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Research Article

Solving Differential Equations by New Optimized MRA and Invariant Solutions

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Abstract. The powerful tools for analyzing problems and equations are offered by the wavelet theory in the numerous scientific fields. In this paper, new father wavelets with two independent variables according to the differential invariants are designed and the novel method based on those are proposed, new father wavelets are produced, the multiresolution analysis (MRA) by these wavelets for solving DEs applied on some examples. The approximate solutions in the form of linear combination of father wavelets and corresponding mother wavelets are provided by this method.

Keywords. Father wavelet; Mother wavelet; Multiresolution analysis (MRA); Invariant solution; Approximation subspace; Wavelet subspace

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1. Introduction

So far, the numerous methods for solving the differential equations are proposed. Here, we merge geometric methods to wavelet methods and propose the new method based on the wavelets with two or more independent variables according to the differential invariants and invariant solutions. Accordingly, we first remind the Lie symmetry method that will be applied for obtaining the differential invariants and invariant solutions.

Sophus Lie, the Norwegian mathematician who founded the theory of continuous groups and their applications to the theory of differential equations. His investigations led to one of the major branches of 20th-century mathematics, the theory of Lie groups and Lie algebras. Lie's principal tool and one of his greatest achievements was the discovery that continuous transformation groups (now called, after him, Lie groups) could be better understood by "linearizing" them, and studying the corresponding generating vector fields (the so-called infinitesimal generators). The generators are subject to a linearized version of the group law, now called the commutator bracket, and have the structure of what is today called a Lie algebra. A German mathematician Hermann Weyl used Lie's work on the group theory in his papers from 1922 and 1923, and Lie groups today play a role in quantum mechanics. However, the subject of Lie groups as it is studied today is vastly different from what the research by Sophus Lie was about and among the 19th-century masters, Lie's work is in detail certainly the least known today [9]. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and Several applications of Lie groups in the theory of differential equations (for many other applications of Lie symmetries see [10], [11]).

From 1909 that Alphered Haar (Hungarian Mathematician) introduce the first wavelet. The wavelets are considered as important function in the functional and harmonic analysis [7] and they have found numerous applications in some fields of the science and technology: seismology, image processing, signal processing, coding theory, biosciences, financial mathematics, fractals and so on [1]. The main problem in constructing new wavelets is that the known wavelets such as Haar, Daubechies, Coiflet, Symlet, CDF, Mexican hat, and Gaussian are not easily extendable to two or more variables (in some cases, they miss some wavelet properties). So far some wavelet methods and algorithms such as MRA, Mallat, Galerkin are introduced [12]. In this paper, we construct new father wavelets with two variables, these wavelets depend on the differential invariants of differential equations. Therefore, we can use MRA with these father wavelets for analyzing differential equations and build the corresponding mother wavelets based on them and invariant solutions of DEs. In fact, we incorporate the results of equivalence methods (like the Lie symmetry methods) into the MRA (as one of the wavelet methods) and this new method is called optimized MRA method with the invariant solutions (OMRA). In practice, the final solution will be obtained in the form of a linear combination of the father wavelets and correspondent mother wavelets. We will show the performance of OMRA method with the example.

The remainder of the paper is organized as follows. In Section 2, we recall some needed results to construct differential invariants, invariant solutions, father wavelets and MRA. In Section 3, OMRA method is proposed. In Section 4, the proposed method will be demonstrated by example. Finally, the conclusions and future works will be presented.

2. Preliminaries

In this section, we remember some needed results, definitions, and theorems of the Lie symmetry methods and wavelet theory. The interested readers invited to see [5], [10] and [9] for the Lie point symmetries and their application to solve differential equations and see [7], [8] and [2] for the wavelets theory.

