problems.

$$\left.\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots, y_m),\\ \frac{dy_2}{dx} &= f_1(x, y_1, y_2, \dots, y_m),\\ &\vdots\\ \frac{dy_m}{dx} &= f_1(x, y_1, y_2, \dots, y_m).\end{aligned}\right\} \tag{1}$$

For $a \le x \le b$ with initial conditions

$$y_1(a) = \alpha_1,\ y_2(a) = \alpha_2, \dots, y_m(a) = \alpha_m\,. \tag{2}$$

The object is to find m function $y_1, y_2, \dots, y_m$ that satisfies the system of differential equations together with all the initial conditions [6].

In our work we used fourth-order Runge-Kutta method as a numerical method to treat the system (1) subject to the conditions (2).

In this work the following algorithm for solving the system (1) using fourth order-Runge-Kutta method has been used.

2.1 Algorithm of Runge-Kutta of Fourth Order Four

Step 1: Set $h = (b-a)/N$.

Step 2: Compute $k_{1i} = hf_i(x_j, y_{1j}, y_{2j}, \dots, y_{mj})$ for each $i = 1, 2, \dots, m$

Step 3: Compute $k_{2i} = hf_i(x_j + h/2, y_{1j} + 1/2k_{11}, y_{2j} + 1/2k_{12}, \dots, y_{mj} + 1/2k_{1m})$ for each $i = 1, 2, \dots, m$

Step 4: Compute $k_{3i} = hf_i(x_j + h/2, y_{1j} + 1/2k_{21}, y_{2j} + 1/2k_{22}, \dots, y_{mj} + 1/2k_{2m})$ for each $i = 1, 2, \dots, m$

Step 5: Compute $k_{4i} = hf_i(x_j + h/2, y_{1j} + 1/2k_{31}, y_{2j} + 1/2k_{32}, \dots, y_{mj} + 1/2k_{3m})$ for each $i = 1, 2, \dots, m$

Step 6: $y_{ij+1} = y_{ij} + 1/6(k_{1i} + 2k_{2i} + 3k_{3i} + k_{4i})$ for each $i = 1, 2, \dots, m$

Step 7: set $x = a + jh$, for $j = 1, 2, \dots, N$

3. Decomposable Kernel

Recalling that the form FIE is as follows,

$$u(x) = f(x) + \lambda \int_a^b k(x,y)u(t)dt, \quad 1 \le x \le b,$$

where $k(x,t)$ and $f(x)$ are given functions and $u(x)$ is unknown fun.

Definition 1. The kernel $k(x,t)$ is called decomposable if it is the sum of a finite number of products of function of x alone by functions of y alone, [11, 12], i.e. it is of form,

$$k(x,t) = \sum_{i=1}^{n} a_i(x)b_i(t). \tag{3}$$

For example for the above used definition consider the following:

Example 1. Consider the kernel $k(x,t) = 1 - x\cos xt$, $(x,y) \in S$ may be approximated by finite number of terms of its Taylor's series about $(x_0,t_0) = 0$ i.e.

$$1 - x\left(1 - \frac{x^2t^2}{2!} + \frac{x^4t^4}{4!} - \dots\right) = 1 - x + \frac{x^3t^2}{2!} - \frac{x^5t^4}{4!} + \dots.$$

Hence, if only 3-terms of the series are considered, we have a degenerate kernel:

$$\begin{aligned} k(x,t) &= 1 - x + \frac{x^3t^2}{2!} - \frac{x^5t^4}{4!} + \dots \\ &\cong \sum_{r=1}^{3} a_r(x)b_r(t) \end{aligned}$$

where $a_1(x) = (1-x)$, $a_2(x) = x^3$, $a_3(x) = x^5$ and $b_1(t) = 1$, $b_2(t) = t^2/2$, $b_3(t) = -t^4/4!$.

4. Bownds Method

Our intention is to approximate the kernel $k(x,t)$ with an appropriate finite sum and, numerically a certain system of differential equations is solved. since we only use an approximation to the kernel $k(x,t)$ with linear case, we of course, expect to experience additional error.

