Communications in Mathematics and Applications
Vol. 9, No. 3, pp. 411–419, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v9i3.809

http://www.rgnpublications.com

Proceedings:
3rd International Conference on Pure and Applied Mathematics
Department of Mathematics, University of Sargodha, Sargodha, Pakistan
November 10-11, 2017

Inequalities of Hardy type for Jackson Nörlund Integrals

Research Article

Naheed Ashraf[1], Khuram Ali Khan[2,*], Shakila Mobeen[1] and Ammara Nosheen[3]

[1] *Department of Mathematics, Govt. Post Graduate College for Women, Sargodha 40100, Pakistan*
[2] *Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[3] *Department of Mathematics, The University of Lahore, Sargodha Campus 40100, Pakistan*
***Corresponding author:** khuramsms@gmail.com

Abstract. In the paper, we prove Jensen's inequality for Jackson Nörlund integrals, and by using Jensen's inequality, Hardy type inequalities with general kernels as well as choosing special kernels are proved. In seek of applications to these inequalities we give Hilbert-Hardy inequality and Pólya-Knop type inequalities.

Keywords. Convex function; Jackson integral; Nörlund sums; Jensen's inequality; Hardy inequality

MSC. Primary 26D07; 26D15; 26D20; 26D9

Received: February 8, 2018 **Accepted:** March 13, 2018

1. Introduction

The progress of the Hardy inequality for both discrete and continuous cases during the period 1906-1928 has its own prehistory. Mathematicians other than G.H. Hardy, such as E. Landau, G. Pólya, I. Schur, and M. Riesz have been contributed in prehistory [8–10]. There is vast area of research to investigate the Hardy type inequalities, its applications, extensions, generalizations and variants [1, 3, 11, 14].

In the theory of time scales, [5] authors gave the following variant of Hardy inequality. Let I_1, I_2 be two time scales and Φ be a convex function, then

$$\int_{I_1} \eta(x)\Phi\left(\frac{1}{\Lambda(x)}\int_{I_2} \lambda(x,y)f(y)\Delta y\right)\Delta x \le \int_{I_2} \xi(y)\Phi(f(y))\Delta y$$

holds, where Λ, λ, f, ξ are defind in [5].

Further q-analogs of Hardy-type inequalities are given in [3], one of which is as follows: Let $0 < r$, $p \le \infty$, then the following inequality hold:

$$\left(\int_0^\infty \left(u(x)\int_0^x v(t)g_1(t)d_qt\right)^r d_qx\right)^{\frac{1}{r}} \le C\left(\int_0^\infty f^p(x)d_qx\right)^{\frac{1}{p}}.$$

It is noted that Hilger time scales theory [4] which unifies continuous and discrete calculus could not cover Hahn quantum calculus [12]. In order to overcome this difficulty Hahn introduce q,w difference operator denoted by $D_{q,w}$ [2], where $q \in (0,1)$ and $w > 0$ are fixed. It combines the two most well known difference operators namely, the Jackson q-difference derivative D_q [7], where $q \in (0,1)$; and the forward difference Δ_w [7], where $w > 0$. In this paper we prove Hardy-type inequalities for Jackson-Nörlund integrals (being anti-derivative of Hahn difference operator) which generalizes Jackson-integral and Nörlund sum (being anti-derivative of q-difference operator and forward-difference operator).

2. Preliminaries

[3] Jackson Nörlund Integration

Let $I = [a,b]$ be a closed interval of R such that $w_0, a, b \in I$ and for $g_1 : I \to R$, we define the q,w integral of g_1 from a to b by

$$\int_a^b g_1(t)d_{q,w}(t) := \int_{w_0}^b g_1(t)d_{q,w}(t) - \int_{w_0}^a g_1(t)d_{q,w}(t),$$

where

$$\int_\infty^x g_1(t)d_{q,w}(t) = (x(1-q)-w)\sum_{k=0}^\infty q^k g_1(xq^k + w[k]_{q,w}),$$

$[k]_{q,w} = \dfrac{w(1-q^k)}{1-q}$, and the series converges at $x = a$ and $x = b$.

