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Research Article

Coupled Best Proximity Point Theorem for Generalized Contractions in Partially Ordered Metric Spaces

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Abstract. In this paper, we obtain coupled best proximity point theorems for generalized contraction in partially ordered metric spaces using P-operator technique. The results presented in this paper generalize and improve some known results in the literature.

Keywords. Garaghty contraction; Partially ordered set; Coupled fixed point; Coupled best proximity points; Weak P-monotone property

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1. Introduction and Preliminaries

The fixed point theory in Banach spaces plays an important role and is useful in mathematics. In particular, a very powerful tool is the Banach fixed point theorem, which was generalized and extended in various directions (see [1–21]). In 1973, Geraghty [5] introduced the interesting class of auxiliary function called Geraghty-contraction and proved remarkable theorem, which is also generalization of Banach contraction principle.

Theorem 1.1 ([5])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an operator. Suppose that there exists $\beta : (0,\infty) \to [0,1)$ satisfying the condition*

$$\beta(t_n) \to 1 \quad \textit{implies} \quad t_n \to 0.$$

If T satisfy the following inequality

$$d(Tx,Ty) \le \beta\big(d(x,y)\big)d(x,y) \quad \textit{for any } x,y \in X.$$

Then T has a unique fixed point.

In 2001, Rhoades [15] introduced the notion of ψ-weakly contractive mappings and proved the following theorem.

Theorem 1.2 ([15])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx,Ty) \le d(x,y) - \psi\big(d(x,y)\big) \quad \textit{for any } x,y \in X,$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0,\infty)$, $\psi(0) = 0$ and $\lim\limits_{t\to\infty} \psi(t) = \infty$. Then T has a unique fixed point.

In 1984, Khan et al. [10] introduced altering distance function and proved fixed point theorem. Following is the definition of an altering distance function.

Definition 1.3 ([10])**.** An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies

(i) ψ is continuous and nondecreasing,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

In the recent past, the idea of altering function has been utilized by many authors. Dutta and Choudhury [4], generalized the results of Rhoades [15] and Khan et al. [10], and also proved the following fixed point theorem for (ψ,ϕ)-weakly contractive mapping.

Theorem 1.4 ([4])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$\psi\big(d(Tx,Ty)\big) \le \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \quad \textit{for any } x,y \in X,$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0,\infty)$, $\psi(0) = 0$ and $\lim\limits_{t\to\infty} \psi(t) = \infty$. Then T has a unique fixed point.

Recently, many authors obtained the results of generalized contraction principle in metric spaces and partially ordered metric spaces (see [7, 8, 20]).

Now we recall the following basic facts and notations. Let A and B be nonempty subsets of a metric space X,

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\},$$

$$A_0 = \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x,y) = d(A,B) \text{ for some } \in A\}.$$

Definition 1.5. An element $x \in A$ is said to be a best proximity point of the non-self mapping $T : A \to B$ if

$$d(x,Tx) = d(A,B).$$

Because of the fact that $d(x,Tx) \geq d(A,B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto (x,Tx)$ is attained at a best proximity point. Moreover, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The notion of P-property was introduced by Sankar Raj [16] as follows.

Definition 1.6 ([16])**.** Let (A,B) be a pair of nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \implies d(x_1,x_2) = d(y_1,y_2),$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

Let Γ be the set of all functions $\beta : (0,\infty) \to [0,1)$ satisfying the following property:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0.$$

Definition 1.7 ([3])**.** Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \Gamma$ such that

$$d(Tx,Ty) \leq \beta\big(d(x,y)\big)d(x,y) \quad \text{for any } x,y \in A.$$

Definition 1.8 ([6])**.** Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. Then F is said to be a mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x,y \in X$

$$x_1,x_2 \in X \quad x_1 \leq x_2 \Rightarrow F(x_1,y) \leq F(x_2,y),$$
$$y_1,y_2 \in X \quad y_1 \leq y_2 \Rightarrow F(x,y_1) \geq F(x,y_2).$$

Definition 1.9 ([13])**.** A mapping $F : A \times A \to B$ is said to be the proximal mixed monotone property if $F(x,y)$ is proximally nondecreasing in x and is proximally nonincreasing in y, that is

$$\begin{cases} x_1 \leq x_2 \\ d(u_1,F(x_1,y)) = d(A,B) \\ d(u_2,F(x_2,y)) = d(A,B) \end{cases} \Rightarrow u_1 \leq u_2,$$

and

$$\begin{cases} y_1 \leq y_2 \\ d(v_1,F(x,y_1)) = d(A,B) \\ d(v_2,F(x,y_2)) = d(A,B) \end{cases} \Rightarrow v_2 \leq v_1,$$

where $x_1,x_2,y_1,y_2,u_1,u_2,v_1,v_2 \in A$.

