Asymptotic Solutions of Fourth Order Near Critically Damped Nonlinear Systems

Md. Habibur Rahman, B. M. Ikramul Haque, and M. Ali Akbar

Abstract. A perturbation technique is developed in this article to solve asymptotic solutions of fourth order near critically damped nonlinear systems based on the Krylov-Bogoliubov-Mitropolskii method. The method is illustrated by an example. The results obtained by the presented technique show excellent coincidence with those obtained by numerical method.

1. Introduction


In this article, we have investigated asymptotic solutions of fourth order near critically damped nonlinear systems with small nonlinearities based on the KBM.
method. The results obtained by the presented method show good agreement with the numerical results.

2. The Method

Let us consider the fourth order weakly nonlinear ordinary differential systems

\[
\frac{d^4x}{dt^4} + c_1 \frac{d^3x}{dt^3} + c_2 \frac{d^2x}{dt^2} + c_3 \frac{dx}{dt} + c_4x = -\varepsilon f(x),
\]

where \(\varepsilon\) is a positive small parameter, \(f(x)\) is the given nonlinear function and \(c_1, c_2, c_3, c_4\) are constants, defined in terms of the eigen-values \(-\lambda_i, i = 1, 2, 3, 4\) of the unperturbed equation of (2.1) as

\[
c_1 = \sum_{i=1}^{4} \lambda_i, \quad c_2 = \sum_{i\neq j}^{4} \lambda_i \lambda_j, \quad c_3 = \sum_{i\neq j\neq k}^{4} \lambda_i \lambda_j \lambda_k \quad \text{and} \quad c_4 = \prod_{i=1}^{4} \lambda_i.
\]

When \(\varepsilon = 0\), the equation (2.1) becomes linear and suppose the eigenvalues \(-\lambda_1\) and \(-\lambda_2\) are almost equal \((\lambda_1 \approx \lambda_2)\) and other two eigenvalues \(-\lambda_3\) and \(-\lambda_4\) are distinct. Therefore, the unperturbed solution is

\[
x(t, 0) = \frac{1}{2} a_{1,0} (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_{2,0} \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right)
+ a_{3,0} e^{-\lambda_3 t} + a_{4,0} e^{-\lambda_4 t},
\]

where \(a_{i,0} (i = 1, 2, 3, 4)\) are arbitrary constants.

When \(\varepsilon \neq 0\), following Alam’s [7, 8] technique we choose the solution of (2.1) in the form

\[
x(t, \varepsilon) = \frac{1}{2} a_1(t) (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2(t) \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3(t) e^{-\lambda_3 t}
+ a_4(t) e^{-\lambda_4 t} + \varepsilon u_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots,
\]

where \(a_i (i = 1, 2, 3, 4)\) satisfy the first order differential equation

\[
\frac{da_i(t)}{dt} = \varepsilon A_i(a_1, a_2, a_3, a_4, t) + \varepsilon^2 \cdots, \quad (i = 1, 2, 3, 4).
\]

Confining only to a first few terms \(1, 2, 3, \ldots, n\) in the series expansions (2.3) and (2.4), we calculate the functions \(u_i\) and \(A_i, i = 1, 2, 3, 4\) such that \(a_i(t), i = 1, 2, 3, 4\) appearing in (2.3) and (2.4) satisfy the given differential equation (2.1) with an accuracy of order \(\varepsilon^{n+1}\). To determine the unknown functions \(u_1, A_1, A_2, A_3, A_4\) it is assumed (as customary in the KBM method) that the correction term, \(u_1\) does not contain secular-type terms \(te^{-\lambda_i t}\), which make them large.