The following theorem is very essential for converting an approximation to VIE into a certain initial value problem,

Theorem 1. *Suppose that*

$$k(x,t) = \sum_{i=1}^{m} A_i(x)B_i(t). \tag{4}$$

The linear equation

$$u(x) = f(x) + \int_a^b k(x,t)u(t)dt \tag{5}$$

has a solution

$$u(x) = f(x) + \sum_{i=1}^{m} A_i(x)Q_i(x) \tag{6}$$

where $Q_i(x)$ are the solution of the system

$$Q_i'(x) = B_i(x)\left(f(x) + \sum_{j=1}^{m} A_j(x)Q_j(x)\right), \tag{7}$$

$$Q_i(a) = 0, \quad i = 1,2,\ldots,m. \tag{8}$$

Proof. See [14] for proof. □

5. Solution of FIE of the Second Kind of Degenerated Type Using Bownd's Method

Recall the equation

$$u(x) = f(x) + \lambda\int_a^b k(x,t)u(t)dt, \quad a \le x \le b. \tag{9}$$

In this search Bownd's method will be used to approximate the solution of equation (9). If the integral eq. has a decomposable kernel then it is not difficult to reduce the integral equation to a system of differential equations (7) and then the method produces an approximate solution. In order to solve eq. (9) we follow this algorithm.

Algorithm (B.M.). *Step* 1: Write the kernel as either an exact or approximate representation

$$k(x,t) = \sum_{i=1}^{m} A_i(x)B_i(t). \tag{10}$$

Step 2: Substituting equation (10) in equation (9) gives

$$u(x) = f(x) + \int_0^x \sum_0^m A_i(x)B_i(t)u(t)dt$$

or

$$u(x) = f(x) + \sum_{i=1}^{m} A_i(x)\int_0^a B_i(t)u(t)dt.$$

Let

$$Q_i(x) = \int_0^b B_i(t)u(t)dt.$$

Then the approximate solution can be written as

$$u(x) = f(x) + \sum_{i=1}^{m} A_i(x)Q_i(x), \tag{11}$$

where

$$Q_i'(x) = B_i(x)\left(f(x) + \sum_{j=1}^{m} A_j(x)Q_j(x)\right);$$

$$Q_i(0) = 0. \tag{12}$$

Step 3: Solve the system (12) by using fourth order Runge-Kutta method to find the values of $Q_i(x)$.

Step 4: Substitute the values of $Q_i(x)$ in eq (11) to find the approximate solution for $u(x)$.

6. Some Basic Definitions and Lemmas

Definition 2 ([7, 15])**.** The space $L_2(a,b)$ is the set of all square integrable functions in the interval $[a,b]$, i.e. k is L_2 function on interval $[a,b]$ if and only if

$$\int_a^b \int_a^b |k(x,t)|^2 dt dx < \infty.$$

Moreover, the operator

$$K = \int_a^b k(x,t)dt$$

is called Fredholm operator.

We say that k is L_2 integable function on interval $[a,b]$, if the integral of k is L_2 function on interval $[a,b]$ [15].

Definition 3 (Dirichlet's formula [5, 14])**.** If the two-dimensional function $\phi(x,s)$ is an L_2 function on interval $[a,b]$, then

$$\int_a^t \int_a^b \phi(x,s) ds ds = \int_a^t \int_s^x \phi(x,s) dx dx. \tag{13}$$

This lemma can be proved by interchanging the order of integration in equation (13), using elementary calculus arguments (see Figure 1).

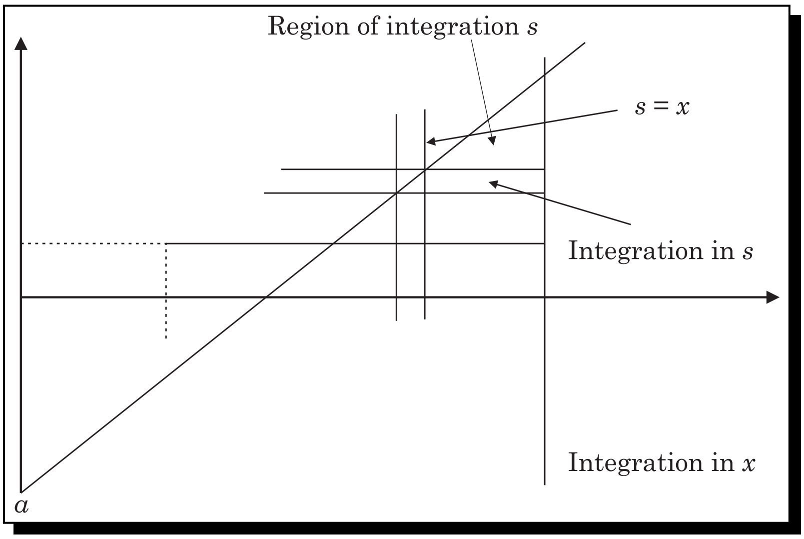

Figure 1. Interchanging the order of integration

Lemma 1. *Let f be L_2 function on the interval $[a,b]$, then any double integral can be reduced to a single integral as follows:*

$$\int_a^b \int_a^y f(t) dt dy = \int_a^b (x-t) f(t) dt. \tag{14}$$

Proof. See [7] for proof. □

Definition 4. Suppose that we have $M_{i0} = 1$ and $M_{0j} = 1$ for all $i, j = 0, 1, \ldots, m$, have $m \in N$ and

$$M_{ij} = \begin{cases} 0 & \text{if } i + j > m \\ \sum\limits_{r=0}^{i} M_{r,j-1}, & \text{otherwise} \end{cases}$$