2.1 The Lie Symmetry Method

In this section, the general procedure for determining symmetries for any system of partial differential equations are recalled (see [5], [10] and [9]). To begin, let us consider the general case of a nonlinear system of partial differential equations of order nth in p independent and q dependent variables is given as a system of equations:

$$\Delta_\nu(x,u^{(n)})=0, \quad \nu=1,\cdots,l, \tag{2.1}$$

involving $x=(x^1,\cdots,x^p)$, $u=(u^1,\cdots,u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group G of infinitesimal transformations acting on the independent and dependent variables of the system (2.1):

$$(\tilde{x}^i,\tilde{u}^j)=(x^i,u^i)+s(\xi^i,\eta^j)+O(s^2), \quad i=1,\cdots,p,\ j=1,\cdots,q,$$

where s is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. A symmetry group of the system (2.1) is a one-parameter Lie group of infinitesimal transformations G acting on an open subset M of the space of independent and dependent variables for the system (2.1) with the property that whenever $u=f(x)$ is a solution of system (2.1) and whenever $g\cdot f$ is defined for $g\in G$, then $u=g\cdot f(x)$ is also a solution of the system (2.1). Indeed, a symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. The infinitesimal generator $\mathbf{v}$ associated with G can be written as $\mathbf{v}=\sum_{i=1}^{p}\xi^i\partial_{x^i}+\sum_{j=1}^{q}\eta^j\partial_{u^j}$. The invariance of the system (2.1) under the infinitesimal transformations leads to the invariance conditions ([10, Theorem 2.31]):

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta_\nu(x,u^{(n)})\big]=0, \quad \Delta_\nu(x,u^{(n)})=0,\ \nu=1,\cdots,l,$$

where $\mathrm{Pr}^{(n)}$ is called the nth-order prolongation of the infinitesimal generator given by $\mathrm{Pr}^{(n)}\mathbf{v}=\mathbf{v}+\sum_{\alpha=1}^{q}\sum_{J}\phi^\alpha_J(x,u^{(n)})\partial_{u^\alpha_J}$, where $J=(j_1,\cdots,j_k)$, $1\le j_k\le p$, $1\le k\le n$ and the sum is over all J's of order $0<\#J\le n$. If $\#J=k$, the coefficient ϕ^α_J of $\partial_{u^\alpha_J}$ will only depend on k-th and lower order derivatives of u and $\phi^J_\alpha(x,u^{(n)})=D_J\left(\phi_\alpha-\sum_{i=1}^{p}\xi^iu^\alpha_i\right)+\sum_{i=1}^{p}\xi^iu^\alpha_{J,i}$, where $u^\alpha_i:=\partial u^\alpha/\partial x^i$ and $u^\alpha_{J,i}:=\partial u^\alpha_J/\partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket. The first advantage of symmetry group methods is to construct new solutions from known solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the PDE to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

We establish the characteristics system for every infinitesimal generator vector field as follows

$$\frac{\mathrm{dx}}{\xi}=\frac{\mathrm{dt}}{\tau}=\frac{\mathrm{du}}{\phi}$$

and obtain the differential invariants corresponding to these vector field. The PDE is expressed in the coordinates (x,t,u), so for reducing this equation, we should search for its form in the specific coordinates. Those coordinates will be constructed by searching for independent invariants (y,v) corresponding to the infinitesimal generator. So using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. For more information and examples (see [10]).

2.2 Wavelets

Wavelets are important functions in the mathematics and other scientific fields. Unless otherwise stated, we assumed that the wavelets belong to $L^2(\mathbb{R}^2)$ (the space of squared integrable functions with integral norm). Here, the mother wavelets, father wavelets and wavelet family are introduced.

Definition 2.3. The wavelet ψ is a function that satisfies the following admissible condition

$$C_\psi = \int_{\mathbb{R}^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|^2} > 0\,,$$

where $F(\psi)(\omega)$ is the Fourier transformation of wavelet ψ and defines as:

$$F(\psi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \exp(-ix.\omega)\psi(x)d\omega\,.$$

C_ψ is called the wavelet coefficient of ψ. Here, we assume that $\omega = (\omega_1,\omega_2)$ and $x = (x_1,x_2)$ belong to $\mathbb{R}^2$. For further information and examples (see [2]).

Definition 2.4. The wavelet ψ is called mother wavelet, if it assured in the following properties:

$$\int_{\mathbb{R}^2} \psi(x)\ \mathrm{dx} = 0,$$

$$\int_{\mathbb{R}^2} |\psi(x)|^2\ \mathrm{dx} < \infty,$$

$$\lim_{|\omega|\to\infty} F(\psi(\omega)) = 0\,.$$

The important point to note here is the first condition equivalent to the admissible condition (for more details see [7]).