Definition 1.10 ([17])**.** Let A, B be subsets of a metric space X. An element $(x,y) \in A \times A$ is called a coupled best proximity point of the mapping $F : A \times A \to B$ if $d\big(x,F(x,y)\big) = d(A,B)$ and $d\big(y,F(y,x)\big) = d(A,B)$.

Definition 1.11 ([6])**.** Let X be a non-empty set. An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping F if $F(x,y) = x$ and $F(y,x) = y$.

In 2006, Gnana Bhaskar and Lakshmikantham [6] obtained the following theorems.

Theorem 1.12 ([6])**.** *Let $(X,\le)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ with*

$$d\big(F(x,y),F(u,v)\big) \le \frac{k}{2}[d(x,u)+d(y,v)] \quad \text{for all } x \ge u,\ y \le v. \tag{1.1}$$

If there exist two elements $x_0, y_0 \in X$ with $x \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$, then there exist $x,y \in X$such that $x = F(x,y)$ and $y = F(y,x)$.

Theorem 1.13 ([6])**.** *Let $(X,\le)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \le x$ for all $n \in \mathbb{N}$,*

(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \le y_n$ for all $n \in \mathbb{N}$.*

Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists $k \in [0,1)$ with

$$d\big(F(x,y),F(u,v)\big) \le \frac{k}{2}[d(x,u)+d(y,v)] \quad \text{for all } x \ge u,\ y \le v. \tag{1.2}$$

If there exist two elements $x_0, y_0 \in X$ with $x \le F(x_0,y_0)$ and $y_0 \ge F(y_0,x_0)$, then there exist $x,y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

It can be proved that the coupled fixed point is in fact unique, provided that the product space $X \times X$ endowed with the partial order mentioned above has the following property:

Every pair of elements has either a lower bound or an upper bound.

It is known [14] that this condition is equivalent to the following.

For every pair of $(x,y),(x^*,y^*) \in X \times X$, there exists a $(z_1,z_2) \in X \times X$ that is comparable to (x,y), (x^*,y^*).

Theorem 1.14 ([6])**.** *In addition to the hypothesis of Theorem* 1.12*, then the uniqueness of the coupled fixed point of F can be obtained.*

The purpose of this paper is to obtain coupled best proximity point theorems for generalized contraction of partially ordered metric spaces by P-operator technique. The results presented in this paper generalize the results of Jingling Zhang et al. [21] and also the various results in the literature.

2. Main Results

In this section, we first recall the concept of weak P-monotone property.

Definition 2.1 ([21]). Let (X,d) be a metric space and (A,B) be a pair of nonempty subsets of X and $A_0 \neq \emptyset$. A pair (A,B) has weak P-monotone property if and only if for any $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$

$$\begin{cases} d(x_1,y_1) = d(A,B) \\ d(x_2,y_2) = d(A,B) \end{cases} \implies d(x_1,x_2) \leq d(y_1,y_2),$$

furthermore, $y_1 \geq y_2$ implies $x_1 \geq x_2$.

Theorem 2.2. *Let (X,d) be a complete metric space and $(X,\leq)$ be a partially ordered set. Let (A,B) be a pair of non-empty closed subsets of X such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ be a mapping with $F(A_0 \times A_0) \subseteq B_0$ there exists $\beta \in \Gamma$ such that*

$$d\,(F(x,y),F(u,v)) \leq \beta\left(\frac{d(x,u)+d(y,v)}{2}\right)\left(\frac{d(x,u)+d(y,v)}{2}\right), \tag{2.1}$$

for all $x \geq u$, $y \leq v$. Assume that $F : A \times A \to B$ be a continuous or that $\bar{A}_0$ is such that if a nondecreasing sequence $x_n \to x \in \bar{A}_0$ and $y_n \to y \in \bar{A}_0$ then $x_n \leq x$ and $y_n \leq y$, for all n. Suppose that the pair (A,B) has the weak P-monotone property. For some $x_0, y_0 \in A_0$ there exists $\hat{x}_0, \hat{y}_0 \in B_0$ such that