$[M]$ is called *a special constant matrix* of natural numbers with dimension $m \times m$, starting from zero dimension. For example take $m = 3$ and $m = 4$ then the special constant matrix $[M]$ becomes respectively:

$$[M]_{3\times 3} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad [M]_{4\times 4} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 3 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Notation. The that lower limit is zero. Also, in our work we take $\lambda = 1$, so we are concerned with linear FIDE of order n of the problem:

$$\frac{d^n}{dx^n} u(x) + \sum_{i=0}^{n-1} P_i(x) u(x) = f(x) + \int_a^b k(x,t) u(t) dt, \quad 0 \le x \le b \tag{15}$$

with given initial condition

$$u(0), u'(0), u''(0), \ldots, u^{(n-1)}(0),$$

where f, P_i and k denote given (continuous) functions.

When $n = 1$ the linear FIDE of first order becomes:

$$\frac{d}{dx} u(x) + P_0(x) u(x) = f(x) + \int_a^b k(x,t) u(t) dt, \quad 0 \le x \le b \tag{16}$$

with given initial condition $u(0)$ where F, P_0 and k denote given (continuous) functions.

7. Reduction to Integral Equation

The reduction of IDEs to an integral equations can be used for the analysis of a variety of linear FIDEs.

Reduction Theorem ([4])**.** *Let f, k be iterated L_2 integrable function on interval $[a,b]$ and $P_i \in c^n[a,b]$ then eq.* (11) *is*

$$\left[D^n + \sum_{i=1}^{n-1} P_i(x) D^i \right] u(x) = f(x) + \int_a^b k(x,y) u(t) dt, \quad x \in [a,b]$$

with initial conditions $u(a) = u_0$, $u'(a) = u_1, \ldots, u^{(n)}(a) = u_n$ can be reduced to linear FIE in the form:

$$u(x) = F_n(x) + \int_a^b k_n(x,t) u(t) dt, \tag{17}$$

where

$$F_n(x) = \sum_{i=0}^{n-1} \frac{u_i x^i}{i!} + \frac{1}{(n-1)!} \int_a^b (x-t)^{n-1} f(t) dt$$

$$+\sum_{k=0}^{n-2}\sum_{j=k+1}^{n-1}\sum_{i=n+k-j}^{n-1}(-1)^k P_j^{(k)}(a)A_{k,n-i-1}\frac{u_{i+j-n-k}x^j}{i!}$$

and

$$k_n(x,t)=\frac{1}{(n-1)!}\int_t^x (x-z_1)^{n-1}k(z_1,t)dz_1$$
$$+\sum_{k=0}^{n-1}\sum_{j=0}^{k}(-1)^k P_{j+n-k-1}^{(k)}(t)B_{n-k-1,j}\frac{(x-t)^k}{k!}$$

$[A]$ and $[B]$ are two special constant matrices of dimensions $n-2\times n-2$ and $n-1\times n-1$, respectively.

We conclude with the following observation that: it has already been shown that a linear FIDEs may be expressed as a linear FIEs of the second kind. Therefore, the methods which solve the second kind linear FIEs (Laplace transform, Degenerate kernel, etc. [3, 5, 8, 11] can be used to treat the linear FIDEs of higher order, as illustrated in the following example.

Example 2. Consider the 3rd order linear FIDE

$$(D^3-D^2-D-1)u(x)=\sin x-\int_a^b e^{x-t}u(t)dt$$

with initial conditions $u(0)=1$, $u'(0)=u''(0)=-1$.