We emphasize that mother wavelets have the admissible condition, n-zero moments and exponential decay properties. The mother wavelet have two parameters: the translation parameter $b = (b_1,b_2)$ and scaling parameter $a > 0$. The mother wavelet corresponding to (a,b) is

$$\psi_{a,b}(x) = \psi\left(\frac{x-b}{a}\right) = \psi\left(\frac{x_1-b_1}{a}, \frac{x_2-b_2}{a}\right).$$

Indeed, the mother wavelet with the parameters (a,b) is the correspondent family wavelet.

Definition 2.5. The wavelet ϕ is called father wavelet, if it satisfies the following properties:

$$\int_{\mathbb{R}^2} \phi(x)\ \mathrm{dx} = 1, \tag{2.2}$$

$$\int_{\mathbb{R}^2} |\phi(x)|^2\ \mathrm{dx} = 1, \tag{2.3}$$

$$\langle \phi(x), \phi(x-n)\rangle = \delta(n) \tag{2.4}$$

where $\langle\ ,\ \rangle$ is scalar product derived from the integral norm of $L^2(\mathbb{R}^2)$.

For more details see [7].

In fact, the father wavelets are considered as scaling functions in the multiresolution analysis, for decomposing $L^2(\mathbb{R}^2)$.

Multiresolution Analysis

Definition 2.6. A multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$ is defined as sequence of non-empty closed subspace $V_j \subset L^2(\mathbb{R}^2)$, $(j \in \mathbb{Z})$ such that

$$\{0\} \subset \ldots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots \subset L^2(\mathbb{R}^2)$$

by the following properties:

(1) $\bigcup\limits_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^2)$ and $\bigcap\limits_{j\in\mathbb{Z}} V_j = \{0\}$,

(2) $f(x) \in V_j$ iff, $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(3) $f(x) \in V_j$ iff, $f(x_1 - 2^s j, x_2 - 2^s k) \in V_j$, for all $(j,k) \in \mathbb{Z}^2$,

(4) There exist a function $\phi(x) \in V_0$ with nonvanishing integral, such that the set $\{\phi_{0,k}(x) = \phi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis of V_0,

where $x = (x_1, x_2) \in \mathbb{R}^2$.

By a given father wavelet $\phi(x)$, the related wavelet family is defined as

$$\phi^s_{j,k}(x) = 2^{s/2}\phi(2^s x_1 - j, 2^s x_2 - k) \tag{2.5}$$

thus $\phi^0_{j,k}(x) = \phi_{j,k}(x) = \phi(x_1 - j, x_2 - k)$, where the function $\phi_{j,k}(x)$ is said to be the scaling function associated with MRA.

Theorem 2.7. *If $M = \{V_j\}$ is a MRA for $L^2(\mathbb{R}^2)$ with $g = \phi(x-k)$ as the scaling function, then for every $j,k \in \mathbb{Z}$, we can build the wavelets ϕ and ψ such that*

$$\{\{\phi_{j,k}\}, \{\psi_{j,k}\}\}_{j,k\in\mathbb{Z}}$$

is an orthonormal basis of $L^2(\mathbb{R}^2)$. The wavelets ϕ and ψ are the father and mother wavelets (respectively).

Proof. For proof and more details, we refer the reader to [8] and [6]. □

Suppose that father wavelet ϕ is given, by obtaining corresponding MRA and constructing corresponding mother wavelet ψ, We can define the following subspaces:

Definition 2.8. Let given a MRA with the scaling function $g = \phi(x-k)$, the father and mother wavelets, ϕ and ψ (respectively). The approximation subspaces V_j and wavelet subspaces W_j (respectively) are defined as follows:

$$V_s := \operatorname{span}\{\phi^s_{j,k} \mid \phi^s_{j,k}(x) = 2^{s/2}\phi(2^s x_1 - j, 2^s x_2 - k)\}, \tag{2.6}$$

$$W_s := \operatorname{span}\{\psi^s_{j,k} \mid \psi^s_{j,k}(x) = 2^{s/2}\psi(2^s x_1 - j, 2^s x_2 - k)\}. \tag{2.7}$$