Differentiating equation (2.3) four times with respect \(t\), substituting the derivatives \(\frac{dx}{dt}, \frac{d^2x}{dt^2}, \frac{d^3x}{dt^3}, \frac{d^4x}{dt^4}\) and \(x\) in the original equation (2.1), utilizing
the relations presented in (2.4) and finally equating the coefficients of $\epsilon$, we obtain

$$
\frac{1}{2} \left( e^{-\lambda_1 t} (D - \lambda_1 + \lambda_2) (D - \lambda_1 + \lambda_3) (D - \lambda_1 + \lambda_4) + e^{-\lambda_2 t} (D - \lambda_2 + \lambda_1) (D - \lambda_2 + \lambda_3) (D - \lambda_2 + \lambda_4) \right) A_1 \\
+ (D + \lambda_4) \left( e^{-\lambda_1 t} (\lambda_1 - \lambda_3 - \frac{3}{2} D) + e^{-\lambda_2 t} (\lambda_2 - \lambda_3 - \frac{3}{2} D) \right) A_2 \\
+ e^{-\lambda_3 t} (D - \lambda_3 + \lambda_1) (D - \lambda_3 + \lambda_2) (D - \lambda_3 + \lambda_4) A_3 \\
+ e^{-\lambda_4 t} (D - \lambda_4 + \lambda_1) (D - \lambda_4 + \lambda_2) (D - \lambda_4 + \lambda_3) A_4 \\
+ \left( e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) (D + \lambda_4) D \left( D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} \right) A_2 \\
- \left( \frac{\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) D \left( D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} \right) A_2 \\
+ (D + \lambda_1) (D + \lambda_2) (D + \lambda_3) (D + \lambda_4) a_1 = -f^{(0)}, \tag{2.5}
$$

where

$$
f^{(0)} = f(x_0) \quad \text{and} \quad x_0 = \frac{1}{2} a_1(t) (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2(t) \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3(t) e^{-\lambda_3 t} + a_4(t) e^{-\lambda_4 t}.
$$

It is assumed, in this article that the functional $f^{(0)}$ can be expanded in power series (Taylor’s series) in the form (see also [7, 8] for details)

$$
f^{(0)} = \sum_{r=0}^{n} F_r(a_3 e^{-\lambda_3 t}, a_4 e^{-\lambda_4 t}) \\
\times \left\{ \frac{1}{2} a_1(t) (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \right\}^r, \tag{2.6}
$$

where $n$ is the order of polynomial of the nonlinear function $f$. This assumption is certainly valid when $f$ is a polynomial function of $x$. Such polynomial functions cover some special and important systems in mechanics. Following Alam’s [6, 7, 8], in this article we assume that $a_4$ does not contain the terms $F_0$ and $F_1$ of $f^{(0)}$, since the system is considered to near critically damped. Substituting the value of $f^{(0)}$ from (2.6) into (2.5) and equating the coefficients of like powers of

$$
\left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right),
$$

we obtain

$$
e^{-\lambda_3 t} (D - \lambda_3 + \lambda_1) (D - \lambda_3 + \lambda_2) (D - \lambda_3 + \lambda_4) A_3 \\
+ e^{-\lambda_4 t} (D - \lambda_4 + \lambda_1) (D - \lambda_4 + \lambda_2) (D - \lambda_4 + \lambda_3) A_4 \\
+ \frac{1}{2} \left\{ e^{-\lambda_1 t} (D - \lambda_1 + \lambda_2) (D - \lambda_1 + \lambda_3) (D - \lambda_1 + \lambda_4) \right\} A_1 \\
+ (D + \lambda_4) \left\{ e^{-\lambda_1 t} \left( \lambda_1 - \lambda_3 - \frac{3}{2} D \right) + e^{-\lambda_2 t} \left( \lambda_2 - \lambda_3 - \frac{3}{2} D \right) \right\} A_2
$$
Solving equation (2.10), we obtain the value of $u$ of (2.1).

Solving equation (2.10), we obtain the value of $u$.