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B),$$

$$\hat{x}_0 \leq F(x_0,y_0) \quad \text{and} \quad \hat{y}_0 \geq F(y_0,x_0).$$

Besides, if for each $(x,y),(x^*,y^*) \in \bar{A}_0 \times \bar{A}_0$, there exists $(z_1,z_2) \in \bar{A}_0 \times \bar{A}_0$ which is comparable to (x,y) and (x^*,y^*). Then there exists $(x^*,y^*) \in A \times A$ such that

$$d(x^*,F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(y^*,F(y^*,x^*)) = d(A,B).$$

Proof. Let us prove that B_0 is closed. Choose $\{y_n\}$ be a sequence converges to $y \in B$. From weak P-monotone property,

$$d(y_n,y_m) \to 0 \quad \text{and} \quad d(x_n,x_m) \to 0,$$

as $n,m \to \infty$ where $x_n,x_m \in A_0$ such that

$$d(x_n,y_n) = d(A,B) \quad \text{and} \quad d(x_m,y_m) = d(A,B).$$

Therefore, $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. Moreover by the continuity of a metric d, we have $d(p,q) = d(A,B)$, for some $q \in B_0$. Hence B_0 is closed.

Choose that $\bar{A}_0$ is the closure of A_0, we claim that $F(\bar{A}_0 \times \bar{A}_0) \subseteq B_0$. Let $\{x_n\},\{y_n\} \subseteq A_0$ be a sequences. If $x,y \in \bar{A}_0 \setminus A_0$ such that $x_n \to x$ and $y_n \to y$. Since F is continuous and B_0 is closed, we have

$$F(x,y) = \lim_{n\to\infty} F(x_n,y_n) \in B_0.$$

Therefore $F(\bar{A}_0 \times \bar{A}_0) \subseteq B_0$. An operator $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to A_0$ is defined by

$$P_{A_0}y = \{x \in A : d(x,y) = d(A,B)\}.$$

P_{A_0} is single valued, since the pair (A,B) has the weak P-monotone property and by the definition of F, we have

$$d\big(P_{A_0}F(x,y),P_{A_0}F(u,v)\big) \leq d\big(F(x,y),F(u,v)\big) \leq \beta\left(\frac{d(x,u)+d(y,v)}{2}\right)\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for some $x \geq u, y \leq v \in \bar{A}_0$. Obviously, $P_{A_0}F$ is increasing. If $x_n, y_n, x, y \in \bar{A}_0$ and $x_n \to x$, $y_n \to y$. Since F is continuous, then we have

$$d\big(P_{A_0}F(x_n,y_n), P_{A_0}F(x,y)\big) \to 0 \;\Rightarrow\; P_{A_0}F(x_n,y_n) \to P_{A_0}F(x,y) \text{ as } n \to \infty.$$

Then $P_{A_0}F$ is continuous. Since the pair (A,B) has weak P-monotone property and F has mixed monotone property, then we have

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(u,y), F(u,y)\big) = d(A,B) \\ F(x,y) \geq F(u,y) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \geq P_{A_0}F(u,y),$$

similarly,

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(x,v), F(x,v)\big) = d(A,B) \\ F(x,y) \geq F(x,v) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \geq P_{A_0}F(x,v),$$

for some $x \geq u, y \leq v \in A_0$. Hence $P_{A_0}F$ is mixed monotone. If $x_0, y_0 \in A_0$, then we have

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B), \tag{2.2}$$

where $\hat{x}_0, \hat{y}_0 \in B_0$. Then, we get

$$\begin{cases} d\big(P_{A_0}F(x_0,y_0), F(x_0,y_0)\big) = d(A,B) \\ d(x_0,\hat{x}_0) = d(A,B) \\ \hat{x}_0 \leq F(x_0,y_0) \in B_0 \end{cases} \implies P_{A_0}F(x_0,y_0) \geq x_0.$$

Similarly, it can be proved that $P_{A_0}F(y_0,x_0) \geq y_0$.