Remark 2.9. We can say W_j is an orthogonal complement of V_j in V_{j+1}, i.e.

$$V_{s+1} = V_s \oplus W_s$$

by following this process, we found that $\oplus_s W_s = L^2(\mathbb{R}^2)$. In other hand, since $\phi(x) \in V_0 \subset V_1$, there exists a sequence $\{a_{j,k}, j,k \in \mathbb{Z}\}$ such that

$$\phi(x) = \sqrt{2} \sum_{j,k} a_{j,k} \phi(2x_1 - j, 2x_2 - k).$$

These equations are called dilation equations, two-scale differential equations, or refinement equations. The mother wavelet ψ satisfies the similar equations as below

$$\psi(x) = \sqrt{2} \sum_{j,k} w_{j,k} \phi(2x_1 - j, 2x_2 - k), \tag{2.8}$$

where the coefficients $w_{j,k}$ are given by

$$w_{j,k} = (-1)^{j+k} \bar{a}_{1-j,1-k}. \tag{2.9}$$

These equations are called the wavelet equations.

Definition 2.10. The coefficients $a_{j,k}$ and $w_{j,k}$ are called the approximate and wavelet coefficients (respectively). In fact, the approximate coefficient $a_{j,k}$ are calculated as follows

$$a_{j,k} = \langle \phi, \phi_{1;j,k} \rangle = \sqrt{2} \iint \phi(x,t)\phi(2x - j, 2t - k)\, \mathrm{dx\ dt} \tag{2.10}$$

where the scaling parameter s is 1.

Theorem 2.11. *Suppose $M = \{V_j\}$ is a MRA for $L^2(\mathbb{R}^2)$ with $g = \phi(x-k)$ as the scaling function and W_j's are the corresponding wavelet subspaces. Then $L^2(\mathbb{R}^2) = \oplus_j W_j$ and every $f \in L^2(\mathbb{R}^2)$ can be uniquely expressed as a sum $\sum w_{j,k} \psi_{j,k}$. In other words, under the above assumptions, the set of all mother wavelets $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R}^2)$.*

Proof. For proof and more details see [3] and [6]. □

3. Optimized MRA Method

The optimized MRA method (OMRA) has 4 following steps:

(1) Apply equivalence algorithms (for example, the Lie symmetry method) on DE, and obtain the differential invariants.

(2) Propose the suitable father wavelet based on the differential invariants.

(3) Apply MRA with father wavelet as the scaling function, then obtain the approximation and wavelet subspaces, the related coefficients and correspondent mother wavelet based on the father wavelet and invariant solutions.

(4) The final solution is in the form of the linear combination of father wavelet (by structure based on differential invariants), the mother wavelet (with structure based on father wavelet and invariant solutions) and wavelet coefficients.

In the following, some OMRA formula are proposed. First, by given suitable father wavelet based on the differential invariants (DI), we obtain approximation coefficients by writing dilation equations as follows

$$\phi(x,t)=\sum_j\sum_k a_{j,k}\phi_{1;j,k}(x,t)=\sqrt{2}\sum_j\sum_k a_{j,k}\phi(2x-j,2t-k), \tag{3.1}$$

where $a_{j,k}$s are the approximation coefficients. Then, we obtain the wavelet coefficients by (2.9).

Now, if father wavelet ϕ completely dependent on DE and its DI, then the corresponding mother wavelet are obtained from (2.8). So, the approximate solution as follows:

$$u(x,t)=\sum_{j,k} a_{j,k}\phi_{j,k}(x,t)+\sum_{j,k} w_{j,k}\psi_{j,k}(x,t), \tag{3.2}$$

After calculating the approximation and wavelet coefficients, we consider the solution as (3.1). Then put it in DE, by solving the resulting ODE for $\psi_{j,k}(x,t)$, the mother wavelet ψ will be obtained as follows:

$$\psi(x,t)=\sum\sum w_{j,k}\psi_{1;j,k}(x,t)=\sqrt{2}\sum_j\sum_k w_{j,k}\psi(2x-j,2t-k).$$

On the other hand, since $\psi_{j,k}$ can be bulit an orthonormal basis for the solution space M (after determining ψ), we can consider the solution of DE as follows:

$$u(x,t)=\sum_{j,k} w_{j,k}\psi_{j,k}(x,t). \tag{3.3}$$

Therefore by having corresponding mother wavelet, we will have approximate solution based on the mother wavelet. We emphasize that in OMRA similar to usual MRA, the corresponding mother wavelets aren't unique.