KBM [9, 10], Sattar [14], Alam [5, 6, 7, 8] imposed the condition that $u_1$ does not contain the fundamental terms (the solution presented in equation (2.2) is called generating solution and its terms are called fundamental terms) of $f^{(0)}$. The solution of (2.8) gives value of the unknown function $A_4$. It is not easy to solve the equation (2.7) for the unknown functions $A_1$, $A_3$ and $A_4$, if the nonlinear function $f$ and the eigenvalues $-\lambda_1$, $-\lambda_2$, $-\lambda_3$, $-\lambda_4$ of the corresponding linear equation (2.1) are not specified. When these are specified the values of $A_1$, $A_3$ and $A_4$ can be found subject to the condition that the coefficients in the solutions of $A_1$, $A_3$ and $A_4$ do not become large (see Akbar et al. [3], Alam [7, 8] for details), as well as $A_1$, $A_3$ and $A_4$ do not contain terms involving $te^{-t}$. In the article, we have imposed the condition that the relation $\lambda_3 \approx 3\lambda_4$ exists between the eigenvalues $\lambda_1$ and $\lambda_4$ (and $\lambda_1 \rightarrow \lambda_2$ since the system is near critically damped). These relations are important, because under these relations the coefficients in the solution of $A_1$, $A_3$ and $A_4$ do not become large. Under these imposed conditions we obtain the values of $A_1$, $A_3$ and $A_4$ from equation (2.7). Substituting the values of $A_1$, $A_2$, $A_3$ and $A_4$ in the equation (2.4), we obtain the results of $\frac{da_i}{dt}$ $(i = 1, 2, 3, 4)$, which are proportional to the small parameter so they are slowly varying functions of time $t$, that is, they are almost constants and by integrating, we obtain the values of $a_i$ $(i = 1, 2, 3, 4)$. It is laborious to solve (2.9) for $u_1$. However, as $\lambda_1 \rightarrow \lambda_2$ it takes simple form

\[
(D + \lambda_1)^2(D + \lambda_3)(D + \lambda_4)u_1 = -\sum_{r=2}^{n} F_r(a_3e^{-\lambda_3 t}, a_4e^{-\lambda_4 t})[e^{-\lambda_1 t}(a_1 - a_2 t)]^r.
\]  

Solving equation (2.10), we obtain the value of $u_1$. Finally, substituting the values of $a_i$ $(i = 1, 2, 3, 4)$ and $u_1$ in the equation (2.3), we obtain the complete solution of (2.1).
3. Example

As an example of the above method, we consider the fourth order nonlinear differential equation
\[ \frac{d^4x}{dt^4} + c_1 \frac{d^3x}{dt^3} + c_2 \frac{d^2x}{dt^2} + c_3 \frac{dx}{dt} + c_4 x = -e x^3. \]  
(3.1)

Here
\[ f(x) = x^3 \]
and
\[ x_0 = \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}. \]

Therefore,
\[ f^{(0)} = \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} \right\}^3. \]  
(3.2)

Therefore, for example (3.1), the equations (2.7)-(2.9) respectively become
\[
e^{-\lambda_1 t} (D - \lambda_3 + \lambda_1)(D - \lambda_3 + \lambda_2)(D - \lambda_3 + \lambda_4)A_3
+ e^{-\lambda_2 t} (D - \lambda_4 + \lambda_1)(D - \lambda_4 + \lambda_2)(D - \lambda_4 + \lambda_3)A_4
+ \frac{1}{2} \left\{ e^{-\lambda_1 t} (D - \lambda_2 + \lambda_2)(D - \lambda_2 + \lambda_3)(D - \lambda_2 + \lambda_3) \right\} A_1
+ (D + \lambda_3) \left\{ e^{-\lambda_1 t} (\lambda_1 - \lambda_3 - \frac{3}{2} D) + e^{-\lambda_2 t} (\lambda_2 - \lambda_3 - \frac{3}{2} D) \right\} A_2
- \left( \frac{(\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t})}{(\lambda_1 - \lambda_2)} \right) \times D \left( D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} \right) A_2
= - \left[ \left( a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} \right)^3 + 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2 \right] \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) \right\}, \]  
(3.3)

\[ (D + \lambda_3) \times D \left( D + \lambda_3 - \frac{\lambda_1 + \lambda_2}{2} \right) A_2 = -3 a_2 (a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t})^2 \]  
(3.4)

and
\[
(D + \lambda_3)(D + \lambda_2)(D + \lambda_3)(D + \lambda_4)u_1
= - \left[ 3(a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t}) \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \right\} + \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \right] \left\{ \frac{1}{2} a_1 (e^{-\lambda_1 t} + e^{-\lambda_2 t}) + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right) \right\}^3. \]  
(3.5)