Properties of Jackson Nörlund Integration

(a) Let $g_1, g_2 : I_1 \to R$ be q, w integrable on $I_1, c \in R$ and $a, b, c \in I_1$, then

- $\displaystyle\int_a^a g_1(t)d_{q,w}(t) = 0;$
- $\displaystyle\int_a^b c g_1(t)d_{q,w}(t) = c\int_a^b g_1(t)d_{q,w}(t);$
- $\displaystyle\int_a^b g_1(t)d_{q,w}(t) = -\int_a^b g_1(t)d_{q,w}(t);$

- $\int_a^b g_1(t)d_{q,w}(t) = \int_a^c g_1(t)d_{q,w}(t) + \int_c^b g_1(t)d_{q,w}(t);$
- $\int_a^b (g_1(t) \pm g_2(t))d_{q,w}(t) = \int_a^b g_1(t)d_{q,w}(t) \pm \int_a^b g_2(t)d_{q,w}(t).$

(b) Every Riemannian Integrable function g_1 on I_1 is q,w-Integrable on I_1.

(c) If $g_1, g_2 : I_1 \to R$ are continuous at w_0, then

$$\int_a^b g_1(t)D_{q,w}g_2(t)d_{q,w}(t) = g_1(t)g_2(t)|_a^b - \int_a^b D_{q,w}g_1(t)g_2(qt+w)d_{q,w}(t).$$

[13] Convex function

Let I be an interval in R. Then $F : I \to R$ is said to be convex if for all $x, y \in I$ and all $a \in [0,1]$, $F(ax + (1-a)y) = aF(x) + (1-a)F(y)$ holds.

Throughout the paper, we assume $q \in\ ]0,1[$, $w \in\ ]0,\infty[$, $w_0 = \dfrac{w}{1-q}$ and I to be an interval in R containing w_0.

3. Jensen's Inequality

Theorem 3.1. *Let $I_1 = [a,b]$, $I_2 = [c,d]$, where $I_1, I_2 \subset R$ are two intervals. Assume $\Psi \in C(I_2, R)$ is convex. Moreover, let $h_1 : I_1 \to R$ be q,w-integrable such that $\int_a^b |h_1(t)|d_{q,w}(t) > 0$ and $g_1 : I_2 \to R$ is q,w-integrable such that $g_1(I_2) \subseteq I_1$, then*

$$\Psi\left(\frac{\int_a^b |h_1(t)|g_1(t)d_{q,w}(t)}{\int_a^b |h_1(t)|d_{q,w}(t)}\right) \le \frac{\int_a^b |h_1(t)|\Psi(g_1)d_{q,w}(t)}{\int_a^b |h_1(t)|d_{q,w}(t)}. \tag{3.1}$$

Proof. Since Ψ is convex, as in [14] for $x \in (c,d)$ there exist $a_x \in R$, such that

$$\Psi(s) - \Psi(x) \ge a_x(s-x) \tag{3.2}$$

holds for all $s \in (c,d)$. Let

$$x = \frac{\int_a^b |h_1(t)|g_1(t)d_{q,w}(t)}{\int_a^b |h_1(t)|d_{q,w}(t)}.$$

(3.1) can be rearranged as

$$\int_a^b |h_1(t)|\Psi(g_1(t))d_{q,w}(t) - \int_a^b |h_1(t)|d_{q,w}(t)(t)\Psi\left(\frac{\int_a^b |h_1(t)|g_1(t)d_{q,w}(t)}{\int_a^b |h_1(t)|d_{q,w}(t)}\right)$$

$$= \int_a^b |h_1(t)|\Psi(g_1(t))d_{q,w}(t) - \int_a^b |h_1(t)|\Psi(x)d_{q,w}(t)$$

$$= \int_a^b |h_1(t)|(\Psi(g_1(t)) - \Psi(x))d_{q,w}(t) \qquad \text{(by using (3.2))}$$

$$\ge a_x \int_a^b |h_1(t)|(g_1(t) - x|d_{q,w}(t)$$

$$= a_x\left[\int_a^b |h_1(t)|g_1(t)d_{q,w}(t) - x\int_a^b |h_1(t)d_{q,w}(t)\right] \qquad \text{(substitute the value of } x\text{)}$$

$$= a_x\left[\int_a^b |h_1(t)|g_1(t)d_{q,w}(t) - \left(\frac{\int_a^b |h_1(t)|(t)d_{q,w}(t)}{\int_a^b |h_1(t)|d_{q,w}(t)}\right)\int_a^b |h_1(t)|d_{q,w}(t)\right]$$

$$= a_x\left(\int_a^b |h_1(t)|g_1(t)d_{q,w}(t) - \int_a^b |h_1(t)|g_1(t)d_{q,w}(t)\right)$$

$$= a_x(0) = 0. \qquad \square$$

In the sequel, we use the following notations: $I_1 = [a,b]$ and $I_2 = [c,d]$, $a_{k'} = aq^{k'} + w[k']_{q,w}$, $b_{k'} = bq^{k'} + w[k']_{q,w}$, $c_k = cq^k + w[k]_{q,w}$, $d_k = dq^k + w[k]_{q,w}$, $(a(1-q)-w) = \widetilde{a}$, $(b(1-q)-w) = \widetilde{b}$, $(c(1-q)-w) = \widetilde{c}$, $(d(1-q)-w) = \widetilde{d}$.

