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Fisher Information Number for Concomitants of Generalized Order Statistics in Morgenstern Family

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Abstract. In this paper, Fisher's information number for concomitants of generalized order statistics in Morgenstern family is obtained. Applications of this result are given for concomitants of order statistics and record values.

1. Introduction

The concept of generalized order statistics was introduced by [9] as a unified approach to a variety of models of ordered random variables. The random variables $X(1,n,m,k), X(2,n,m,k), \cdots, X(n,n,m,k)$ are called generalized order statistics based on an absolutely continuous distribution function F with density function f, if their joint density function is given by

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^m f(x_i)\right)(1-F(x_n))f(x_n),$$

on the cone $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$ of $\mathbf{R}^n$, with parameters $n \in \mathrm{N}$, $k > 0$, $m \in \mathbf{R}$ such that $\gamma_r = k + (n-r)(m+1) > 0$ for all $1 \leq r \leq n$.

Now, Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be independent and identically distributed random pairs from some continuous bivariate distributions. If $X_{(r)}$ denotes the rth-order statistic, then the Y's associated with $X_{(r)}$, denoted by $Y_{[r]}$, is called the concomitant of the rth-order statistic. There is a similar concept of concomitants for the record values. Suppose that $(X_1,Y_1),(X_2,Y_2),\cdots$ is a sequence of bivariate random pairs from a continuous distribution. If $\{R_r, r \geq 1\}$ is the sequence of record values in the sequence of X's, then the Y which corresponds with the rth-record will be called the concomitant of the rth-record, denoted by $R_{[r]}$.

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The concomitants are of interest in selection and prediction problems. An excellent review on concomitants of order statistics is given by [4].

The Morgenstern family discussed in [7] provides a flexible family that can be used in such contexts, which is specified by the distribution function (df)

$$F_{X,Y}(x,y)=F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \tag{1.1}$$

where $-1\le\alpha\le 1$, and $f_X(x)$, $f_Y(y)$, and $F_X(x)$, $F_Y(y)$ are marginal pdf and cdf of X and Y, respectively. Concomitants can also be defined in the case of generalized order statistics (see [9] and [1]).

For the Morgenstern family with df given by (1.1), the density function of the concomitant of r-th generalized order statistic $Y_{[r,n,m,k]}$, $1\le r\le n$, is given by [2], as follows:

$$g_{[r,n,m,k]}(y)=f_Y(y)\left[1+C^*(r,n,m,k)\alpha(1-2F_Y(y))\right], \tag{1.2}$$

where

$$C^*(r,n,m,k)=\frac{2\prod_{j=1}^{r}\gamma_j}{(\gamma_1+1)(\gamma_2+1)\ldots(\gamma_r+1)}-1\,.$$

The Fisher information number (with respect to a translation parameter) for a continuous random variable X with probability density function $f_X(x)$ is defined as

$$I_f(X)=\int_{-\infty}^{+\infty}\left[\frac{\partial\ln f_X(x)}{\partial x}\right]^2 f_X(x)dx. \tag{1.3}$$

This is Fisher's information for location parameter, and also called shift-invariant Fisher information. Recently, it has been used to develop a unifying theory physical law called the principle of "extreme physical information" (see [5] and [6]). Also measures of multivariate dependence based on Fisher information number matrix are presented by [12] and [13]. [3] established superadditivity of Fisher information number and [8] investigated the statistical meaning of Carlen's superadditivity. Superadditivity provides a coefficient of association between two continuous random variables X and Y as follows:

$$\rho^*(X,Y)=\frac{I_f(X)+I_f(Y)}{I_f(XX)+I_f(YY)},$$

where $I_f(XX)$ and $I_f(YY)$ are elements of the main diagonal of Fisher's information number-matrix for the a random pair (X,Y) with joint pdf $f_{X,Y}(x,y)$. It is to easy to see that $0\le\rho^*(X,Y)\le 1$, and $\rho^*(X,Y)=1$ iff X and Y are independent random variables. Examples of $\rho^*(X,Y)$ for bivariate normal distribution and bivariate inverted Dirichlet distribution are given by [11].