Using equation (18) with $n=3$, we obtain the IE of the second kind:

$$u(x)=F_3(x)+\int_a^b k_3(x,t)u(t)dt$$

where

$$F_3(x)=\sum_{i=0}^{2}\frac{u_ix^j}{i!}+\frac{1}{2!}\int_a^b (x-t)^2\sin t\,dt+\sum_{k=0}^{1}\sum_{j=k+1}^{2}\sum_{i=3+k-j}^{2}(-1)^k P_j^{(k)}(0)A_{k,2-i}\frac{u_{i+j-3-k}x^i}{i!}$$
$$=1-x-\frac{x^2}{2}+\cos x+\frac{x^2}{2}-1+\left(\frac{-x^2}{2}-\left(x-\frac{x}{2}\right)\right)$$
$$=\cos x$$

and

$$k_n(x,t)=\frac{1}{2!}\int_t^x (x-z_1)^2 e^{z_1-t}dz_1-\sum_{k=0}^{2}\sum_{j=0}^{k}(-1)^j P_{j+2-k}^{(k)}(t)B_{2-k,j}\frac{(x-t)^k}{k!}$$
$$=e^{x-t}-\frac{(x-t)^2}{2}-(x-t)-1+1+(x-1)+\frac{(x-t)^2}{2}$$
$$=e^{x-t}.$$

Hence, we have the linear FIE of second kind

$$u(x)=\cos x-\int_a^b e^{(x-t)}u(t)dt.$$

We solve it by any methods of linear FIEs, using Laplace transform [7], we obtain

$$u(x)=\cos x-\sin x.$$

8. Method of Successive Approximation

In this method the technique starts with substituting initial condition for the solution $u(x)$ in eq. (11) as the zeroth approximation to $u_0(x)$ to obtain a first approximation $u_1(x)$ [13].

Recall equation (15)

$$\frac{d^n}{dx^n}u(x)+\sum_{i=0}^{n-1}P_i(x)u(x)=f(x)+\int_a^b k(x,t)u(t)dt,$$

$$u^{(n)}(x)+\sum_{i=0}^{n-1}p_i(x)u^{(i)}(x)=f(x)+\lambda\int_a^b k(x,t)u(t)dt, \quad 0\le x\le b \tag{18}$$

with initial condition

$$u(0)=u_0,u'(0)=u_1,\ldots,u^{(n-1)}(0)=u_{n-1}$$

Eq. (15) can be written in the following form

$$\frac{d^n}{dx^n}u(x)=f(x)+\int_a^b k(x,t)u(t)dt-+\sum_{i=0}^{n-1}P_i(x)u(x). \tag{19}$$

Successive approximation method with the first iteration gives.

$$\frac{d^n}{dx^n}u_1(x)=f(x)+\int_a^b k(x,t)u_0(t)dt-+\sum_{i=0}^{n-1}P_i(x)u_0(x). \tag{20}$$

Substituting the 0th approximation $u_0(x)=0$ in to (20) gives

$$\frac{d^n}{dx^n}u_1(x)=f(x) \tag{21}$$

with initial condition

$$u_1(0)=u_0,u_1'(0)=u_1,\ldots,u_1^{(n-1)}(0)=u_{n-1}$$

after integration equation (21) n-times, yields

$$u_1(x)=F(x)+\sum_{i=0}^{n-1}\frac{x^i}{i!}u_i.$$

Then substituting the first approximation $u_1(x)$ again in eq. (19) yields second approximate $u_2(x)$

$$\frac{d^n}{dx^n}u_2(x)=f(x)+\int_a^b k(x,t)u_1(t)dt-\sum_{i=0}^{n-1}P_i(x)u_1(x)$$

with initial condition

$$u_2(0)=u_0,u_2'(0)=u_1,\ldots,u_2^{(n-1)}(0)=u_{n-1}.$$

As before and after using the initial condition, we have $u_2(x)$. After m iteration, a method of Successive approximation gives the following:

$$\frac{d^n}{dx^n}u_m(x)=f(x)+\int_a^b k(x,t)u_{m-1}(t)dt-\sum_{i=0}^{n-1}P_i(x)u_{m-1}(x) \tag{22}$$

with initial condition

$$u_m(0)=u_0,u_m'(0)=u_1,\ldots,u_m^{(n-1)}(0)=u_{n-1}.$$

where $m=1,2,3,\ldots$ and $0\le x\le b$.

Then determine whether $u_m(x)$ approaches the solution $u(x)$ as m increases.

It turns out that if $f(x)$ is continuous for $0 \le t \le x$ and if $k(x,t)$ is continuous for $0 \le t \le x$, then it can be proved that the sequence $u_m(x)$ will converge to the solution $u(x)$ of equation (15) [11].

Example 3. Consider the equation

$$u'(x) = 1 - \int_a^b u(t)dt, \quad 0 \le x \le 1$$

with initial condition $u(0) = 0$.