3.1 Inequalities with General Kernels

Theorem 3.2. *Let*

$$I_1 = [a,b] \text{ and } I_2 = [c,d] \text{ be two interval on } R, \tag{3.3}$$

$$\lambda : I_1 \times I_2 \to R \text{ is such that } \Lambda(x) = \int_c^d \lambda(x,y)d_{q,w}(y) < \infty,\ x \in I_1, \tag{3.4}$$

$$\eta : I_1 \to R \text{ is such that } \xi(y) = \int_a^b \frac{\lambda(x,y)\eta(x)}{\Lambda(x)}d_{q,w}(x) < \infty,\ y \in I_2. \tag{3.5}$$

If $\Psi \in C(I_1,R)$ is convex,where $I_1 \subset R$ is an interval, then

$$\int_a^b \eta(x)\Psi\left(\frac{1}{\Lambda(x)}\int_c^d \lambda(x,y)g_1(y)d_{q,w}(y)\right)_{q,w}(x) \le \int_c^d \xi(y)\Psi(g_1(y))d_{q,w}(y) \tag{3.6}$$

holds for all q,w-integrable $g_1 : I_2 \to R$ such that $g_1(I_2) \subset I_1$.

Proof. Consider the left hand side of (3.6) and use Jensen's inequality (3.1)

$$\int_a^b \eta(x)\Psi\left(\frac{1}{\Lambda(x)}\int_c^d \lambda(x,y)g_1(y)d_{q,w}(y)\right)d_{q,w}(x) \le \int_a^b \frac{\eta(x)}{\Lambda(x)}\left(\int_c^d \lambda(x,y)\Psi(g_1(y))d_{q,w}(y)\right)d_{q,w}(x)$$

apply definition of Jackson Nörlund Integrals

$$= \int_a^b \eta(x)\left(\frac{1}{\Lambda(x)}\left\{\int_{w_0}^d \lambda(x,y)\Psi(g_1(y))d_{q,w}(y) - \int_{w_0}^c \lambda(x,y)\Psi(g_1(y))d_{q,w}(y)\right\}\right)d_{q,w}(x)$$

$$= \widetilde{b}\sum_{k'=0}^{\infty} q^{k'}\eta(b_{k'})\left(\frac{1}{\Lambda(b_{k'})}\left\{\widetilde{d}\sum_{k=0}^{\infty} q^k\lambda\left(b_{k'},d_k\right)\Psi(g_1(d_k)) - \widetilde{c}\sum_{k=0}^{\infty} q^k\lambda\left(b_{k'},c_k\right)\Psi(f(c_k)\right\}\right)$$

$$- \widetilde{a}\sum_{k'=0}^{\infty} q^{k'}\eta(a_{k'})\left(\frac{1}{\Lambda(a_{k'})}\left\{\widetilde{d}\sum_{k=0}^{\infty} q^k\lambda\left(a_{k'},d_k\right)\Psi(g_1(d_k) - \widetilde{c}\sum_{k=0}^{\infty} q^k\lambda\left(a_{k'},c_k\right)\Psi(f(c_k))\right\}\right)$$

switch the sums to get

$$= \widetilde{d}\sum_{k=0}^{\infty} q^k\left\{\frac{\widetilde{b}\sum_{k'=0}^{\infty} q^{k'}\Psi(b_{k'},d_k)\eta(b_{k'})}{\Lambda(b_{k'})} - \frac{\widetilde{a}\sum_{k'=0}^{\infty} q^{k'}\lambda(a_{k'},d_k)\eta(a_{k'})}{\Lambda(a_{k'})}\right\}\Psi(g_1(d_k))$$

