Existence of Coincidence and Fixed Point Theorems for Non-linear Hybrid Map on Generalized Space

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Abstract. In a recent paper Pathak et al. [20] established the coincidence and fixed point theorems for nonlinear hybrid contraction map under f-weak compatible continuous maps on metric spaces. In this paper we prove coincidence and fixed point theorems for nonlinear hybrid contraction maps on generalized metric spaces for multi-valued and single maps. Proved results of this paper to be a substantial generalization of the corresponding theorem of the recent paper [20].

1. Introduction

There are many coincidence and fixed point theorems for nonlinear hybrid contraction maps of a closed and bounded subset $CB(X)$ for a complete metric space $X$. However, in many applications, the maps involved may refer to Hadzic [5], Jungck [8], Kaneko et al. [9–11], Kannan [12], Pathak et al. [17–22], so it is interest to determine sufficient conditions on nonlinear hybrid maps which sure the existence of a fixed point. Subsequently, a number of generalizations of the multi-valued contraction principle for non-linear hybrid contraction maps obtained may refer to Khan [13], Kubiak [14], Nadler [15], Naimpally et al. [16], Rhoades et al. [23], Sessa [24], Smithson [29]. In this paper we consider the hybrid of maps, viz., contractive conditions involving multi-valued and single maps on a generalized metric space satisfying very general contractive type conditions which include several general conditions studied by Hematulin and Singh [6], Pathak et al. [20, 21], Singh et al. [26]. The result of this paper is a substantial generalization of the corresponding Theorem 1.1 of the recent paper of Pathak, Khan and Cho [20].

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Theorem 1.1. Let $(X,d)$ be a complete metric space, let $f : X \to X$ and $P : X \to CB(X)$ be $f$-weak compatible continuous maps such that $P(X) \subset f(X)$ and
\[ H(Px,Py) \leq h[aL_1(x,y) + (1-a)N_1(x,y)] \text{ for all } x, y \text{ in } X, \]
where $0 \leq h < 1$, $0 \leq a < 1$,
\[ L_1(x,y) = \max\left\{ d(fx,fy), d(fx,Px), d(fy,Py), \frac{1}{2} [d(fx,Py) + d(fy,Px)] \right\} \]
and
\[ N_1(x,y) = \left[ \max[d^2(fx,fy), d(fx,Px) \cdot d(fy,Py), d(fx,Py) \cdot d(fy,Px), \right. \]
\[ \left. \frac{1}{2} [d(fx,Px) \cdot d(fy,Py)], \frac{1}{2} [d(fx,Py) \cdot d(fy,Px)] \right]^{\frac{1}{2}}. \]
Then there exists a point $z \in X$ such that $fz \in Pz$, i.e. the point $z$ is a coincidence point of $f$ and $P$.

2. Preliminaries

In a sequel, we use the following notations and definitions.

Definition 2.1 (Czerwik [1–4]). Let $X$ be (nonempty) a set and $s \geq 1$ a given real number. A function $d : X \times X \to R^+$ (nonnegative real) is called a b-metric provided that for all $x, y, z \in X$,

\[ \text{(bm-1) } d(x,y) = 0, \text{ iff } x = y, \]
\[ \text{(bm-2) } d(x,y) = d(y,x), \]
\[ \text{(bm-3) } d(x,z) \leq s[d(x,y) + d(y,z)]. \]

The pair $(X,d)$ is called a b-metric space.

We remark that a metric space is evidently a b-metric space. However, Czerwik [1,2] has shown that a b-metric on $X$ need not be a metric on $X$ (see also [3,27]). The following example shows that a b-metric on $X$ need not be a metric on $X$.