This initial value problem may be written as

$$u(x) = x - \int_a^b (x-t)u(t)dt \tag{23}$$

when applying successive method, we have:

Start with $u_0(x) = 0$ in the integral to obtain $u_1(x)$

$$u_1(x) = x - \int_a^b (x-t)dt,$$

$$u_1(x) = x,$$

$$u_2(x) = x - \int_a^b (x-t)u_1(t)dt,$$

$$u_2(x) = x - \int_a^b (x-t)t\,dt$$

$$= x - \frac{x^3}{6} = x - \frac{x^3}{3!}.$$

Now,

$$u_3(x) = x - \int_a^b (x-t)\left(t - \frac{t^3}{3!}\right)dt$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^3}{3} - \frac{x^5}{30}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Continue this process to obtain the nth approximation $u_n(x)$ as:

$$u_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Hence, the solution to equation (23) is

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x.$$

9. Adomian's Method [10]

Adomian's decomposition method is an approximation method which can be applied to the solution of linear or nonlinear both differential and integral equations; in this method the solution is found as an infinite series which converges rapidly to an accurate solution. Moreover it can be considered as an extension to the successive approximation method. While being much more powerful, such a method is more efficient and easily computable.

The Adomian's method consists of representing u as a series $u = \sum_{k=0}^{\infty} u_k$ where the terms u_k are calculated by the following algorithm:

$$u_0 = f(x)$$
$$u_1 = A_0$$
$$u_2 = A_1$$
$$\vdots$$
$$u_k = A_{k-1}$$

A_k is called *Adomain's polynomials*, which are defined by:

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[\sum_{m=0}^{\infty}\lambda^m u_m\right]_{\lambda=0}, \quad k = 0,1,2,\ldots. \tag{24}$$

Generally, it is possible to obtain exactly A_k as a function of $u_0, u_1, \ldots, u_k$.

According to the above algorithm, we see that the solution u can be determined by the calculation of A_n^s and we have:

$$u = \sum_{k=0}^{\infty} u_k = f(x) + \sum_{k=0}^{\infty} A_k.$$

Recall equation (15)

$$\frac{d^n}{dx^n}u(x) + \sum_{i=0}^{n-1} P_i(x)\frac{d^i}{dx^i}u(x) = f(x) + \int_a^b k(x,t)u(t)dt.$$

With given initial

$$u(0), u'(0), u''(0), \ldots, u^{(n-1)}(0).$$

Let $\frac{d^n}{dx^n}u_0(x) + \sum_{i=0}^{n-1} P_i(x)\frac{d^i}{dx^i}u_0(x) = f(x)$ which represents the ordinary differential equation with initial conditions and it can be solved by using one of the analytic methods depending on the kind of this equation, we get the initial solution $u_0(x) = f(x)$.

Using equation (24), we get:

$$A_k = \int_a^b k(x,t)\frac{d^0}{d\lambda^0}\left[\sum_{m=0}^{\infty}\lambda^m u_m\right]_{\lambda=0} dt = \frac{d^n}{dx^n}u_1(x) + \sum_{i=0}^{n-1} P_i(x)\frac{d^i}{dx^i}u_1(x). \tag{25}$$

As before and after using the initial conditions, we have $u_1(x)$.

After k iterations, the method of Adomain gives the following:

$$A_k = \frac{1}{k!}\int_a^b k(x,t)\frac{d^k}{d\lambda^k}\left[\sum_{m=0}^{\infty}\lambda^m u_m\right]_{\lambda=0} dt = \frac{d^n}{dx^n}u_k(x) + \sum_{i=0}^{n-1} P_i(x)\frac{d^i}{dx^i}u_k(x).$$

As before and after using the initial condition, we obtain k-term approximation to the solution $u(x)$ by:

$$\phi_k(x) = \sum_{k=0}^{k} u_i$$

with

$$\lim_{k\to\infty}\phi_k(x)=u(x).$$

Example 4. Consider the equation

$$u'(x)=1-x^3+\int_a^b 2xu(t)dt,\quad 0\le x\le 1$$

with initial condition $u(0)=0$.

This initial value problem may be written as

$$u(x)=x-\frac{x^4}{4}+\int_a^b (x^2-t^2)u(t)dt$$

when applying Adomain's method, we have

$$u_0(x)=x-\frac{x^4}{4}$$

$$u_1(x)=A_0=\int_a^b (x^2-t^2)u_0(t)dt=\int_a^b (x^2-t^2)\left(t^2-\frac{t^2}{4}\right)dt$$

$$=\frac{x^4}{4}-\frac{x^7}{70},$$

$$u_2(x)=A_1=\int_a^b (x^2-t^2)u_1(t)dt=\int_a^b (x^2-t^2)\left(\frac{t^4}{4}-\frac{t^7}{70}\right)dt$$