Solving equation (3.4), we obtain
\[ A_2 = a_2 \left[ n_1 a_1^2 e^{-2 \lambda_1 t} + n_2 a_3 a_4 e^{-(\lambda_1 + \lambda_2)t} + n_3 a_3^2 e^{-2 \lambda_2 t} \right], \]  
(3.6)
where
\[ n_1 = \frac{1}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}, \quad n_2 = \frac{12}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 + \lambda_4)}, \quad n_3 = \frac{1}{(\lambda_1 - \lambda_3)(\lambda_2 + \lambda_4)}. \]

Now substituting the value of $A_2$ from (3.6) into (3.3) and in order to separate the equation (3.3) for determining the unknown functions $A_1, A_3$ and $A_4$, we use...
the conditions as discussed in the method (see also Akbar et al. [3], Alam [6, 7, 8]. It is interesting to note that our solution approaches toward critically damped solution (see Alam [8]) if \( \lambda_1 \to \lambda_2 \). However, equation (3.3) has not an exact solution unless \( \lambda_1 \to \lambda_2 \). Under these imposed conditions and by equating like terms on both sides of the equation (3.3), we obtain

\[
e^{-\lambda_1 t}(D - \lambda_1 + \lambda_2)(D - \lambda_1 + \lambda_3)(D - \lambda_1 + \lambda_4)A_1
\]

\[
= -a_2a_3^2n_1\lambda_2\lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)te^{-(\lambda_1+2\lambda_3)t} - \frac{1}{2}a_2a_3a_4n_2\lambda_4(2\lambda_2^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_4)te^{-(\lambda_1+\lambda_2+\lambda_4)t} - a_2a_4^2n_3\lambda_4(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)te^{-(\lambda_1+2\lambda_3)t} (3.7)
\]

\[
e^{-\lambda_1 t}(D - \lambda_3 + \lambda_1)(D - \lambda_3 + \lambda_2)(D - \lambda_3 + \lambda_4)A_3
\]

\[
= \left[a_2n_1(3\lambda_1 + 2\lambda_2 - \lambda_3 - 3\lambda_4 + \lambda_3(\lambda_1 + \lambda_2 + 2\lambda_3)) - \frac{3}{2}a_1\right]a_3^2e^{-(\lambda_1+2\lambda_3)t} + \left[a_2n_2(\lambda_1 + \lambda_2 + 2\lambda_3)(2\lambda_1 + \lambda_3 + 3\lambda_4) - 3a_1\right]a_3a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} + \left[a_2n_3(\lambda_1 + \lambda_3\lambda_1(\lambda_1 - \lambda_3 + 3\lambda_4) + \lambda_4(\lambda_1 + \lambda_2 - 2\lambda_3 + 4\lambda_4)) - \frac{3}{2}a_1\right]a_3^2e^{-(\lambda_2+2\lambda_3)t} + \left[a_2n_4(\lambda_1 + \lambda_2 + 2\lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_4) - \frac{3}{2}a_1\right]a_3a_4e^{-(\lambda_1+2\lambda_3)t} + \left[a_2n_5(\lambda_1 + \lambda_2 + 3\lambda_4) - 3a_1\right]a_3a_4e^{-(\lambda_1+2\lambda_3)t} + \left[3a_2a_3a_4e^{-2\lambda_2+2\lambda_3} + a_3^2a_4^2e^{-(\lambda_1+2\lambda_3)} + a_3^2e^{-3\lambda_3}t\right] (3.8)
\]

and

\[
e^{-\lambda_4 t}(D - \lambda_4 + \lambda_1)(D - \lambda_4 + \lambda_2)(D - \lambda_4 + \lambda_3)A_4 = 0 \quad (3.9)
\]