2. Main Results

Theorem 2.1. *If $Y_{[r,n,m,k]}$ is the concomitant of rth-generalized order statistics from* (1.1)*, then the Fisher information number of $Y_{[r,n,m,k]}$ for $1\le r\le n$, $\alpha\neq 0$ is given* *by*

$$I_g(Y_{[r,n,m,k]})=I_f(Y)+\alpha C^*(r,n,m,k)[\tau_f(u)-4\phi_f(u)]+[2\alpha C^*(r,n,m,k)]^2\delta_f(u), \tag{2.1}$$

where $u=F_Y(y)$,

$$\tau_f(u)=\int_0^1\left[\frac{f'_Y(F_Y^{-1}(u))}{f_Y(F_Y^{-1}(u))}\right]^2(1-2u)du,$$

$$\phi_f(u)=\int_0^1 f'_Y(F_Y^{-1}(u))du,$$

$$\delta_f(u)=\int_0^1\frac{[f_Y(F_Y^{-1}(u))]^2}{1+C^*(r,n,m,k)\alpha(1-2u)}du.$$

Proof. By (1.2) and (1.3), we have

$$\begin{aligned} I_g(Y_{[r,n,m,k]}) = &\int_{-\infty}^{+\infty}\left[\frac{\partial \ln f_Y(y)}{\partial y}\right]^2 f_Y(y)dy \\ &+\int_{-\infty}^{+\infty}\left[\frac{\partial \ln f_Y(y)}{\partial y}\right]^2 f_Y(y)[C^*(r,n,m,k)\alpha(1-2F_Y(y))]dy \\ &+[2\alpha C^*(r,n,m,k)]^2\int_{-\infty}^{+\infty}\frac{[f_Y(y)]^3}{1+C^*(r,n,m,k)\alpha(1-2F_Y(y))}dy \\ &-4C^*(r,n,m,k)\alpha\int_{-\infty}^{+\infty} f_Y(y)f'_Y(y)dy. \end{aligned} \tag{2.2}$$

Using the transformation $u=F_Y(y)$ on the right of (2.2) the result follows. □

Now, we consider the following two special cases.

Case 1. If we take $m=0$ and $k=1$, then $C^*(r,n,0,1)=\frac{n-2r+1}{n+1}$, and the Fisher information number for the concomitant of rth-order statistic is given by

$$\begin{aligned} I_g(Y_{[r]}) = I_f(Y)+\alpha\left(\frac{n-2r+1}{n+1}\right)[\tau_f(u)-4\phi_f(u)] \\ +\left[2\alpha\left(\frac{n-2r+1}{n+1}\right)\right]^2\delta_f(u). \end{aligned} \tag{2.3}$$

Remark 2.2. If n is odd and $r=\frac{n+1}{2}$, or $\alpha=0$, then $I_g(Y_{[r]})=I_f(Y)$.

Remark 2.3. If λ is a rational number such that we change r to $r\lambda$ and n to $(n+1)\lambda-1$, then $I_g(Y_{[r]})=I_g(Y_{[r\lambda]})$.

Interesting applications of (2.3) are given by the following examples.

Example 2.4. Let (X_i,Y_i), $i=1,2,\cdots,n$ be a random sample from the copula model of Morgenstern family (see [10, Chapter 2]) with pdf

$$f_{X,Y}(x,y)=[1+\alpha(1-2x)(1-2y)],\quad 0\le x,y\le 1\,. \tag{2.4}$$

Then, by using (2.3), we have

$$I_g(Y_{[r]}) = 2\alpha\left(\frac{n-2r+1}{n+1}\right)\log\left[\frac{1+\alpha\left(\frac{n-2r+1}{n+1}\right)}{1-\alpha\left(\frac{n-2r+1}{n+1}\right)}\right] = t_{\alpha,n}(r).$$