$$=\frac{x^7}{70}-\frac{x^{10}}{2800},$$

$$u_3(x)=A_2=\int_a^b (x^2-t^2)u_2(t)dt=\int_a^b (x^2-t^2)\left(\frac{t^7}{70}-\frac{t^{10}}{2800}\right)dt$$

$$=\frac{x^{10}}{2800}-\frac{x^{13}}{101000}.$$

Continuing this process, we obtain nth approximation u_n the exact solution

$$u(X)=\sum_{n=0}^{\infty}u_n=f(x)+\sum_{n=0}^{\infty}A_n=x.$$

10. Aitkens Elementary Approximation [9]

This method deals with the linear Fredholm Integral equation of the second kind having three approximate solutions $u_0(t)$, $u_1(t)$, $u_2(t)$ we can extrapolate (elementary) to an improved and estimate, this can be done by considering the following formula

$$\alpha(t)=u_2(t)-\frac{(u_2(t)-u_2(t))^2}{u_3(t)-2u_2(t)+u_t(t)},$$

where $u_0(t)$, $u_1(t)$ and $u_2(t)$ are solutions of the Adomian's method of Successive approximation method or any method which uses iterated kernel method, and $\alpha(t)$ is approximate exact solution we can simplify it as

$$\alpha(t)=\frac{(u_3(t)u_1(t)-u_2^2(t))^2}{u_3(t)-2u_2(t)+u_t(t)}. \tag{26}$$

The general form of it

$$\alpha(t) = \frac{(u_{n+3}(t)u_{n+1}(t) - u_{n+2}^2(t))^2}{u_{n+3}(t) - 2u_{n+2}(t) + u_{n+1}(t)}. \tag{27}$$

11. Iterated Kernels [2]

Recall Fredholm integral equation

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt$$

we can modify it by this formula

$$\Psi(x) = f(x) + \sum_{n=1}^{\infty} \Psi_n(x)\lambda^n, \tag{28}$$

where the function $\Psi_n(x)$ may be found from the formulas

$$\Psi_1(x) = \int_a^b k(x,t)f(t)dt$$

$$\Psi_2(x) = \int_a^b k(x,t)\Psi_1(t)dt = \int_a^b k_2(x,t)f(t)dt$$

$$\Psi_3(x) = \int_a^b k(x,t)\Psi_2(t)dt = \int_a^b k_3(x,t)f(t)dt$$

and so on

$$\Psi_n(x) = \int_a^b k(x,t)\Psi_{n-1}(x)dt. \tag{29}$$

Here

$$k_2(x,t) = \int_a^b k(x,t)k_1(s,t)ds$$

$$k_3(x,t) = \int_a^b k(x,t)k_2(s,t)ds$$

and generally

$$k_n(x,t) = \int_a^b k(x,t)k_{n-1}(s,t)ds$$

where $n = 2,3,\ldots$ and $k_1(x,t) = k(x,t)$.

The function $k_n(x,t)$ which is determined from formulas defined in equation (**??**) is called iterated kernels.

12. Derivation of the New Technique

This section has been based successfully on Adomian's method as an extrapolated formula to find an exact solution where three values of unknown function to achieve this method we follow the flowing procedure.

First, recall equation (9)

$$u(x)$$

Second step we find $u_0(t)$, $u_1(t)$, $u_2(t)$ and

$$u_0(t) = f(x)$$

$$u_1(t) = \int_a^b k(t,s)u_0(s)ds$$

$$u_2(t) = \int_a^n k(t,s)u_1(s)ds$$

$$u_3(t) = \int_a^n k(t,s)u_2(s)ds$$

Third step suppose that the exact solution is $\varsigma(t)$ then the first approximate $\varsigma_0(t)$.

$$\varsigma_0 = u_0 + u_1 \tag{30}$$

and the second approximate $\varsigma_1(t)$

$$\varsigma_1 = u_0 + u_1 + u_2 \tag{31}$$

Finally approximate $\varsigma_2(t)$ is

$$\varsigma_1 = u_0 + u_1 + u_2 + u_3 \tag{32}$$

We can simplify (31),(32) and recall (30)

$$\varsigma_0 = u_0 + u_1,$$

$$\varsigma_1 = \varsigma_0 + u_2,$$

$$\varsigma_2 = \varsigma_1 + u_3 .$$

Now use $\varsigma_0(t)$, $\varsigma_1(t)$ and $\varsigma_2(t)$ and substituting them in Aitken's formula we get

$$\varsigma(t) = \frac{\varsigma_1^2 - \varsigma_0\varsigma_2}{\varsigma_2 - 2\varsigma_1 + \varsigma_0} . \tag{33}$$