The particular solutions of equations (3.7)-(3.9) yield respectively

\[
A_1 = I_1a_2a_3^2te^{-(\lambda_1-\lambda_2+2\lambda_3)t} + I_2a_2a_3^2te^{-(\lambda_1-\lambda_2+2\lambda_3)t} + I_3a_2a_3a_4te^{-(\lambda_1-\lambda_2+\lambda_3)t} + I_4a_2a_3a_4te^{-(\lambda_1-\lambda_2+\lambda_3)t} + I_5a_2a_3a_4te^{-(\lambda_1-\lambda_2+2\lambda_3)t} + I_6a_2a_3a_4te^{-(\lambda_1-\lambda_2+2\lambda_3)t} \quad (3.10)
\]

\[
A_3 = (M_1a_2 + M_2a_1)a_2^2e^{-(\lambda_1+\lambda_2)t} + (M_3a_2 + M_4a_1)a_2^2a_3ae^{-(\lambda_1+\lambda_2)t} + (M_5a_2 + M_6a_1)a_2^2e^{-(\lambda_1+2\lambda_4-\lambda_3)t} + (M_7a_2 + M_8a_1)a_2^2e^{-(\lambda_1+2\lambda_4-\lambda_3)t} + (M_9a_2 + M_{10}a_1)a_3a_4e^{-(\lambda_1+\lambda_4)t} + (M_{11}a_2 + M_{12}a_1)a_3a_4e^{-(\lambda_1+\lambda_4)t} + M_{13}a_3a_4^2e^{-(\lambda_1+\lambda_4)t} + M_{15}a_3a_4^2e^{-(\lambda_1+\lambda_4)t} + M_{16}a_3^2a_4^2e^{-(\lambda_1+\lambda_4)t} \quad (3.11)
\]
and

\[ A_4 = 0 \]  

(3.12)

where

\[
\begin{align*}
  r_1 &= -n_1 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2 + 2 \lambda_3), \\
  r_2 &= -\frac{1}{2} n_2 \lambda_2 (2 \lambda_2^2 + 2 \lambda_3 \lambda_4 + \lambda_1 \lambda_3 + \lambda_2 \lambda_4 + \lambda_1 \lambda_4 + \lambda_2 \lambda_4), \\
  r_3 &= -n_3 \lambda_2 \lambda_4 (\lambda_1 + \lambda_3 - 2 \lambda_3 + 4 \lambda_4), \\
  I_1 &= -r_1 \left( \frac{1}{2 \lambda_3 (\lambda_1 + \lambda_3) (\lambda_1 + 2 \lambda_3 - \lambda_4)} \right), \\
  I_2 &= -r_2 \left( \frac{1}{2 \lambda_3 (\lambda_1 + \lambda_3) (\lambda_1 + 2 \lambda_3 - \lambda_4)} \right) + \frac{1}{(\lambda_1 + \lambda_3 + \lambda_4)} + \frac{1}{(\lambda_1 + 2 \lambda_3 - \lambda_4)}, \\
  I_3 &= -r_3 \left( \frac{1}{(\lambda_1 + \lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)} \right), \\
  I_4 &= -r_4 \left( \frac{1}{(\lambda_1 + \lambda_3 + \lambda_4)(\lambda_3 + \lambda_4)} \right) + \frac{1}{(\lambda_1 + \lambda_3 + \lambda_4)} + \frac{1}{(\lambda_3 + \lambda_4)}, \\
  I_5 &= -r_5 \left( \frac{1}{2 \lambda_4 (\lambda_1 + \lambda_4) (\lambda_1 + 2 \lambda_4 - \lambda_3)} \right), \\
  I_6 &= -r_6 \left( \frac{1}{2 \lambda_4 (\lambda_1 + \lambda_4) (\lambda_1 + 2 \lambda_4 - \lambda_3)} \right) + \frac{1}{(\lambda_1 + \lambda_4 + \lambda_3)} + \frac{1}{(\lambda_1 + 2 \lambda_4 - \lambda_3)}, \\
  m_1 &= n_1 \{ (\lambda_1 + 2 \lambda_3)(\lambda_1 + 2 \lambda_3 - \lambda_4) + \lambda_3 (\lambda_1 + \lambda_2 + 2 \lambda_3) \}, \\
  m_2 &= -\frac{3}{2}, \\
  l_1 &= \frac{1}{2} n_1 \left( \lambda_1 + \lambda_3 \right) (2 \lambda_1 + \lambda_3 + 3 \lambda_4), \\
  l_2 &= -3, \\
  p_1 &= n_3 \{ (\lambda_1 + \lambda_4)(\lambda_1 - \lambda_3 + 3 \lambda_4) + \lambda_4 (\lambda_1 + \lambda_2 - 2 \lambda_3 + 4 \lambda_4) \}, \\
  p_2 &= -\frac{3}{2}, \\
  q_1 &= n_1 (\lambda_2 + 2 \lambda_3)(\lambda_2 + 2 \lambda_3 - \lambda_4), \\
  q_2 &= -\frac{3}{2}, \\
  h_1 &= \frac{1}{2} n_2 (\lambda_2 + \lambda_3)(2 \lambda_2 + \lambda_3 + 3 \lambda_4), \\
  h_2 &= -3, \\
  s_1 &= n_3 (\lambda_2 + \lambda_4)(\lambda_2 - \lambda_3 + 3 \lambda_4), \\
  s_2 &= -\frac{3}{2}, \\
  M_1 &= -n_1 \left( \frac{m_1}{2 \lambda_3 (\lambda_1 + 2 \lambda_3 - \lambda_2) (\lambda_1 + 2 \lambda_3 - \lambda_4)} \right), \\
  M_2 &= -n_2 \left( \frac{m_2}{2 \lambda_3 (\lambda_1 + 2 \lambda_3 - \lambda_2) (\lambda_1 + 2 \lambda_3 - \lambda_4)} \right), \\
  M_3 &= -n_3 \left( \frac{l_1}{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)(\lambda_1 + \lambda_3)} \right), \\
  M_4 &= -n_4 \left( \frac{l_2}{(\lambda_3 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)(\lambda_1 + \lambda_3)} \right), \\
  M_5 &= -n_5 \left( \frac{p_1}{2 \lambda_4 (\lambda_1 + \lambda_4)(\lambda_1 + 2 \lambda_4 - \lambda_2)} \right).
\end{align*}
\]
The solution of the equation (3.5) is