$$- \widetilde{c}\sum_{k=0}^{\infty} q^k\left\{\frac{\widetilde{b}\sum_{k'=0}^{\infty} q^{k'}\lambda(b_{k'},c_k)\eta(b_{k'})}{\Lambda(b_{k'})} - \frac{\widetilde{a}\sum_{k'=0}^{\infty} q^{k'}\lambda(a_{k'},c_k)\eta(a_{k'})}{\Lambda(a_{k'})}\right\}\Psi(g_1(c_k))$$

$$= \int_c^d \left\{ \int_a^b \frac{\lambda(x,y)\eta(x)d_{q,w}(x)}{\Lambda(x)} \right\} (\Psi(g_1(y))d_{q,w}(y)$$

$$= \int_c^d \xi(y)(\Psi(g_1(y))d_{q,w}(y).$$ □

Corollary 3.3. *Assume* (3.3), (3.4) *and* (3.5). *If* $p > 1$, *then*

$$\int_a^b \eta(x)\left(\frac{1}{\Lambda(x)}\int_c^d \lambda(x,y)g_1(y)d_{q,w}(y)\right)^p d_{q,w}(x) \le \int_c^d \xi(y)(g_1(y))^p d_{q,w}(y)$$

holds for all q,w*-integrable* $g_1 : I_2 \to R$.

Proof. Use $\Psi(x) = x^p$ in Theorem 3.2. □

Corollary 3.4. *Assume* (3.3), (3.4) *and* (3.5). *If* $p > 1$, *then*

$$\int_a^b \eta(x) e^{\frac{p}{\Lambda(x)}\int_c^d \lambda(x,y)\ln g_2(y)d_{q,w}(y)} d_{q,w}(y) \le \int_c^d \xi(y)(g_2(y))^p d_{q,w}(y)$$

holds for all q,w*-integrable* $g : I_2 \to (0,\infty)$.

Proof. Use $\Psi(r) = e^r$ and $g_1 = \ln(g_2^p)$ in Theorem 3.2. □

Corollary 3.5. *Assume* (3.3), (3.4) *and* (3.5). *Then*

$$\int_a^b \eta(x) e^{\frac{1}{\Lambda(x)}\int_c^d \lambda(x,y)\ln g_2(y)d_{q,w}(y)} d_{q,w}(y) \le \int_c^d \xi(y)g_2(y)d_{q,w}(y)$$

holds for all q,w*-integrable* $g : I_2 \to (0,\infty)$.

Proof. Use $p = 1$ in the above corollary. □

Theorem 3.6. *Let*

$$0 \le a < b \le \infty,\ I = I_1 = I_2 = [a,b] \text{ be an interval on } R, \tag{3.7}$$

$$u : I_1 \to R_+ \text{ is such that } v(y) = \int_a^b \frac{y\lambda(x,y)u(x)}{(qx+w)\Lambda(x)} d_{q,w}(x) < \infty,\ y \in I_2. \tag{3.8}$$

If $\Psi \in C(I,R)$ *is convex, then*

$$\int_a^b u(x)\Psi((A_k g_1)(x))\frac{d_{q,w}(x)}{(qx+w)} \le \int_a^b v(y)\Psi(g_1(y))\frac{d_{q,w}(y)}{y}$$

holds for all q,w*-integrable* $g_1 : I \to R$ *such that* $g_1(I) \subset I$, *where*

$$(A_k g_1)(x) = \frac{1}{\Lambda(x)}\int_a^b \lambda(x,y)g_1(y)d_{q,w}(y).$$

Proof. Replace $\eta(x)$ by $\dfrac{u(x)}{qx+w}$ in the Theorem 3.2. In this case $\xi(y) = \dfrac{v(y)}{y}$.
An application of Theorem 3.2 completes the proof. □

Corollary 3.7. *Assume* (3.7) *and* (3.8). *If* $p > 1$, *then*

$$\int_a^b u(x)((A_k g_1)(x))^p \frac{d_{q,w}(x)}{(qx+w)} \leq \int_a^b v(y)(g_1(y))^p \frac{d_{q,w}(y)}{y}$$

holds for all q,w*-integrable* $g_1 : I \to R$ *such that* $g_1(I) \subset I$.