So that $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to \bar{A}_0$ is a contraction satisfying all the conditions in Theorem 1.14. Using Theorem 1.14, it can be obtained that $P_{A_0}F$ has a unique coupled fixed point (x^*, y^*). That is,

$$P_{A_0}F(x^*,y^*) = x^* \in A_0 \quad \text{and} \quad P_{A_0}F(y^*,x^*) = y^* \in A_0,$$

this implies that

$$d(x^*, F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(F(y^*,x^*), y^*) = d(A,B). \qquad \square$$

Remark 2.3. If in particular, consider the function $\beta(t) = k$ where $k \in [0,1)$ then the inequality (2.1) reduces to (1.1) and if remove the weak P-monotone property then Theorem 2.2 is the Theorem 1.12.

Theorem 2.4. *Let (X,d) be a complete metric space and $(X,\leq)$ be a partially ordered set. Let (A,B) be a pair of non-empty closed subsets of X such that $A_0 \neq \emptyset$. Suppose that $F : A \times A \to B$ be a mapping with $F(A_0 \times A_0) \subseteq B_0$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{d(x,u)+d(y,v)}{2} - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for all $x \geq u$, $y \leq v$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that ψ is positive in $(0,\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. Assume that $F : A \times A \to B$ be a continuous or that $\bar{A}_0$ is such that if a nondecreasing sequence $x_n \to x \in \bar{A}_0$ and $y_n \to y \in \bar{A}_0$ then $x_n \leq x$ and $y_n \leq y$, for all n. Suppose that the pair (A,B) has the weak P-monotone property. For some

$x_0, y_0 \in A_0$ there exists $\hat{x}_0, \hat{y}_0 \in B_0$ such that

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B),$$

$$\hat{x}_0 \le F(x_0,y_0) \quad \text{and} \quad \hat{y}_0 \ge F(y_0,x_0).$$

Besides, if for each $(x,y),(x^*,y^*) \in \bar{A}_0 \times \bar{A}_0$, there exists $(z_1,z_2) \in \bar{A}_0 \times \bar{A}_0$ which is comparable to (x,y) and (x^*,y^*). Then there exists $(x^*,y^*) \in A \times A$ such that

$$d(x^*,F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(y^*,F(y^*,x^*)) = d(A,B).$$

Proof. In Theorem 2.2, it can be obtained that, B_0 is closed and $F(\bar{A}_0 \times \bar{A}_0) \subseteq B_0$. An operator $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to A_0$ is defined by

$$P_{A_0} y = \{x \in A : d(x,y) = d(A,B)\}.$$

P_{A_0} is single valued, since the pair (A,B) has weak P-monotone property and by the definition of F, we have

$$d\big(P_{A_0}F(x,y), P_{A_0}F(u,v)\big) \le d\big(F(x,y),F(u,v)\big) \le \frac{d(x,u)+d(y,v)}{2} - \psi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for some $x \ge u, y \le v \in \bar{A}_0$. Obviously, $P_{A_0}F$ is continuous and nondecreasing. Since the pair (A,B) has weak P-monotone property and F has mixed monotone property, then we have

$$\begin{cases} d\big(P_{A_0}F(x,y),F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(u,y),F(u,y)\big) = d(A,B) \\ F(x,y) \ge F(u,y) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \ge P_{A_0}F(u,y),$$

similarly,

$$\begin{cases} d\big(P_{A_0}F(x,y),F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(x,v),F(x,v)\big) = d(A,B) \\ F(x,y) \ge F(x,v) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \ge P_{A_0}F(x,v),$$

for some $x \ge u, y \le v \in A_0$. Hence $P_{A_0}F$ is mixed monotone. If $x_0, y_0 \in A_0$, then we have

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B), \tag{2.3}$$

where $\hat{x}_0, \hat{y}_0 \in B_0$. Then we get

$$\begin{cases} d\big(P_{A_0}F(x_0,y_0),F(x_0,y_0)\big) = d(A,B) \\ d(x_0,\hat{x}_0) = d(A,B) \\ \hat{x}_0 \le F(x_0,y_0) \in B_0 \end{cases} \implies P_{A_0}F(x_0,y_0) \ge x_0.$$

Similarly, it can be proved that $P_{A_0}F(y_0,x_0) \ge y_0$.