Remark 3.1. In OMRA, the approximation and wavelet subspaces are defined like MRA with the scaling parameter $s=1$. Therefore, the solution space $M^{(n)}$ is decomposed as

$$M^{(n)}=\{\oplus_{j,k}V_{j,k}\}\oplus_{j,k}\{\oplus_{j,k}W_{j,k}\}.$$

For more details see [6].

4. Example

In this section, we demonstrate OMRA by example. In practice, we apply OMRA on the heat equation and obtain solutions. Finally, the OMRA results will be proposed.

First, the Lie symmetry method results for the heat equation $u_t=u_{xx}$ proposed in Table 1 (for detailed calculations about implementation of the Lie symmetry method on the heat equation, see [10]):

Table 1. The Lie symmetry method results for the heat equation

Symmetry group	Differential invariants	dim($\mathfrak{g}$)	Invariant solutions
Translation	$x-ct$, u	2	$k\exp(-c(x-ct))+l$
Scaling	$(x/\sqrt{t})$, (u/t^a)	3	$k\cdot erf(x/\sqrt{2t})+l$
Galilean Boost	t, $u\exp(x^2/4t)$	2	$\frac{k}{\sqrt{t}}\exp(-\frac{x^2}{4t})$

erf is the error function

In Table 1, the symmetry groups are the translation with factor (c), scaling with factor (a) and Galilean boost (respectively). We offer two following father wavelets

$$\phi_1(x,t)=\frac{4}{3\pi}\exp\left(-\frac{x^2+t^2}{0.75}\right),\quad \phi_2(x,t)=\exp\left(-\frac{x^2+15t^2}{5}\right)\cos(x)\cos(t/3).$$

By a little calculation, it can be seen that offered functions have properties (2.2)-(2.4) of the father wavelets. These wavelets are related to the Galilean boost differential invariants. Figures 1 and 2 show the graphs of father wavelets. The some properties are clear from these figures.

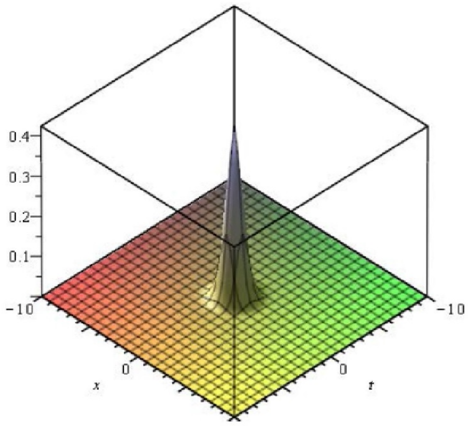

Figure 1. The graph of ϕ_1

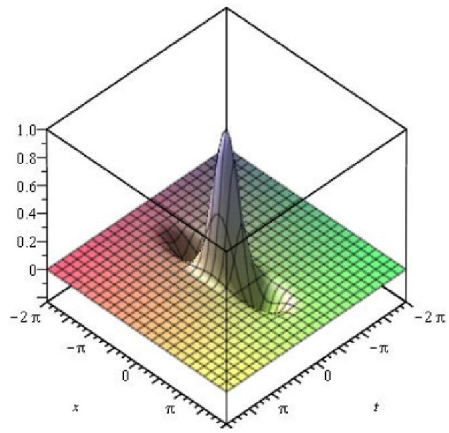

Figure 2. The graph of ϕ_2

Now, we will apply OMRA by these father wavelets on the heat equation. First, by considering the father wavelet $\phi_1(x,t)$, we have

$$\phi_{1;j,k}(x,t)=\sqrt{2}\,\phi(2x-j,2t-k)=\frac{4\sqrt{2}}{3\pi}\exp\left(-\frac{(2x-j)^2+(2t-k)^2}{0.75}\right)$$