Proof. Use $\Psi(r) = r^p$ in the Theorem 3.6. □

3.2 Inequalities with Special Kernels

Corollary 3.8. *Assume* (3.7) *and* (3.8) *with the kernel* λ *such that*

$$\lambda(x,y) = 0 \quad \text{if } a \leq y \leq (qx+w) \leq b. \tag{3.9}$$

If $\Psi \in C(I,R)$ *is convex, then*

$$\int_a^b u(x)\Psi((A_k g_1)(x))\frac{d_{q,w}(x)}{qx+w} \leq \int_a^b v(y)\Psi(g_1(y))\frac{d_{q,w}(y)}{y},$$

where

$$\Lambda(x) = \int_{(qx+w)}^b \lambda(x,y)d_{q,w}(y),$$

$$v(y) = y\int_a^y \frac{\lambda(x,y)u(x)}{(qx+w)\Lambda(x)}d_{q,w}(x),$$

$$(A_k g_1)(x) = \frac{1}{\Lambda(x)}\int_{qx+w}^b \lambda(x,y)g_1(y)d_{q,w}(y).$$

Proof. Use (3.9) in Theorem 3.6. □

Theorem 3.9. *Assume* (3.7) *and* $\eta : I \to R_+$ *is such that*

$$\widetilde{\xi}(y) = \int_y^b \frac{\eta(x)}{(qx+w)-a}d_{q,w}(x) < \infty, \qquad y \in I.$$

If $\Psi \in C(I,R)$ *is convex, then*

$$\int_a^b \eta(x)\Psi\left((\widetilde{A}g_1)(x)\right)d_{q,w}(x) \leq \int_a^b \widetilde{\xi}(y)\Phi(g_1(y))d_{q,w}(y)$$

holds for all q,w*-integrable* $g_1 : I \to R$ *such that* $g_1(I) \subset I$, *where*

$$(\widetilde{A}g_1)(x) = \frac{1}{(qx+w)-a}\int_a^{(qx+w)} g_1(y)d_{q,w}(y).$$

Proof. Let Λ and $A_k g_1$ be defined as in the statement of Theorem 3.2 and Theorem 3.6, respectively. The statement follows from Theorem 3.2 by using

$$\lambda(x,y) = \begin{cases} 1 & \text{if } a \leq y \leq qx+w \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

Since in this case we have

$$\Lambda(x) = \int_a^{qx+w} d_{q,w}(y) = qx+w-a$$

and thus $A_k = \widetilde{A}$ and $\xi = \widetilde{\xi}$. □

3.3 Hardy-Hilbert-type Inequality

Theorem 3.10. *Assume* (3.7) *with* $a = 0$ *and* $b = \infty$. *If we define*

$$\Lambda_1(x) = \int_0^\infty \frac{(\frac{y}{x})^{\frac{-1}{p}}}{x+y} d_{q,w}(y)$$

and

$$\Lambda_2(y) = \int_0^\infty \frac{(\frac{y}{x})^{\frac{-1}{p}}}{x+y} d_{q,w}(x),$$

then

$$\int_0^\infty (\Lambda_1(x))^{1-p} \left(\int_0^\infty \frac{g_1(y)}{x+y} d_{q,w}(y)^p \right)^p \le \int_0^\infty \Lambda_2(y)(g_1(y))^p d_{q,w}(y)$$

holds for all q,w*-integrable* $g_1 : I \to R_+$.

Proof. Use $\eta(x) = \dfrac{K_1(x)}{x}$ and

$$\lambda(x,y) = \begin{cases} \dfrac{(\frac{y}{x})^{\frac{-1}{p}}}{x+y} & \text{if } x = 0,\ y = 0,\ x+y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

in Corollary 3.3 to obtain

$$\int_0^\infty (K_1(x))^{1-p} \left(\int_0^\infty \frac{\left(\frac{y}{x}\right)^{\frac{-1}{p}}}{x+y} d_{q,w} y \right)^p \frac{d_{q,w} x}{x} \le \int_0^\infty \xi(y)(g_1(y))^p d_{q,w} y,$$

where

$$\begin{aligned} \xi(y) &= \int_0^\infty \frac{\lambda(x,y)\eta(x)}{\Lambda_1(x)} d_{q,w} x \\ &= \int_0^\infty \frac{\lambda(x,y)}{x} d_{q,w} x \\ &= \frac{1}{y} \int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-\frac{1}{p}}}{x+y} d_{q,w} x \\ &= \frac{\Lambda_2(y)}{y}. \end{aligned}$$

□

3.4 Pólya-Knopp type Inequalities

Corollary 3.11. *Assume* (3.7) *with* $a \ge 0$, $b = \infty$. *If* $\Psi \in C(I,R)$ *is convex, then*

$$\int_a^\infty \eta(x)\Psi\left(\frac{1}{(qx+w)-a} \int_a^{(qx+w)} g_1(y) d_{q,w}(y) \right) d_{q,w}(x) \le \int_a^\infty \left(\int_y^\infty \frac{\eta(x) d_{q,w}(x)}{(qx+w)-a} \right) \Psi(g_1(y)) d_{q,w}(y) \tag{3.10}$$

holds for all q,w*-integrable* $g_1 : I \to R$.