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Example 2.1. Let $X = \{x_1,x_2,x_3\}$ and $d : X \times X \to R^+$ such that
\[ d(x_1,x_2) = x \geq 3, \quad d(x_1,x_3) = d(x_2,x_3) = 1, \quad d(x_n,x_n) = 0, \quad d(x_n,x_k) = d(x_k,x_n). \]
Then
\[ d(x_n,x_k) \leq \frac{1}{3} [d(x_n,x_i) + d(x_i,x_k)], \quad n,k,i, = 1,2,3. \]
Then $(X,d)$ is a b-metric space.
Definition 2.2 (Czerwik [2]). Let \((X,d)\) be a b-metric space. The Hausdorff b-metric \(H\) on \(CL(X)\), the collection of all nonempty closed subsets of \((X,d)\) is defined as follows:

\[
H(A,B) := \left\{ \max \left\{ \sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A) \right\}, \text{if the maximum exists, otherwise } \infty \right\}.
\]

In all that follows \(Y\) is an arbitrary nonempty set and \((X,d)\) a b-metric space unless otherwise specified.

We cite the following lemmas from Czerwik [1,2], and Singh et al. [27].

Lemma 2.1. For any \(A,B,C \in CL(X)\),
(i) \(d(x,B) \leq d(x,y)\) for any \(y \in B\),
(ii) \(d(A,B) \leq H(A,B)\),
(iii) \(d(x,B) \leq H(A,B), x \in A\)
(iv) \(H(A,C) \leq s[H(A,B) + H(B,C)]\),
(v) \(d(x,A) \leq sd(x,y) + sd(y,A), x,y \in X\).

Lemma 2.2. Let \(A,B \in CL(X)\) and \(k > 1\). Then for each \(a \in A\), there exists a point \(b \in B\) such that \(d(a,b) \leq kH(A,B)\).

3. Coincidence Point Theorems

We start with following theorem.

Theorem 3.1. Let \((X,d)\) be a complete b-metric space, let \(f : Y \to Y\) and \(P,Q : Y \to CL(X)\) be maps such that \(P(Y) \cup Q(Y) \subset f(Y)\)

\[
H(Px,Qy) \leq h[aL(x,y) + (1 - a)N(x,y)]
\]

for all \(x,y\) in \(X\), where \(0 \leq h, a < 1\),

\[
L(x,y) = \max \left\{ d(fx,fy), d(fx,Px), d(fy,Qy), \frac{1}{2} [d(fx,Qy) + d(fy,Px)] \right\}
\]
and
\[ N(x, y) = \left[ \max \left\{ d^2(f x, f y), d(f x, P x) \cdot d(f y, Q y), d(f x, Q y) \cdot d(f y, P x), \right. \right. \]
\[ \left. \left. \frac{1}{2} [d(f x, P x) \cdot d(f y, P x)] \right\} \right]^{1/2}. \]  \hspace{1cm} (3.3)

If \( s \sqrt{n} < 1 \), one of \( P(Y), Q(Y) \) or \( f(Y) \) is a complete subspace of \( X \), then \( f x \in P x \cap Q x \) has a solution. Indeed, for any \( x_0 \in Y \), there exists a sequence \( \{x_n\} \) in \( Y \) such that the sequence \( \{x_n\} \) converges to \( f z \) for some \( z \in Y \), and \( f z \in P z \cap Q z \).

**Proof.** If \( s = 1 \) then the conclusion follows from metric space setting, so we need to take \( s > 1 \). Pick \( x_0 \in Y \). We construct sequences \( \{x_n\} \) in \( Y \) and \( \{f x_n\} \) in \( X \) in the following manner. Since \( P(Y) \subseteq f(Y) \), we can find a point \( x_1 \in Y \) such that \( f x_1 \in P x_0 \). Noting that \( Q(Y) \) is also a subspace of \( f(Y) \), for a suitable point \( x_2 \in Y \), we can choose a point \( f x_2 \in Q x_1 \) such that
\[ d(f x_1, f x_2) \leq kH(P x_0, Q x_1), \quad \text{where} \quad k = h^{-1/2}. \]