\[
M_0 = -\frac{P_2}{2\lambda_4(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_4 - \lambda_2)},
\]
\[
M_7 = -\frac{q_1}{2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)},
\]
\[
M_8 = -\frac{q_2}{2\lambda_3(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)},
\]
\[
M_9 = \frac{h_1}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)},
\]
\[
M_{10} = \frac{h_2}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)},
\]
\[
M_{11} = \frac{s_1}{2\lambda_4(\lambda_2 + \lambda_4)(\lambda_2 + 2\lambda_4 - \lambda_1)},
\]
\[
M_{12} = \frac{s_2}{2\lambda_4(\lambda_2 + \lambda_4)(\lambda_2 + 2\lambda_4 - \lambda_1)},
\]
\[
M_{13} = \frac{1}{3(\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(3\lambda_3 - \lambda_4)},
\]
\[
M_{14} = \frac{2\lambda_3(3\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)}{3},
\]
\[
M_{15} = \frac{(\lambda_3 + \lambda_4)(2\lambda_4 + \lambda_3 - \lambda_1)(2\lambda_4 + \lambda_3 - \lambda_2)}{3},
\]
\[
M_{16} = \frac{1}{2\lambda_4(3\lambda_4 - \lambda_1)(3\lambda_4 - \lambda_2)}.
\]