So that $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to \bar{A}_0$ is a contraction satisfying all the conditions in Theorem 1.14. Using Theorem 1.14, it can be obtained $P_{A_0}F$ has a unique coupled fixed point (x^*,y^*), that is

$$P_{A_0}F(x^*,y^*) = x^* \in A_0 \quad \text{and} \quad P_{A_0}F(y^*,x^*) = y^* \in A_0,$$

this implies that

$$d(x^*,F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(F(y^*,x^*),y^*) = d(A,B). \qquad \square$$

Corollary 2.5. *Let (X,d) be a complete metric space and $(X,\le)$ be a partially ordered set. Let (A,B) be a pair of non-empty closed subsets of X such that $A_0 \ne \emptyset$. Suppose that $F : A \times A \to B$*

be a mapping with $F(A_0 \times A_0) \subseteq B_0$ *such that*

$$d\big(F(x,y),F(u,v)\big) \le \frac{k}{2}\big(d(x,u)+d(y,v)\big),$$

for all $x \ge u,\ y \le v$, *where* $k \in [0,1)$. *Assume that* $F : A \times A \to B$ *be continuous or that* $\bar{A}_0$ *is such that if a nondecreasing sequence* $x_n \to x \in \bar{A}_0$ *and* $y_n \to y \in \bar{A}_0$ *then* $x_n \le x$ *and* $y_n \le y, \forall n$. *Suppose that the pair (A,B) has the weak P-monotone property. For some* $x_0, y_0 \in A_0$ *there exists* $\hat{x}_0, \hat{y}_0 \in B_0$ *such that*

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B),$$

$$\hat{x}_0 \le F(x_0,y_0) \quad \text{and} \quad \hat{y}_0 \ge F(y_0,x_0).$$

Besides, if for each $(x,y),(x^*,y^*) \in \bar{A}_0 \times \bar{A}_0$, *there exists* $(z_1,z_2) \in \bar{A}_0 \times \bar{A}_0$ *which is comparable to* (x,y) *and* (x^*,y^*). *Then there exists* $(x^*,y^*) \in A \times A$ *such that*

$$d(x^*,F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(y^*,F(y^*,x^*)) = d(A,B).$$

Proof. It follows by taking $\psi(t) = (1-k)t$, where $0 \le k < 1$ in Theorem 2.4. □

Remark 2.6. In Corollary 2.5, removing weak P-monotone property and take $A = B$, we get Theorem 1.12.

Theorem 2.7. *Let* (X,d) *be a complete metric space and* $(X,\le)$ *be a partially ordered set. Let* (A,B) *be a pair of non-empty closed subsets of* X *such that* $A_0 \ne \emptyset$. *Suppose that* $F : A \times A \to B$ *be a mapping with* $F(A_0 \times A_0) \subseteq B_0$ *such that*

$$\psi\big(d\big(F(x,y),F(u,v)\big)\big) \le \phi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for all $x \ge u,\ y \le v$, *where* ψ *is an altering distance function and* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function with the condition* $\psi(t) > \phi(t)$ *for all* $t > 0$. *Assume that* $F : A \times A \to B$ *be a continuous or that* $\bar{A}_0$ *is such that if a nondecreasing sequence* $x_n \to x \in \bar{A}_0$ *and* $y_n \to y \in \bar{A}_0$ *then* $x_n \le x$ *and* $y_n \le y$, *for all* n. *Suppose that the pair* (A,B) *has the weak P-monotone property. For some* $x_0, y_0 \in A_0$ *there exists* $\hat{x}_0, \hat{y}_0 \in B_0$ *such that*

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B),$$

$$\hat{x}_0 \le F(x_0,y_0) \quad \text{and} \quad \hat{y}_0 \ge F(y_0,x_0).$$

Besides, if for each $(x,y),(x^*,y^*) \in \bar{A}_0 \times \bar{A}_0$, *there exists* $(z_1,z_2) \in \bar{A}_0 \times \bar{A}_0$ *which is comparable to* (x,y) *and* (x^*,y^*). *Then there exists* $(x^*,y^*) \in A \times A$ *such that*

$$d(x^*,F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(y^*,F(y^*,x^*)) = d(A,B).$$

