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Research Article

On the Solution of a System of Integral Equations via Matrix Version of Banach Contraction Principle

Muhammad Usman Ali[1], Tayyab Kamran[2], Hassan Houmani[3] and Mihai Postolache[4]

[1] *Department of Mathematics, COMSATS Institute of Information Technology, Attock, Pakistan*
[2] *Department of Mathematics, Quaid-i-Azam University, Islambad, Pakistan*
[3] *Department of Mathematics and Informatics, University Politehnica of Bucharest, