Abstract. In this paper, we introduce split Jacobsthal and split Jacobsthal-Lucas quaternions. We obtain generating functions and Binet’s formulas for these quaternions. We also investigate some properties of them.

Keywords. Jacobsthal numbers; Jacobsthal-Lucas numbers; Split quaternions; Recurrence relations

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1. Introduction

The quaternion numbers have been introduced by William Rowan Hamilton in the mid nineteenth century. Quaternions are four-dimensional hyper-complex numbers.

A quaternion is defined by

\[ p = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_3, \]

where \( p_0, p_1, p_2 \) and \( p_3 \) are real numbers, and the units \( e_1, e_2, e_3 \) satisfy the rules

\[ e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1, \]

\[ e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3. \]  

(1)

For more details on quaternions, one can see, for example [5,17].
The split quaternions, in other words coquaternions, have been introduced by James Cockle in 1849. Split quaternions form a four-dimensional non-commutative associative algebra over the real numbers with basis \( \{ 1, e_1, e_2, e_3 \} \).

A split quaternion \( q \) is of the form
\[
q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = (q_0, q_1, q_2, q_3),
\]
where \( q_0, q_1, q_2 \) and \( q_3 \) are real numbers, and the units \( e_1, e_2, e_3 \) satisfy the rules
\[
e_1^2 = -1, \quad e_2^2 = e_3^2 = e_1 e_2 e_3 = 1,
e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = -e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3.
\]

The conjugate of split quaternion \( q \) denoted by \( \overline{q} \) is
\[
\overline{q} = q_0 + q_1 e_1 - q_2 e_2 - q_3 e_3,
\]
and the norm of \( q \) is
\[
N(q) = q \overline{q} = q_0^2 + q_1^2 e_1^2 - q_2^2 - q_3^2 e_3^2.
\]

The Fibonacci sequence is defined recursively by the relation
\[
F_n = F_{n-1} + F_{n-2}
\]
with initial conditions \( F_0 = 0 \) and \( F_1 = 1 \). Similarly, the Lucas sequence is defined as
\[
L_n = L_{n-1} + L_{n-2},
\]
where \( L_0 = 2 \) and \( L_1 = 1 \).

The Jacobsthal sequence is defined by the recurrence relation
\[
J_n = J_{n-1} + 2J_{n-2}
\]
with initial conditions \( J_0 = 0 \) and \( J_1 = 1 \). Also, the Jacobsthal-Lucas sequence is defined recursively by the relation
\[
j_n = j_{n-1} + 2j_{n-2},
\]
where \( j_0 = 2 \) and \( j_1 = 1 \).

The generating functions of the Jacobsthal and Jacobsthal-Lucas sequences are given by
\[
G(t) = \frac{t}{1-t-2t^2}
\]
and
\[
g(t) = \frac{2-t}{1-t-2t^2},
\]
respectively. Moreover, the Binet’s formulas for these sequences are defined as
\[
J_n = \frac{2^n - (-1)^n}{3}
\]
and
\[
j_n = 2^n + (-1)^n,
\]
respectively. There have been many studies on the Jacobsthal and Jacobsthal-Lucas sequences (see, for example [3,7,9,16]).

Horadam [6] defined the Fibonacci and Lucas quaternions as
\[
Q_n = F_n + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3
\]
and
\[
K_n = L_n + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3,
\]
respectively, where \( F_n \) is the \( n \)th Fibonacci number, \( L_n \) is the \( n \)th Lucas number, and \( e_1, e_2, e_3 \) satisfy the rules (1).

Akyigit et al. [1] defined the split Fibonacci and Lucas quaternions as

$$Q_n = F_n + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3$$

and

$$T_n = L_n + L_{n+1}e_1 + L_{n+2}e_2 + L_{n+3}e_3,$$

respectively, where $F_n$ is the $n$th Fibonacci number, $L_n$ is the $n$th Lucas number, and $e_1, e_2, e_3$ satisfy the rules (2).

Polatli et al. [10] studied the split $k$-Fibonacci and $k$-Lucas quaternions, and in [15], Tokeser et al. introduced the split Pell and Pell-Lucas quaternions.

The Jacobsthal and Jacobsthal-Lucas quaternions are defined by Szynal-Liana and Włoch [12] as

$$JQ_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$$

and

$$JLQ_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3,$$

respectively, where $J_n$ is the $n$th Jacobsthal number, $j_n$ is the $n$th Jacobsthal-Lucas number, and $e_1, e_2, e_3$ satisfy the rules (1).


The main objective of this paper is to introduce split Jacobsthal quaternions and split Jacobsthal-Lucas quaternions. We also aim to obtain some properties of these quaternions including generating functions, Binet’s formulas, determinantal representations, matrix representations, Cassini’s identities, Catalan’s identities, and d’Ocagne’s identities.

### 2. Split Jacobsthal and Split Jacobsthal-Lucas Quaternions

The $n$th split Jacobsthal quaternion and $n$th split Jacobsthal-Lucas quaternion are defined, for $n \geq 0$, by

$$SJQ_n = J_n + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3$$

and

$$S JLQ_n = j_n + j_{n+1}e_1 + j_{n+2}e_2 + j_{n+3}e_3,$$

respectively, where $J_n$ is the $n$th Jacobsthal number, $j_n$ is the $n$th Jacobsthal-Lucas number, and $e_1, e_2, e_3$ are split quaternionic units which satisfy the rules (2).
It is easy to see that
\[ SJQ_n = SJQ_{n-1} + 2SJQ_{n-2} \] (5)
and
\[ SJLQ_n = SJLQ_{n-1} + 2SJLQ_{n-2}. \] (6)

The generating functions for the split Jacobsthal and Jacobsthal-Lucas quaternions are given in the following theorem.

**Theorem 1.** The generating functions of the split Jacobsthal and split Jacobsthal-Lucas quaternions are
\[ J(t) = SJQ_0(1-t) + SJQ_1t \]
\[ JL(t) = SJLQ_0(1-t) + SJLQ_1t, \]
respectively.

**Proof.** Let us write
\[ J(t) = \sum_{n=0}^{\infty} SJQ_n t^n = SJQ_0 + SJQ_1t + SJQ_2t^2 + SJQ_3t^3 + \ldots + SJQ_n t^n + \ldots. \]
Then, we have
\[ tJ(t) = SJQ_0t + SJQ_1t^2 + SJQ_2t^3 + \ldots + SJQ_{n-1}t^n + \ldots \]
and
\[ 2t^2 J(t) = 2SJQ_0t^2 + 2SJQ_1t^3 + \ldots + 2SJQ_{n-2}t^n + \ldots. \]
Thus, we obtain
\[ (1-t-2t^2)J(t) = SJQ_0 + (SJQ_1 - SJQ_0)t + \sum_{n=2}^{\infty} (SJQ_n - SJQ_{n-1} - 2SJQ_{n-2})t^n \]
\[ = SJQ_0 + (SJQ_1 - SJQ_0)t \]
which completes the proof of eq. (7).

Eq. (8) can be proved similarly. \qed

The following theorem gives Binet’s formulas for the split Jacobsthal and Jacobsthal-Lucas quaternions.

**Theorem 2.** The \( n \)th term of the split Jacobsthal quaternion and the \( n \)th term of the split Jacobsthal-Lucas quaternion are
\[ SJQ_n = \alpha^* 2^n - \beta^*(-1)^n \] (9)
and
\[ SJLQ_n = \alpha^* 2^n + \beta^*(-1)^n, \] (10)
respectively, where \( \alpha^* = (1,2,4,8) \) and \( \beta^* = (1,-1,1,-1). \)
Proof. The characteristic equation of the recurrence relations (5) and (6) is \( t^2 - t - 2 = 0 \), and the roots of this equation are 2 and \(-1\). From the recurrence relation and initial values \( SJQ_0 = (0, 1, 1, 3), \ SJQ_1 = (1, 1, 3, 5) \), Binet’s formula for \( SJQ_n \) is obtained as

\[
SJQ_n = c_1 2^n + c_2 (-1)^n = \frac{1}{3} [(1, 2, 4, 8) 2^n - (1, -1, 1, -1)(-1)^n],
\]

where \( c_1 = \frac{SJQ_0 + SJQ_1}{3} = \frac{\alpha^*}{3} \) and \( c_2 = \frac{2SJQ_0 - SJQ_1}{3} = \frac{-\beta^*}{3} \).

Thus, we get

\[
SJQ_n = \alpha^* 2^n - \beta^*(-1)^n.
\]

Similarly, from the recurrence relation and initial values \( SJLQ_0 = (2, 1, 5, 7), \ SJLQ_1 = (1, 5, 7, 17) \), Binet’s formula for \( SJLQ_n \) is obtained as

\[
SJLQ_n = (1, 2, 4, 8) 2^n + (1, -1, 1, -1)(-1)^n = \alpha^* 2^n + \beta^*(-1)^n.
\]

**Theorem 3.** For \( n \geq 1 \), let \( P_n \) be \( n \times n \) tridiagonal matrix defined by

\[
P_n = \begin{pmatrix}
P_{11} & P_{12} & 0 & 0 & \cdots & 0 \\
-2 & 1 & 2 & 0 & \cdots & 0 \\
0 & -1 & 1 & 2 & \ddots & 0 \\
0 & 0 & -1 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 \\
0 & \ldots & \ldots & 0 & -1 & 1
\end{pmatrix}
\]

and for \( P_{11} = SJQ_1 \) and \( P_{12} = SJQ_0 \), let \( P_0 = SJQ_0 \), and for \( P_{11} = SJLQ_1 \) and \( P_{12} = SJLQ_0 \), let \( P_0 = SJLQ_0 \). Then

\[
det P_n = SJQ_n,
\]

where \( P_{11} = SJQ_1 \) and \( P_{12} = SJQ_0 \), and

\[
det P_n = SJLQ_n,
\]

where \( P_{11} = SJLQ_1 \) and \( P_{12} = SJLQ_0 \).

**Proof.** We prove the theorem for \( P_{11} = SJQ_1 \) and \( P_{12} = SJQ_0 \). The other condition can be done similarly.

We use mathematical induction on \( n \). For \( n = 1 \) and \( n = 2 \), we have

\[
det P_1 = P_{11} = SJQ_1 \quad \text{and} \quad det P_2 = P_{11} + 2P_{12} = SJQ_2.
\]

Let us assume that the equality holds for \( n - 1 \) and \( n - 2 \), that is,

\[
det P_{n-1} = SJQ_{n-1} \quad \text{and} \quad det P_{n-2} = SJQ_{n-2}.
\]

Finally, for \( n \), we get

\[
det P_n = det P_{n-1} + 2 det P_{n-2} = SJQ_{n-1} + 2SJQ_{n-2} = SJQ_n.
\]
Theorem 4. Let $n$ be positive integer. Then
\[
\begin{pmatrix} 1 & 2 \\
1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} SJQ_2 & SJQ_1 \\
SJQ_1 & SJQ_0 \end{pmatrix} = \begin{pmatrix} SJQ_{n+1} & SJQ_n \\
SJQ_n & SJQ_{n-1} \end{pmatrix}
\]
and
\[
\begin{pmatrix} 1 & 2 \\
1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} SJLQ_2 & SJLQ_1 \\
SJLQ_1 & SJLQ_0 \end{pmatrix} = \begin{pmatrix} SJLQ_{n+1} & SJLQ_n \\
SJLQ_n & SJLQ_{n-1} \end{pmatrix}.
\]

This theorem can be proved easily by using mathematical induction on $n$. Moreover, the consequence of this theorem, which gives the Cassini's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions, is the following theorem.

Theorem 5. For positive integer $n$, we have
\[
SJQ_{n+1}SJQ_{n-1} - SJQ_n^2 = (-2)^{n-1}\lambda
\]
and
\[
SJLQ_{n+1}SJLQ_{n-1} - SJLQ_n^2 = (-1)^n2^{n-1}9\lambda,
\]
where $\lambda = (1, -5, -3, -9)$.

Proof. By taking determinants of the matrices defined in Theorem 4, the proof can be done easily.

Now we give the Catalan's identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions in the following theorem.

Theorem 6. For $r \leq n$, let $n$ and $r$ be positive integers. Then
\[
SJQ_{n+r}SJQ_{n-r} - SJQ_n^2 = (-2)^{n-r} \frac{1}{3}(\mu_12^r - \mu_2(-1)^r)J_r
\]
and
\[
SJLQ_{n+r}SJLQ_{n-r} - SJLQ_n^2 = (-1)^{n-r+1}2^{n-r}(\mu_2 + \mu_14^r - (\mu_1 + \mu_2)(-2)^r),
\]
where $\mu_1 = (1, -13, 1, -13)$ and $\mu_2 = (1, 11, -11, -1)$.

Proof. By using the Binet's formula, we have
\[
SJQ_{n+r}SJQ_{n-r} - SJQ_n^2
= \frac{\alpha^*2^{n+r} - \beta^*(-1)^{n+r}}{3} - \frac{\alpha^*2^n - \beta^*(-1)^n}{3}
= \frac{1}{3} \left[ \alpha^*\beta^*(-2)^n + \beta^*\alpha^*(-2)^n - \alpha^*\beta^*(-1)^{n-r}2^{n+r} - \beta^*\alpha^*(-1)^{n-r}2^{n-r} \right]
= \frac{1}{3} \left[ \beta^*\alpha^*(-1)^r(2^{-r} - (-1)^r) - \alpha^*\beta^*(-1)^{n-r}2^{n}(2^r - (-1)^r) \right]
= \frac{2^r - (-1)^r}{3} \left[ \beta^*\alpha^*(-1)^r - \alpha^*\beta^*2^r \right].
\]
Since $\alpha^* = (1, 2, 4, 8)$ and $\beta^* = (1, -1, 1, -1)$, and also by considering eq. (3), we obtain the desired result.

The other identity can be proved similarly by using the Binet’s formula.
Note that if we set \( r = 1 \) in Theorem 6, the Cassini’s identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions, which are given in Theorem 5 can be obtained again.

The following theorem gives the d’Ocagne’s identities involving the split Jacobsthal and Jacobsthal-Lucas quaternions.

**Theorem 7.** Let \( m \) and \( n \) be two positive integers. Then

\[
SJQ_m SJQ_{n+1} - SJQ_n SJQ_{m+1} = (-1)^{n+1} 2^n \rho J_{m-n}
\]

and

\[
SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1} = (-2)^n 3 \rho (2^{m-n} - (-1)^{m-n}),
\]

where \( \rho = (1, 3, -7, -5) \).

**Proof.** By using the Binet’s formula \((10)\), we have

\[
SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1}
\]

\[
= (\alpha^* 2^n + \beta^*(-1)^m)(\alpha^* 2^{n+1} + \beta^*(-1)^{m+1}) - (\alpha^* 2^n + \beta^*(-1)^n)(\alpha^* 2^{n+1} + \beta^*(-1)^{m+1})
\]

\[
= \alpha^* 2^{n+1} + \beta^*(-1)^m + \beta^* 2^n + \alpha^*)(-1)^m 2^n - \alpha^* 2^{n+1} + \beta^*(-1)^{m+1} \rho (2^{m-n} - (-1)^{m-n}).
\]

Since \( \alpha^* = (1, 2, 4, 8) \) and \( \beta^* = (1, -1, 1, -1) \), we obtain

\[
SJLQ_m SJLQ_{n+1} - SJLQ_n SJLQ_{m+1} = (-2)^n 3 \rho (2^{m-n} - (-1)^{m-n}).
\]

In a similar way, the first identity can be proved. \( \square \)

### 3. Results

In this section, we derive some identities of the split Jacobsthal quaternions and split Jacobsthal-Lucas quaternions.

**Theorem 8.** Let \( m, n \) and \( r \) be positive integers. Then

\[
2SJQ_{n-1} + SJQ_{n+1} = SJQ_n,
\]

\( \text{(11)} \)

\[
9SJQ_n^2 - SJLQ_n^2 = (-2)^{n+2}(1, -1, -5, -7),
\]

\( \text{(12)} \)

\[
SJQ_{m+n} + (-2)^n SJQ_{m-n} = J_n SJQ_m,
\]

\( \text{(13)} \)

\[
SJLQ_{m+n} + (-2)^n SJLQ_{m-n} = J_n SJLQ_m,
\]

\( \text{(14)} \)

\[
SJQ_{m+n} = J_{n+1} SJQ_m + 2J_n SJQ_{m-1},
\]

\( \text{(15)} \)

\[
SJLQ_{m+n} = \frac{1}{3}(J_{n+1} SJLQ_m + 2J_n SJLQ_{m-1}),
\]

\( \text{(16)} \)

\[
SJQ_{2n} = J_{n+1} SJQ_n + 2J_n SJQ_{n-1},
\]

\( \text{(17)} \)

\[
SJQ_{2n+1} = J_{n+1} SJQ_{n+1} + 2J_n SJQ_n,
\]

\( \text{(18)} \)

\[
SJQ_{m+n} SJLQ_{m+r} SJLQ_{m+n} = (-1)^{m+n} 2^{m+n+1}(1, -1, -5, -7) J_{r-n}.
\]

\( \text{(19)} \)
Proof. Throughout the proof, we consider \( \alpha^* = (1, 2, 4, 8) \) and \( \beta^* = (1, -1, 1, -1) \).