Indeed equation (33), identifies a new procedure to approximate the exact solution of Fredholm integral equation (8) by the knowledge of only three function values $\varsigma_0(t)$, $\varsigma_1(t)$ and $\varsigma_2(t)$ respectively, in general and for an n.

$$\varsigma(t) = \frac{\varsigma_{n+1}^2 - \varsigma_n\varsigma_{n+2}}{\varsigma_{n+2} - 2\varsigma_{n+1} + \varsigma_n} .$$

13. Numerical Examples

Example 5. Consider the following FIDE

$$u'(x) + 3u(x) = 10x + 15x^2 - \frac{35}{12}x^4 + \int_a^b (x+y)u(y)dy, \quad 0 \le x \le 1 .$$

Table 1 presents the results from a computer program that solves this problem for winch the analytical solution is $u(x) = 5x^2$ over the interval $x = 0$ to $x = 1$ with $h = 0.1$, and reduction to FIE $u(x) = 5x^2$

$$u(x) = 5x^2 + 5x^3 - \frac{7}{12}x^5 + \int_a^b \left(\frac{x^2}{2} - 3 + xy - \frac{3y^2}{2}\right) u(y)dy .$$

Table 1

x	Exact	Bownd's	Successive	Adomian's	Improved
0.0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.050000000	0.049988313	0.050000000	0.050000000	0.050000000
0.2	0.200000000	0.199961740	0.200000030	0.200000030	0.200000000
0.3	0.450000000	0.449871053	0.450001157	0.450001157	0.450000009
0.4	0.800000000	0.799591681	0.800015113	0.800015113	0.800000156
0.5	1.250000000	1.248900941	1.250109800	1.250109800	1.250001338
0.6	1.800000000	1.797463797	1.800549249	1.800549249	1.800007583
0.7	2.450000000	2.444825596	2.452119493	2.452119493	2.450032249
0.8	3.200000000	3.190410542	3.206752015	3.206752015	3.200111036
0.9	4.050000000	4.033524986	4.068548968	4.068548968	4.050324878
1.0	5.000000000	4.973364852	5.045266790	5.045266790	5.000834809
L.S.E.		0.001107414	0.002443542	0.002443542	8.1588e-007
R.T.		0.770000000	7.530000000	14.50000000	21.64000000

Example 6. Consider the following FIDE:

$$u'(x) = 1 - \int_a^b u(t)dt, \quad 0 \le x \le 1.$$

Table 2 presents the results from a computer program that solves this problem for which the analytical solution is $u(x) = \sin x$ over the interval $x = 0$ to $x = 1$ with $h = 0.1$, and reduction to VIE gives $u(x) = x - \int_a^b (x-t)u(t)dt$, $0 \le x \le 1$. The exact solution is $u(x) = \sin x$.

Table 2

x	Exact	Bownd's	Successive	Adomian's	Improved
0.0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	0.099833416	0.099833333	0.099833416	0.099833416	0.099833416
0.2	0.198669330	0.198669165	0.198669330	0.198669330	0.198669330
0.3	0.295520206	0.295519963	0.295520206	0.295520206	0.295520206
0.4	0.389418342	0.389418026	0.389418342	0.389418342	0.389418342
0.5	0.479425538	0.479425158	0.479425538	0.479425538	0.479425538
0.6	0.564642473	0.564642039	0.564642473	0.564642473	0.564642473
0.7	0.644217687	0.644217211	0.644217687	0.644217687	0.644217687
0.8	0.717356090	0.717355588	0.717356090	0.717356090	0.717356090
0.9	0.783326909	0.783326396	0.783326909	0.783326909	0.783326909
1.0	0.841470984	0.841470478	0.841470984	0.841470984	0.841470984
L.S.E.		1.5281e-012	2.7279e-020	2.7279e-020	6.8197e-031
R.T.		0.440000000	1.260000000	0.880000000	1.320000000

Example 7. Consider the following VIDE:

$$u'(x)+xu(x)=e^x+x+\int_0^x xu(y)dy, \quad 0\le x\le 1.$$

Table 3 presents the results from a computer program that solves this problem for which the analytical solution is $u(x)=e^x$ over the interval $x=0$ to $x=1$ with $h=0.1$, and reduction to VIE gives

$$u(x)=e^x+\frac{1}{2}x^2+\int_a^b\left(\frac{x^2}{2}-\frac{y^2}{2}-y\right)u(y)dy, \quad 0\le x\le 1.$$

The exact solution is $u(x)=e^x$.