The solution of the equation (3.5) is

\[
\begin{align*}
u_1 &= -3a_3e^{-(2\lambda_1+\lambda_3)t}[d_0a_1^2 + 2d_1a_1a_2 + d_3a_2^2 + (d_2a_2^2 - 2d_0a_1a_2)t + d_0a_2^2t^2] \\
&\quad - 3a_4e^{-(2\lambda_1+\lambda_3)t}[d_4a_1^2 + 2d_5a_1a_2 + d_7a_2^2 + (d_6a_2^2 - 2d_4a_1a_2)t + d_4a_2^2t^2] \\
&\quad - e^{-\lambda_3t}[da_3^2 + 3a_1a_2(d_9a_1 + d_{10}a_2) + d_{12}a_2^2 + a_2(d_1a_2^2 - 3d_6a_1^2) \\
&\quad - 6d_6a_1a_2)t + 3a_2^2(d_9a_1 + d_{10}a_2)t^2 - d_8a_2^2t^3]
\end{align*}
\]

(3.13)

where

\[
d_0 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_4)},
\]
\[
d_1 = -\frac{1}{2\lambda_1(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_4)} \left( \frac{1}{2\lambda_1} + \frac{2}{\lambda_1 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_3 + \lambda_4 - \lambda_1} \right),
\]
\[
d_2 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_4)} \left( \frac{1}{\lambda_1} + \frac{4}{\lambda_1 + \lambda_3} + \frac{2}{\lambda_1 + \lambda_3 + \lambda_4 - \lambda_1} \right),
\]
\[
d_3 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_3)^2(2\lambda_1 + \lambda_3 - \lambda_4)} \times \left[ \frac{1}{2\lambda_1^2} + \frac{1}{\lambda_1(\lambda_1 + \lambda_3)} + \frac{6}{\lambda_1 + \lambda_3 + \lambda_4 - \lambda_1} \right] + \frac{2}{(2\lambda_1 + \lambda_3 - \lambda_4)^2} + \frac{1}{(2\lambda_1 + \lambda_3 - \lambda_4)} \left( \frac{1}{\lambda_1} + \frac{4}{\lambda_1 + \lambda_3} \right),
\]
Asymptotic Solutions of Fourth Order Near Critically Damped Nonlinear Systems

\[ d_4 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_4)^2(2\lambda_1 - \lambda_3 + \lambda_4)}, \]
\[ d_5 = -\frac{1}{2\lambda_1(\lambda_1 + \lambda_4)^2(2\lambda_1 - \lambda_3 + \lambda_4)} \left( \frac{1}{2\lambda_1} + \frac{2}{\lambda_1 + \lambda_4} + \frac{1}{(2\lambda_1 - \lambda_3 + \lambda_4)} \right), \]
\[ d_6 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_4)^2(2\lambda_1 - \lambda_3 + \lambda_4)} \left( \frac{1}{\lambda_1} + \frac{4}{\lambda_1 + \lambda_4} + \frac{2}{(2\lambda_1 - \lambda_3 + \lambda_4)} \right), \]
\[ d_7 = \frac{1}{2\lambda_1(\lambda_1 + \lambda_4)^2(2\lambda_1 - \lambda_3 + \lambda_4)} \left[ \frac{1}{2\lambda_1^2} + \frac{2}{\lambda_1(\lambda_1 + \lambda_4)} + \frac{6}{(\lambda_1 + \lambda_4)^2} \right. \\
\left. + \frac{2}{(2\lambda_1 - \lambda_3 + \lambda_4)^2} + \frac{1}{(2\lambda_1 - \lambda_3 + \lambda_4)} \left( \frac{1}{\lambda_1} + \frac{4}{\lambda_1 + \lambda_4} \right) \right], \]
\[ d_9 = \frac{1}{4\lambda_1^2(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)}, \]
\[ d_{10} = \frac{1}{4\lambda_1^2(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} \left[ \frac{2}{(3\lambda_1 - \lambda_3)^2} + \frac{2}{(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} + \frac{2}{(3\lambda_1 - \lambda_3)^3} + \frac{2}{(3\lambda_1 - \lambda_3)^2 + \frac{3}{2\lambda_1}} \right], \]
\[ d_{11} = -\frac{1}{4\lambda_1^2(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} \left[ \frac{6}{(3\lambda_1 - \lambda_3)^2} + \frac{6}{(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} + \frac{9}{2\lambda_1^2} + \frac{6}{(3\lambda_1 - \lambda_3)^2} + \frac{6}{(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} + \frac{6}{\lambda_1(3\lambda_1 - \lambda_4)} \right], \]
\[ d_{12} = \left\{ \begin{array}{l}
\left[ \frac{6}{(3\lambda_1 - \lambda_3)^2} + \frac{6}{(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} + \frac{6}{(3\lambda_1 - \lambda_4)^2} \right] + \frac{1}{\lambda_1} \left( \frac{6}{(3\lambda_1 - \lambda_3)^2} + \frac{6}{(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)} + \frac{3}{3\lambda_1 - \lambda_3} \right) + \frac{3}{\lambda_1} \right\}. \]