Proof. In Theorem 2.2, it can be obtained that, B_0 is closed and $F(\bar{A}_0 \times \bar{A}_0) \subseteq B_0$. An operator $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to A_0$ is defined by

$$P_{A_0} y = \{x \in A : d(x,y) = d(A,B)\}.$$

P_{A_0} is single valued, since the pair (A,B) has weak P-monotone property and by the definition of F, we have

$$\psi\big(d\big(P_{A_0}F(x,y),P_{A_0}F(u,v)\big)\big) \le \psi\big(d\big(F(x,y),F(u,v)\big)\big) \le \phi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for some $x \geq u, y \leq v \in \bar{A}_0$. Since

$$\phi\big(d(x,u)+d(y,v)\big) \to 0 \Leftrightarrow d(x,u)+d(y,v) \to 0,$$
$$\psi\big(d\big(P_{A_0}F(x,y), P_{A_0}F(u,v)\big)\big) \to 0 \Leftrightarrow d\big(P_{A_0}F(x,y), P_{A_0}F(u,v)\big) \to 0.$$

Then $P_{A_0}F$ is continuous and nondecreasing. Since the pair (A,B) has weak P-monotone property and F has mixed monotone property, then we have

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(u,y), F(u,y)\big) = d(A,B) \\ F(x,y) \geq F(u,y) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \geq P_{A_0}F(u,y),$$

similarly,

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(x,v), F(x,v)\big) = d(A,B) \\ F(x,y) \geq F(x,v) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \geq P_{A_0}F(x,v),$$

for some $x \geq u, y \leq v \in A_0$. Hence $P_{A_0}F$ is mixed monotone. If $x_0, y_0 \in A_0$, then we have

$$d(x_0,\hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0,\hat{y}_0) = d(A,B), \tag{2.4}$$

where $\hat{x}_0, \hat{y}_0 \in B_0$. Then, we obtain

$$\begin{cases} d\big(P_{A_0}F(x_0,y_0), F(x_0,y_0)\big) = d(A,B) \\ d(x_0,\hat{x}_0) = d(A,B) \\ \hat{x}_0 \leq F(x_0,y_0) \in B_0 \end{cases} \implies P_{A_0}F(x_0,y_0) \geq x_0.$$

Similarly, we can show that $P_{A_0}F(y_0,x_0) \geq y_0$.

So that $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to \bar{A}_0$ is a contraction satisfying all the conditions in Theorem 1.14. Using Theorem 1.14, it can be obtained $P_{A_0}F$ has a unique coupled fixed point (x^*, y^*), that is

$$P_{A_0}F(x^*,y^*) = x^* \in A_0 \quad \text{and} \quad P_{A_0}F(y^*,x^*) = y^* \in A_0,$$

this implies that

$$d(x^*, F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(F(y^*,x^*), y^*) = d(A,B). \qquad \square$$

Remark 2.8. Considering ψ to be the identity mapping and $\phi(t) = kt$ in Theorem 2.7, then it can be obtained that Corollary 2.5.

Theorem 2.9. *Let (X,d) be a complete metric space and $(X,\leq)$ be a partially ordered set. Let (A,B) be a pair of non-empty closed subsets of X such that $A_0 \neq \emptyset$. Suppose that $F : A \times A \to B$ be a mapping with $F(A_0 \times A_0) \subseteq B_0$ such that*

$$\psi\big(d\big(F(x,y), F(u,v)\big)\big) \leq \frac{1}{2}\psi\big(d(x,u)+d(y,v)\big) - \phi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for all $x \geq u$, $y \leq v$, where ψ and ϕ are altering distance functions. Assume that $F : A \times A \to B$ be a continuous or that $\bar{A}_0$ is such that if a nondecreasing sequence $x_n \to x \in \bar{A}_0$ and $y_n \to y \in \bar{A}_0$ then $x_n \leq x$ and $y_n \leq y$, for all n. Suppose that the pair (A,B) has the weak P-monotone property. For some $x_0, y_0 \in A_0$ there exists $\hat{x}_0, \hat{y}_0 \in B_0$ such that

$$d(x_0,\hat{x}_0) = d(A,B) \quad \textit{and} \quad d(y_0,\hat{y}_0) = d(A,B),$$

$$\hat{x}_0 \le F(x_0, y_0) \quad and \quad \hat{y}_0 \ge F(y_0, x_0).$$

Besides, if for each $(x,y),(x^*,y^*) \in \bar{A}_0 \times \bar{A}_0$, *there exists* $(z_1,z_2) \in \bar{A}_0 \times \bar{A}_0$ *which is comparable to* (x,y) *and* (x^*,y^*). *Then there exists* $(x^*,y^*) \in A \times A$ *such that*