Proof. The statement follows from Theorem 3.9. □

Example 3.12. Use $\Psi(x) = x^p$, $p > 1$ in (3.10), to get the following inequality

$$\int_a^\infty \eta(x)\left(\frac{1}{(qx+w)-a}\int_a^{(qx+w)} g_1(y)d_{q,w}(y)\right)^p d_{q,w}(x) \le \int_a^\infty \left(\int_y^\infty \frac{\eta(x)d_{q,w}(x)}{(qx+w)-a}\right) g_1^p(y)d_{q,w}(y).$$

Example 3.13. Use $\Psi(r) = e^r$ and $g_1 = \ln(g_2^p)$ for $p \ge 1$ in (3.10), to get the following inequality

$$\int_a^\infty \eta(x)e^{\frac{p}{(qx+w)-a}\int_a^{(qx+w)} \ln(g_2(y))d_{q,w}(y)} d_{q,w}(x) \le \int_a^\infty \left(\int_y^\infty \frac{\eta(x)d_{q,w}(x)}{(qx+w)-a}\right) g_2^p(y)d_{q,w}(y).$$

Remark 3.14. By taking $w = 0$ or $q = 1$ in the above inequalities, one can deduce inequalities for Jackson integrals [7] or for Nörlund sums [6] respectively, which are also new upto knowledge of authors.

Acknowledgement

The results of this papers are presented in “Third International Conference on Pure and Applied Mathematics 2017” hosted by Department of Mathematics, University of Sargodha, Sargodha-Punjab, Pakistan.

Competing Interests

The authors declare that they have no competing interests.

Authors’ Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Abramovich, K. Krulić, J. Pečarić and L.E. Persson, Some new refined Hardy type inequaliteis with general kernels and measures, *Aequat. Math.* **79** (2010), 157 – 172.

[2] M.H. Annaby, A.E. Hamza, K.A. Aldwoah, Hahn difference operator and associated Jackson-Nörlund integral, *J. Optim. Theory Appl.* **154** (2012), 133 – 153.

[3] A.O. Baiarystanov, L.E. Persson, S. Shaimardan and A. Temirkhanova, Some new Hardy-type inequalities in q-analysis, *Jour. Math. Ineq.* **10**(3) (2016), 761 – 781.

[4] M.Bohner and A. Peterson, *Dynamic Equation on Time Scales: An Introduction with Application*, Birkhäuser, Boston (2001).

[5] M. Bohner, A. Nosheen, J. Pečariȁ and A. Younus, Some dynamic Hardy-type inequalities with general kernal, *Jour. Math. Ineq.* **8**(1) (2014), 185 – 199.

[6] A.M.C. Brito da Cruz, N. Martins and D.F.M. Torres, A symmetric quantum calculus, *Springer Proceedings in Math. Stat.* **47** (2013), 359 – 366.

[7] P. Cheung, V. Kac, *Quntum Calculus*, Edwards Brothers Inc., Ann Arbor, MI, USA (2000).

[8] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press (1934).

[9] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality. About Its History and Some Related Result*, Vydavatelsky Servis Publishing House, Pilsen (2007).

[10] A. Kufner, L. Maligranda and L.-E. Persson, The prehistory of the Hardy inequality, *Springer, Amer. Math. Monthly* **113** (2006), 715–732.

[11] S. Kaijser, L. Nikolova, L.E. Persson and A. Wedestig, Hardy-type inequalities via convexity, *Math. Ineq. Appli.* **8**(3) (2005), 403 – 417.

[12] A.B. Malinowska and D.F.M. Torres, The Hahn quantum variational calculus, *J. Optim. Theory Appl.* **147**(3) (2010), 419 – 442.

[13] C. Niculescu and L.E. Persson, *Convex Functions and Their Applications, A Contemporary Approach*, Springer-Verlag (2006).

[14] U.M. Özkan, M.Z. Sarikaya and H. Yildirim, Existences of certain integral inequalities on time scales, *Appl. Math. Letters* **21**(10) (2008), 993 – 1000.