Now, we can easily show that $I_g(Y_{[r]})$ for the copula model of Morgenstern family has the following properties

(i) $t_{\alpha,n}(r) = t_{-\alpha,n}(r)$;

(ii) $I_g(Y_{[r]}) = I_g(Y_{[n-r+1]})$, $1 \le r \le n$;

(iii) $I_g(Y_{[r]}) \ge I_f(Y)$;

(iv) $t_{\alpha,n}(r)$ is a convex function of r and α;

(v) $\min_r I_g(Y_{[r]}) = I_g(Y_{[\frac{n+1}{2}]}) = 0$ if n is odd and $\alpha \neq 0$;

(vi) $\max_r I_g(Y_{[r]}) = I_g(Y_{[1]}) = I_g(Y_{[n]}) \le 2\alpha\log\left(\frac{1+\alpha}{1-\alpha}\right)$.

Also, for $2 < r < n-1$ and $n > 3$, ρ^*(a measures of dependence) between $X_{(r)}$ and $Y_{[r]}$ is given by

$$\rho^*(X_{(r)}, Y_{[r]}) = \frac{\frac{n(n-1)(n-3)}{(r-2)(n-r-1)} + t_{\alpha,n}(r)}{\frac{n(n-1)(n-3)}{(r-2)(n-r-1)} + E_{X_{(r)}}\left[\frac{2\alpha^2(1-2x)^4+2}{\alpha(1-2x)^3}\log\left(\frac{1+\alpha(1-2x)}{1-\alpha(1-2x)}\right) - \frac{4}{(1-2x)^2}\right]},$$

where the density function of $X_{(r)}$ is

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}(x)^{r-1}(1-x)^{n-r}.$$

Table 1 provides the values of $\rho^*(X_{(r)}, Y_{[r]})$ as a function of n, r $(2 < r < n-1)$ and α, for $n = 6(2)12$ and $\alpha = \pm.25, \pm.5, \pm.75, \pm1$. As it can be seen from Table 1, for fixed n and r, $\rho^*(X_{(r)}, Y_{[r]})$ decreases as $|\alpha|$ increases.

Table 1. $\rho^*(X_{(r)}, Y_{[r]})$ for the copula model of Morgenstern family

		α			
n	r	±0.25	±0.5	±0.75	±1
6	3	0.9974	0.9897	0.9763	0.9559
6	4	0.9974	0.9897	0.9763	0.9559
8	3	0.9984	0.9937	0.9853	0.9713
8	4	0.9976	0.9906	0.9786	0.9609
8	5	0.9976	0.9906	0.9786	0.9609
8	6	0.9984	0.9937	0.9853	0.9713
10	3	0.9990	0.9960	0.9903	0.9800
10	4	0.9983	0.9933	0.9845	0.9709
10	5	0.9980	0.9920	0.9818	0.9670
10	6	0.9980	0.9920	0.9818	0.9670
10	7	0.9983	0.9933	0.9845	0.9709
10	8	0.9990	0.9960	0.9903	0.9800

Example 2.5. Let (X_i, Y_i), $i = 1, 2, \cdots, n$ be a random sample from Gumbel's type II bivariate exponential distribution(G_2BVE) with cdf

$$F(x,y) = \left(1 - \exp\left(\frac{-x}{\theta_1}\right)\right)\left(1 - \exp\left(\frac{-y}{\theta_2}\right)\right)\left[1 + \alpha \exp\left(\frac{-x}{\theta_1} - \frac{y}{\theta_2}\right)\right],$$
$$x, y > 0,\ \theta_1, \theta_2 > 0.$$