thus

$$\phi(x,t)=\sqrt{2}\sum_j\sum_k a_{j,k}\phi(2x-j,2t-k)$$

from (2.10) (for the approximation coefficients), we get

$$a_{j,k}=\sqrt{2}\iint\frac{16\sqrt{2}}{9\pi^2}\exp\left(-\frac{5x^2+5t^2-4xj-4tk+j^2+k^2}{0.75}\right)\,\mathrm{dx\,dt}$$

so, the approximation coefficients as follows

$$a_{j,k}=\frac{32}{9\pi^2}\exp\left(-\frac{j^2+k^2}{0.75}\right)\iint\exp\left(-\frac{5x^2+5t^2-4(xj+tk)}{0.75}\right)\,\mathrm{dx\,dt},$$

where $j,k=0,1,2$ (for nth-order DE, we assume that $j,k=0,1,2,\cdots,n$). After calculating nine approximation coefficients $a_{j,k}$, the approximation matrix $A=[a_{j,k}]_{j,k}$ as follows

$$\begin{bmatrix}4.134 & 3.17 & 1.42\\ 3.17 & 2.42 & 1.09\\ 1.42 & 1.09 & 0.49\end{bmatrix}$$

therefore (based on the wavelet coefficients from (2.9)) the matrix $W = [w_{j,k}]_{j,k}$ as follows

$$\begin{bmatrix} 2.42 & -3.17 & 2.42 \\ -3.17 & 4.134 & -3.17 \\ 2.42 & -3.17 & 2.42 \end{bmatrix}.$$

Now, we consider solution as (3.2) and put it in the heat equation. The resulting PDE for $\psi_{j,k}(x,t)$ is

$$\psi^t - \psi^{xx} = \left(\frac{86x^2 - 8t - 8}{0.75}\right)\exp\left(-\frac{4x^2 + 4t^2}{0.75}\right), \tag{4.1}$$

by solving this PDE, the mother wavelet ψ will be obtained as follows:

$$\psi(x,t) = \psi_h + \psi_p\,,$$

where ψ_h, ψ_p are the homogeneous and particular solutions of (4.1), respectively (for more details and information about the analytical methods for solving differential equations, we refer the reader to [4]). Here, we can consider ψ_h as an invariant solution of the heat equation. For example, the invariant solutions of translation and Galilean boost (respectively) are

$$k\exp(-c(x-ct)) + l, \quad \frac{k}{\sqrt{t}}\exp\left(-\frac{x^2}{4t}\right).$$

where $k, l = cte$ (for more details and calculations see [10]). On the other hand, we have the following special solution

$$\psi_p = -1.025\exp(-5.33(x^2 + t^2)).$$

Since the mother wavelet ψ made by the father wavelet ϕ, the following combination of father wavelet ϕ is always a particular solution for the PDE in terms of ψ

$$\psi_p := -\frac{a_{0,0}}{w_{0,0}}\phi_{0,0}\,,$$

where $\phi_{0,0} = \sqrt{2}\phi(2x, 2t)$.

Second, by using of the father wavelet $\phi_2(x,t)$, we have

$$\phi_{1;j,k}(x,t) = \sqrt{2}\ \phi(2x-j, 2t-k) = \sqrt{2}\ \exp\left(-\frac{(2x-j)^2 + 15(2t-k)^2}{5}\right)\cos(2x-j)\cos((2t-k)/3)$$

thus $\phi(x,t) = \sqrt{2}\sum_j\sum_k a_{j,k}\phi(2x-j, 2t-k)$,

where

$$a_{j,k} = \sqrt{2}\iint\left\{\exp\left(-\frac{5x^2 + 285t^2 - 4xj - 60tk + j^2 + 15k^2}{5}\right)\right.$$
$$\left.\cos(x)\cos(t/3)\cos(2x-j)\cos\left(\frac{2t-k}{3}\right)\right\}\ \mathrm{dx\ dt}$$

and $j, k = 0, 1, 2$. Now, we should calculate nine approximation coefficients $a_{j,k}$. After calculation, the matrix $A = [a_{j,k}]_{j,k}$ are obtained as follows

$$\begin{bmatrix} 0.25618 & 0.024 & 0.00002 \\ 0.0195 & 0.0189 & 0.00015 \\ -0.9346 & -0.00002 & 0.000006 \end{bmatrix}.$$