In general, we can choose a sequence \( \{x_n\} \) in \( Y \) such that \( f x_{2n+1} \in P x_{2n}, f x_{2n+2} \in Q x_{2n+1}, f x_{2n+3} \in P x_{2n+2} \) and
\[ d(f x_{2n+1}, f x_{2n+2}) \leq kH(P x_{2n}, Q x_{2n+1}), \]
\[ \leq kh[aL(x_{2n}, x_{2n+1}) + (1 - a)N(x_{2n}, x_{2n+1})] \]  \hspace{1cm} (3.4)

where
\[ L(x_{2n}, x_{2n+1}) \leq \max \left\{ d(f x_{2n}, f x_{2n+1}), d(f x_{2n}, f x_{2n+1}), d(f x_{2n+1}, f x_{2n+2}), \right. \]
\[ \left. \frac{1}{2} [d(f x_{2n}, f x_{2n+2}) + d(f x_{2n+1}, f x_{2n+1})] \right\} \]
\[ \leq \max \left\{ d(f x_{2n}, f x_{2n+1}), d(f x_{2n+1}, f x_{2n+2}), \right. \]
\[ \left. \frac{1}{2} s [d(f x_{2n}, f x_{2n+1}) + d(f x_{2n+1}, f x_{2n+2})] \right\} \]  \hspace{1cm} (3.5)

and
\[ N(x_{2n}, x_{2n+1}) \]
\[ \leq \left[ \max \left\{ d^2(f x_{2n}, f x_{2n+1}), d(f x_{2n}, f x_{2n+1}) \cdot d(f x_{2n+1}, f x_{2n+2}), 0, 0 \right\} \right]^{1/2}. \]  \hspace{1cm} (3.6)

Now by equation (3.4), (3.5) and (3.6), we get
\[ d(f x_{2n+1}, f x_{2n+2}) \leq kh[a d(f x_{2n}, f x_{2n+1}) + (1 - a)0]. \]

Suppose that \( d(f x_{2n+1}, f x_{2n+2}) > k h a d(f x_{2n}, f x_{2n+1}) \) for some \( n \in N \). Then we obtain \( d(f x_{2n+1}, f x_{2n+2}) < d(f x_{2n}, f x_{2n+1}) \), which is a contradiction, and so \( d(f x_{2n+1}, f x_{2n+2}) \leq a \sqrt{h d}(f x_{2n}, f x_{2n+1}). \)
Similarly $d(fx_{2n+2},fx_{2n+3}) \leq \sqrt{n}d(fx_{2n+1},fx_{2n+2})$. Therefore in general $d(fx_{n+1},fx_{n+2}) \leq \sqrt{n}d(fx_{n},fx_{n+1})$, for all $n \in \mathbb{N}$. Since $a < 1$, $s \sqrt{n} < 1$ and $X$ is complete, it follows from (3.4) that $\{fx_n\}$ is a Cauchy sequence. If we assume that $f(Y)$ is a complete subspace of $X$, then the sequence $\{x_n\}$ and its subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ have a limit in $f(Y)$. Call it $u$. Then there exists a point $z \in Y$ such that $fz = u$. By (3.1) and Lemma 2.2, we have

$$d(fz,fx_{2n+2}) \leq kH(Pz,Qx_{2n+1})$$

$$= kh[aL(z,x_{2n+1}) + (1-a)N(z,x_{2n+1})] \quad (3.7)$$

where

$$L(z,x_{2n+1}) \leq \max \left\{ d(fz,fx_{2n+1}), d(fz,Pz), d(fx_{2n+1},fx_{2n+2}), \right. \]

$$\left. \quad \frac{1}{2} \left[ d(fz,fx_{2n+2}) + d(fx_{2n+1},Pz) \right] \right\}$$

and

$$N(z,x_{2n+1}) \leq \left[ \max \left\{ d^2(fz,fx_{2n+1}), d(fz,Pz) \cdot d(fx_{2n+1},fx_{2n+2}), \right. \]

$$\left. \quad d(fz,fx_{2n+2}) \cdot d(fx_{2n+1},Pz), \right. \]

$$\left. \quad \frac{1}{2} \left[ d(fz,Pz) \cdot d(fx_{2n+1},Pz) \right] \right\}^{1/2}.$$ 