Then by using (2.3), we have

$$I_g(Y_{[r]}) = \frac{4\alpha(n-2r+1)}{\theta_2^2(n+1)} + \frac{[\alpha(n-2r+1)-(n+1)]^2}{2\alpha\theta_2^2(n+1)(n-2r+1)} \cdot \log\left[\frac{1+\alpha\left(\frac{n-2r+1}{n+1}\right)}{1-\alpha\left(\frac{n-2r+1}{n+1}\right)}\right]. \tag{2.5}$$

Now, we can see that $I_g(Y_{[r]})$ in this example has the following properties

(1) $I_g(Y_{[r]}) \geq I_f(Y) = \frac{1}{\theta_2^2}$ if $0 \leq \alpha \leq 1$ and $1 \leq r \leq \frac{n+1}{2}$.

(2) If $r < \frac{n+1}{2}$ $(r > \frac{n+1}{2})$ then $I_g(Y_{[r]})$ is increasing (decreasing) in α.

(3) $I_g(Y_{[r]})$ is a convex function of n, α, and r.

(4) if $0 < \alpha \leq 1$ $(-1 \leq \alpha < 0)$, then $I_g(Y_{[r]})$ is decreasing(increasing) in r.

Case 2. If we take $m = -1$ and $k = 1$, then $C^*(r,n,-1,1) = 2^{1-r} - 1$, and Fisher information number for the concomitant of rth-record value is as follows:

$$I_g(R_{[r]}) = I_f(Y) + \alpha(2^{1-r}-1)[\tau_f(u) - 4\phi_f(u)] + [2\alpha(2^{1-r}-1)]^2\delta_f(u). \tag{2.6}$$

Remark 2.6. If $r = n = 2^c - 1$, then $I_g(Y_{[2^c-1]}) = I_g(R_{[c]})$.

As an application of the representation (2.5), we consider the following two examples.

Example 2.7. Let (X_i, Y_i), $i = 1, 2, \cdots$ be a sequence of independent observations for the copula model of Morgenstern family, then by using (2.5), we have

$$I_g(R_{[r]}) = 2\alpha(2^{1-r}-1)\log\left[\frac{1+\alpha(2^{1-r}-1)}{1-\alpha(2^{1-r}-1)}\right] = W_\alpha(r).$$

Now, by mathematical computations, we can easily show that $I_g(R_{[r]})$ for the copula model of Morgenstern family has the following properties:

(1) $W_\alpha(r) = W_{-\alpha}(r)$.

(2) $0 \leq I_g(R_{[r]}) < \infty$.

(3) for $r > 1$, $I_g(R_{[r]})$ is decreasing (increasing) in α for $-1 \leq \alpha < 0$ $(0 < \alpha \leq 1)$.

(4) $\lim_{r\to\infty} \Delta(r) = \lim_{r\to\infty}(I_g(R_{[r+1]}) - I_g(R_{[r]})) = 0$.

(5) $W_\alpha(r)$ is a convex function of α for all $r > 1$.

(6) $I_g(R_{[r]}) \geq I_g(Y_{[r]})$ for fixed values of r, n, α.

Example 2.8. Let (X_i, Y_i), $i = 1, 2, \cdots$ be a sequence of independent observations from G_2BVE, then by using (2.5), we have

$$I_g(R_{[r]}) = \frac{4\alpha(2^{1-r}-1)}{\theta_2^2} + \frac{[\alpha(2^{1-r}-1)-1]^2}{2\theta_2^2\alpha(2^{1-r}-1)} \log\left[\frac{1+\alpha(2^{1-r}-1)}{1-\alpha(2^{1-r}-1)}\right]. \tag{2.7}$$

Now, we can see that $I_g(Y_{[r]})$ for G_2BVE has the following properties

(1) $I_g(R_{[r]}) \geq I_f(Y)$ if $-1 \leq \alpha < 0$ and $1 < r$.

(2) $\lim_{r\to\infty}(I_g(R_{[r+1]}) - I_g(R_{[r]})) = 0$.

(3) If $-1 \leq \alpha < 0$ then $I_g(R_{[r]})$ is increasing in r.

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