Therefore from (2.9), the matrix $W = [w_{j,k}]_{j,k}$ as follows

$$\begin{bmatrix} 0.0189 & -0.0195 & 0.00131 \\ -0.024 & 0.25618 & -0.024 \\ 0.0189 & 0.0195 & 0.0189 \end{bmatrix}.$$

Similar to the first case, by considering the solution as (3.2) and putting it in the heat equation, the resulting PDE for $\psi_{j,k}(x,t)$ is appeared as follows:

$$\psi^t - \psi^{xx} = (19.35)\left\{\frac{64x^2}{25} + \frac{16x}{5}\sin(2x)\cos(2t/3) + \left(\frac{24t-25}{5}\right)\cos(2x)\cos(2t/3) - 1.6\right\}\exp\left(-\frac{4x^2+60t^2}{5}\right), \tag{4.2}$$

by solving this PDE, the mother wavelet ψ will obtained as follows:

$$\psi(x,t) = \psi_h + \psi_p\,,$$

where ψ_h, ψ_p are the homogeneous and particular solutions of (4.2), respectively (see [4]). Here, again we can consider ψ_h as an invariant solution of the heat equation. While for the particular solution, we have

$$\psi_s = -19.16\exp\left(-\frac{4x^2+60t^2}{5}\right)\cos(2x)\cdot\cos(2t/3).$$

Finally, according to the above discussions and calculations, the final solutions for the heat equation with OMRA based on invaraint solutions (under symmetry groups such as the translation and Galilean boost) as follows:

$$u_1 = \sum_{i,j,k} c_{i,j}\{k\exp(A_1) - 1.025\exp(A_2) + l\},$$

$$u_2 = \sum_{i,j,k} c_{i,j}\left\{\frac{k}{\sqrt{t}}\exp(A_3) - 1.025\exp(A_2)\right\},$$

$$u_3 = \sum_{i,j,k} c_{i,j}\{k\exp(A_1) - 19.16\exp(A_4)\cos(4x-2i)\cos(4t-2j)/3) + l\},$$

$$u_4 = \sum_{i,j,k} c_{i,j}\left\{\frac{k}{\sqrt{t}}\exp(A_3) - 19.16\exp(A_4)\cos(4x-2i)\cos(4t-2j)/3)\right\},$$

where

$$A_1 = 8(t-j) + 2(i-2x),$$

$$A_2 = -5.33((2x-i)^2 + (2t-j)^2),$$

$$A_3 = -\frac{(2x-i)^2}{4(2t-j)},$$

$$A_4 = -\frac{4(2x-i)^2 + 60(2t-j)^2}{5},$$

and $c_{i,j}$'s are the coefficients that can obtained by putting u_l (for $l = 1,\dots,4$) in the heat equation and initial or boundary conditions, also note that all summations should be calculated for $(i,j,k = 0,1,2)$. (It is worth noting that, we consider $c = 2$ for the corresponding solutions to the translation.)

As a result, according to the above discussions and calculations, by using the father wavelets (built based on the differential invariants), the correspondent mother wavelets (constructed by invariant solutions) and implementation of modified MRA (i.e. OMRA), we obtained the new approximate solutions (in the form of linear combination of father and mother wavelets) for the heat equation, that include solutions derived by other methods (like invariant solutions of the Lie symmetry method).

5. Conclusions and Future Works

In this paper, we proposed the new method based on the wavelets for analyzing PDEs. We constructed new father and mother wavelets with two independent variables according to the differential invariants, then proposed corresponding MRA, applied it on the differential equations and here after the PDE was solved and the solutions were obtained. Indeed, OMRA is improved MRA by differential invariants (in the structure of father wavelets) and invariant solutions (in the structure of mother wavelets). In future works, we will propose suitable father wavelets for every differential invariant and symmetry group by implementing WTM on other PDEs. Moreover, by implementing OMRA on other PDEs, we hope to generalize WTM for solving both linear and non-linear PDEs at every order and every number of independent variables with every initial condition, and we will propose the proper father wavelets for every symmetry group.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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