Making $n \to \infty$, we have

$$L(z,x_{2n+1}) \leq \max \left\{ d(fz,fz), d(fz,Pz), d(fz,fz), \frac{1}{2} \left[ d(fz,fz) + 0 \right] \right\}$$

$$\leq \max\{0,d(fz,Pz),0,0\}$$

$$= d(fz,Pz) \quad (3.8)$$

and

$$N(z,x_{2n+1}) \leq \left[ \max \left\{ d^2(fz,fz), d(fz,Pz) \cdot d(fz,fz), d(fz,fz) \cdot d(fz,Pz), \right. \]

$$\left. \quad \frac{1}{2} \left[ d(fz,fz) \cdot d(fz,fz) \right] \right\}^{1/2}$$

$$\leq \left[ \max\{0,0,0,0,0\} \right]^{1/2} \quad (3.9)$$

respectively. Thus we have from (3.7), (3.8), (3.9)

$$d(fz,Pz) \leq khad(fz,Pz)$$

$$= a\sqrt{h}d(fz,Pz).$$
Which implies \( d(fz, Pz) = 0 \), because \( a \sqrt{k} \leq 1 \) therefore \( fz \in Pz \), since \( Pz \) is closed. Similarly \( fz \in Qz \), Thus \( fz \in Pz \cap Qz \). This completes the proof. \( \square \)

**Remark 3.1.** Take \( P = Q \) the identity maps, in Theorem 3.1, we obtain generalizations of several coincidence results existing in the literature (see, for instance [6], [25], [26]).

### 4. Fixed Point Theorems

We apply coincidence theorem of the previous section to study fixed point theorem.

**Theorem 4.1.** Let all the hypotheses of Theorem 3.1 be satisfied with \( Y = X \). If \( f \) is \((IT)\)-commuting with each of \( P \) and \( Q \) at their common coincidence point \( z \), and if \( u = fz \) is fixed point of \( f \), then \( f, P \) and \( Q \) have a common fixed point, i.e.,

\[
u = fu \in Pu \cap Qu.
\]

**Proof.** It comes from Theorem 1.3 that there exist \( z, u \in X \) such that \( u = fz \in Pz \) and \( u = fz \in Qz \). Since \( u = fu \), the \((IT)\)-commutativity of \( f \) and \( P \) implies that \( u = fu = fzfz \subseteq Pfz = Pu \). Similarly \( u = fu \in Qu \). So \( u = fu \in Pu \cap Qu \). This completes the proof. \( \square \)

**Remark 4.1.** Let all the hypotheses of Theorem 4.1 be satisfied with \( Y = X \). If \( f \) is \((IT)\)-commuting with each of \( P = Q \) at their common coincidence point \( z \), and if \( u = fz \) is fixed point of \( f \), then \( f, P = Q \) have a common fixed point, i.e.,

\[
u = fu \in Pu.
\]

**Remark 4.2.** If we take \( k = h^{-1/2} \) in proof Theorem 3.1 at \( k > 1 \) then it to be \( skh < 1 \). So \( k > 1 \) in Theorems 3.1 and 4.1. Then we can take \( s \sqrt{k} < 1 \) at \( skh < 1 \). If we change condition \( N(x, y) \) with condition \( N_1(x, y) \) then condition \( s \sqrt{k} < 1 \) will change with condition \( skh^{2/3} < 1 \) and make some corrections. So we can take \( skh < 1 \) at \( s \sqrt{k} < 1 \), where \( k > 1 \).

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**References**


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