(11): By using the Binet’s formula (9), we have

\[
2SJQ_{n-1} + SJQ_{n+1} = 2 \frac{\alpha^* 2^{n-1} - \beta^*(-1)^{n-1}}{3} + \frac{\alpha^* 2^{n+1} - \beta^*(-1)^{n+1}}{3} = \frac{1}{3}(3\alpha^* 2^n + 3\beta^*(-1)^n)
\]

\[
= \alpha^* 2^n + \beta^*(-1)^n.
\]

From the Binet’s formula (10), the proof of the identity (11) is completed.

(12): From the Binet’s formulas (9) and (10), we have

\[
9SJQ_n^2 - SJQ_n J_{n,\alpha} = \frac{\alpha^* 2^{2n} - \beta^*(-1)^n \alpha^* 2^n - \beta^*(-1)^n}{3} - (\alpha^* 2^n + \beta^*(-1)^n)(\alpha^* 2^n + \beta^*(-1)^n)
\]

\[
= (-2)^{n+1}(\alpha^* \beta^* + \beta^* \alpha^*)
\]

\[
= (-2)^{n+2}(1, -1, -5, -7).
\]

(13): By using the Binet’s formula (9), we have

\[
SJQ_{m+n} + (-2)^n SJQ_{m-n} = \frac{\alpha^* 2^{m+n} - \beta^*(-1)^{m+n}}{3} + (-2)^n \frac{\alpha^* 2^{m-n} - \beta^*(-1)^{m-n}}{3}
\]

\[
= \frac{1}{3}(2^n + (-1)^n)(\alpha^* 2^m - \beta^*(-1)^m).
\]

From the eqs. (4) and (9), we obtain the desired result.

The proof of the identity (14) can be done similarly by using the Binet’s formula (10).

(15): From the definition of the split Jacobsthal quaternion and the identity \( J_{m+n} = J_m J_{n+1} + 2J_{m-1} J_n \) (see [9]), we have

\[
SJQ_{m+n} = J_{m+n} + J_{m+n+1} e_1 + J_{m+n+2} e_2 + J_{m+n+3} e_3
\]

\[
= J_{n+1}(J_m + J_{m+1} e_1 + J_{m+2} e_2 + J_{m+3} e_3) + 2J_n(J_{m-1} + J_m e_1 + J_{m+1} e_2 + J_{m+2} e_3)
\]

\[
= J_{n+1}SJQ_m + 2J_nSJQ_{m-1}.
\]

The identity (16) can be proved similarly by using the identity \( j_{m+n} = j_m j_{n+1} + 2j_{m-1} j_n \). The identities (17) and (18) can be proved by taking, respectively, \( m = n \) and \( m = n + 1 \) into eq. (15).

(19): By using the Binet’s formulas (9) and (10), we have

\[
SJQ_{m+n}SJLQ_{m+r} - SJQ_{m+r}SJLQ_{m+n} = \frac{\alpha^* 2^{m+n} - \beta^*(-1)^{m+n}}{3}(\alpha^* 2^{m+r} + \beta^*(-1)^{m+r}) - \frac{\alpha^* 2^{m+r} - \beta^*(-1)^{m+r}}{3}(\alpha^* 2^{m+n} + \beta^*(-1)^{m+n})
\]

\[
= \frac{1}{3}[\alpha^* \beta^*(-1)^{m+r} 2^{m+n} - \beta^* \alpha^*(-1)^{m+n} 2^{m+r} - \alpha^* \beta^*(-1)^{m+n} 2^{m+r} - \beta^* \alpha^*(-1)^{m+r} 2^{m+n}]
\]

\[
= \frac{2^{r-n} - (-1)^{r-n}}{3}(1, -1, -5, -7)(\alpha^* \beta^* + \beta^* \alpha^*).
\]

By considering \( \alpha^* \), \( \beta^* \), and the Binet’s formula (3), we get

\[
SJQ_{m+n}SJLQ_{m+r} - SJQ_{m+r}SJLQ_{m+n} = (-1)^{m+n} 2^{m+n+1}(1, -1, -5, -7)J_{r-n}.
\]
4. Conclusion

In this study, the split Jacobsthal and Jacobsthal-Lucas quaternions were introduced. Some results including Binet’s formulas, generating functions and determinantal representations for these quaternions were given. Moreover, some well-known identities, such as Catalan’s, Cassini’s and d’Ocagne’s identities, involving the split Jacobsthal and Jacobsthal-Lucas quaternions were obtained.

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Competing Interests

The author declares that he has no competing interests.

Authors’ Contributions

The author wrote, read and approved the final manuscript.

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