$$d(x^*, F(x^*,y^*)) = d(A,B) \quad and \quad d(y^*, F(y^*,x^*)) = d(A,B).$$

Proof. In Theorem 2.2, it can be obtained that, B_0 is closed and $F(\bar{A}_0 \times \bar{A}_0) \subseteq B_0$. An operator $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to A_0$ is defined by

$$P_{A_0} y = \{x \in A : d(x,y) = d(A,B)\}.$$

P_{A_0} is single valued, since the pair (A,B) has weak P-monotone property and by the definition of F, we have

$$\psi\big(d\big(P_{A_0}F(x,y), P_{A_0}F(u,v)\big)\big) \le \psi\big(d\big(F(x,y),F(u,v)\big)\big) \le \frac{1}{2}\psi\big(d(x,u)+d(y,v)\big) - \phi\left(\frac{d(x,u)+d(y,v)}{2}\right),$$

for some $x \ge u, y \le v \in \bar{A}_0$. Since

$$d(x,u)+d(y,v) \to 0 \Rightarrow \psi\big(d(x,u)+d(y,v)\big) - \phi\big(d(x,u)+d(y,v)\big) \to 0,$$

$$\psi\big(d\big(P_{A_0}F(x,y), P_{A_0}F(x,y)\big)\big) \to 0 \Leftrightarrow d\big(P_{A_0}F(x,y), P_{A_0}F(x,y)\big) \to 0.$$

Then $P_{A_0}F$ is continuous and nondecreasing. Since the pair (A,B) has weak P-monotone property and F has mixed monotone property, then we have

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(u,y), F(u,y)\big) = d(A,B) \\ F(x,y) \ge F(u,y) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \ge P_{A_0}F(u,y),$$

similarly,

$$\begin{cases} d\big(P_{A_0}F(x,y), F(x,y)\big) = d(A,B) \\ d\big(P_{A_0}F(x,v), F(x,v)\big) = d(A,B) \\ F(x,y) \ge F(x,v) \subseteq B_0 \end{cases} \implies P_{A_0}F(x,y) \ge P_{A_0}F(x,v),$$

for some $x \ge u, y \le v \in A_0$. Hence $P_{A_0}F$ is mixed monotone. If $x_0, y_0 \in A_0$, then we have

$$d(x_0, \hat{x}_0) = d(A,B) \quad \text{and} \quad d(y_0, \hat{y}_0) = d(A,B), \tag{2.5}$$

where $\hat{x}_0, \hat{y}_0 \in B_0$. Then we get

$$\begin{cases} d\big(P_{A_0}F(x_0,y_0), F(x_0,y_0)\big) = d(A,B) \\ d(x_0, \hat{x}_0) = d(A,B) \\ \hat{x}_0 \le F(x_0,y_0) \in B_0 \end{cases} \implies P_{A_0}F(x_0,y_0) \ge x_0.$$

Similarly, it can be proved that $P_{A_0}F(y_0,x_0) \ge y_0$.

So that $P_{A_0} : F(\bar{A}_0 \times \bar{A}_0) \to \bar{A}_0$ is a contraction satisfying all the conditions in Theorem 1.14. Using Theorem 1.14, we can obtain $P_{A_0}F$ has a unique coupled fixed point (x^*, y^*), that is

$$P_{A_0}F(x^*,y^*) = x^* \in A_0 \quad \text{and} \quad P_{A_0}F(y^*,x^*) = y^* \in A_0,$$

this implies that

$$d(x^*, F(x^*,y^*)) = d(A,B) \quad \text{and} \quad d(F(y^*,x^*), y^*) = d(A,B). \qquad \square$$

Remark 2.10. If ψ to be the identity mappings and $\phi(t) = (1-k)t$, where $0 \le k < 1$ in Theorem 2.9, then it can be obtained the Corollary 2.5.

3. Conclusion

Main purpose of this paper was to established existence and uniqueness of coupled best proximity point theorems in the setting of partially ordered metric spaces such that the non-self mapping satisfies contractive and weakly contractive condition using weak P-monotone property.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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