Table 3

x	Exact	Bownd's	Successive	Adomian's	Improved
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105166372	1.105170918	1.105170918	1.105170918
0.2	1.221402758	1.221369680	1.221402758	1.221402758	1.221402758
0.3	1.349858807	1.349750702	1.349858807	1.349858807	1.349858807
0.4	1.491824697	1.491569995	1.491824697	1.491824697	1.491824697
0.5	1.648721270	1.648220871	1.648721270	1.648721270	1.648721270
0.6	1.822118800	1.821243622	1.822118801	1.822118801	1.822118800
0.7	2.013752707	2.012341098	2.013752714	2.013752714	2.013752707
0.8	2.225540928	2.223395797	2.225540968	2.225540968	2.225540928
0.9	2.459603111	2.456488611	2.459603306	2.459603306	2.459603112
1.0	2.718281828	2.713919435	2.718282621	2.718282621	2.718281834
L.S.E.		3.6419e-005	6.6926e-013	6.6926e-013	3.4362e-017
R.T.		0.710000000	3.850000000	7.960000000	14.33000000

14. Conclusions

In this work, Bownd's, Successive, Adomian's and the improved methods are used to solve linear FIDEs. Several examples are included for illustration. The following points have been identified:

1. The results of Bownd's method depend on the numerical methods that we use for solving the ODEs.
2. Through the solution of integrals and FIEs of second kind, we see that the improved method gives the best results when it is compared with other iterative methods (see Example 5, 6 and 7.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

References

[1] G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer (1989), URL: https://www.springer.com/gp/book/9789027725257.

[2] N. Al-Gardi, *New Techniques in the Numerical Solution of Fredholm and Volterra Integral Equations*, PhD thesis, University of Salahaddin, Iraq (1999).

[3] K. E.Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press (1997), URL: https://www.cambridge.org/core/books/numerical-solution-of-integral-equations-of-the-second-kind/91E77DDA93701D3B024830429043A037.

[4] J. M. Bownds and B. Wood, On numerically solving nonlinear integral equations with fewer computations, *Journal on Numerical Analysis* **13**(5) (1976), 705 – 719, URL: http://ftp.math.utah.edu/pub/tex/bib/toc/siamjnumeranal.html#13(5):October:1976.

[5] H. Brunner and P. J. van der Houwen, *The Numerical Solution of Volterra Equations*, CWI Monographs, North-Holland Publishing, Amsterdam, The Netherlands (1986).

[6] R. L. Burden and J. D. Faires, *Numerical Analysis*, 9th editions, Brooks/Cole Publishing Company (2011), URL: https://fac.ksu.edu.sa/sites/default/files/numerical_analysis_9th.pdf.

[7] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press (1985), DOI: 10.1017/CBO9780511569609.

[8] L. M. Delves and J. Walsh, *Numerical Solution of Integral Equation*, Oxford University Press (1974).

[9] C. F. Gerald and P. O. Wheatley, *Applied Numerical Analysis*, 7th edition, Addison-Wesley Publishing, Company (1984), URL: http://www.cse.iitm.ac.in/~vplab/downloads/opt/Applied%20Numerical%20Analysis.pdf.

[10] M. Hadizadeh and K. Maleknejad, The numerical study of nonlinear Volterra integro-differential equations by Adomian's method, *Journal of Sciences, Islamic Republic of Iran* **9**(1) (1998), 51 – 58, URL: https://jsciences.ut.ac.ir/article_31365_bec3005aa1516be1889585f6ab6e0f68.pdf.

[11] A. J. Jerri, *Introduction to Integral Equations with Applications*, Marcel Dekker (1985).

[12] H. H. Kagiwada and R. Kalaba, *Integral Equations Via Imbedding Methods*, Applied Mathematics and Computation, No. 6 (1974), p. 395, Addison-Wesley Publishing Company, Reading, MA.

[13] M. Krasnov, A. Kiselev and G. Makarenko, *Problems and Exercise in Integral Equations*, MIR Publishers, Moscow (1971), URL: https://archive.org/details/ProblemsAndExercisesInIntegralEquationsKrasnovKiselevMakarenko/page/n9/mode/2up.

[14] P. Linz, Analytical and numerical methods for Volterra equations, in *Studies in Applied and Numerical Mathematics*, SIAM Philadelphia (1985), DOI: 10.1137/1.9781611970852.

[15] D. Porter and D. S. G. Stirling, *Integral Equations a Practical Treatment From Spectral Theory to Applications*, Cambridge University Press (1990), DOI: 10.1017/CBO9781139172028.