Putting the values of \( A_1, A_2, A_3 \) and \( A_4 \) from equations (3.10), (3.6), (3.11), (3.12) into equation (2.4) and integrating, we obtain

\[ a_1(t) = a_{1,0} + \varepsilon \left[ a_{2,0}a_{3,0} \frac{I_1(1 - e^{-(\lambda_1 + \lambda_2 - 2\lambda_3)t})}{\lambda_1 - \lambda_2 + 2\lambda_3} \right] \]
\[ + I_1 \left( t e^{-(\lambda_1 + \lambda_2 - 2\lambda_3)t} + \frac{e^{-(\lambda_1 + \lambda_2 - 2\lambda_3)t} - 1}{\lambda_1 - \lambda_2 + 2\lambda_3} \right) \]
\[ - I_1 \left( t e^{-(\lambda_1 + \lambda_2 - 2\lambda_3)t} + \frac{e^{-(\lambda_1 + \lambda_2 - 2\lambda_3)t} - 1}{\lambda_1 - \lambda_2 + 2\lambda_3} \right) \]
Therefore, we obtain the first approximate solution of the equation (3.1) is

\[
x(t, \varepsilon) = \frac{1}{2} a_1 e^{-\lambda_1 t} + a_2 \left( \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} \right)
+ a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + \varepsilon u_1,
\]

where \(a_1, a_2, a_3, a_4\) are given by the equation (3.14) and \(u_1\) is given by the equation (3.13).
4. Results and Discussion

In this article, an analytical approximate solution of fourth order non-oscillatory nonlinear systems has been found based on the KBM method. In order to test the accuracy of an approximate analytical solution obtained by a certain perturbation technique, we compared the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer the work of Murty et al. [12].

First, $x(t, \epsilon)$ is calculated by (3.15) by using the imposed conditions that $\lambda_1 \rightarrow \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ in which $a_1, a_2, a_3, a_4$ are calculated by the equation (3.14) and $u_1$ is calculated by the equation (3.13) for different sets of initial conditions and for various values of $t$. The corresponding numerical solution of (3.1) is also computed by fourth order Runge-Kutta method. The approximate analytic solution and numerical solutions are plotted in the figure Figure 1 and Figure 2. From the figures we observe that the analytical solution and the numerical solution show excellent agreement.

![Figure 1](image.png)

5. Conclusion

An asymptotic method, based on the theory of Krylov-Bogoliubov-Mitropolskii, is developed for solving the fourth order near critically damped nonlinear systems...
under some conditions with small nonlinearities, when the four eigenvalues of the corresponding linear equation are real and negative numbers. The relations $\lambda_1 \rightarrow \lambda_2$ and $\lambda_3 \approx 3\lambda_4$ among the eigenvalues are imposed to solve the systems. The results obtained by this method agree with those obtained by the numerical method.

References


Asymptotic Solutions of Fourth Order Near Critically Damped Nonlinear Systems


Md. Habibur Rahman, Department of Mathematics, Khulna University of Engineering & Technology (KUET), Khulna-9203, Bangladesh
E-mail: mhabib_75@yahoo.com

B. M. Ikramul Haque, Department of Mathematics, Khulna University of Engineering & Technology (KUET), Khulna-9203, Bangladesh
E-mail: bmih06@yahoo.com

M. Ali Akbar, Department of Applied Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh
E-mail: ali_math74@yahoo.com

Received May 7, 2009
Accepted July 15, 2009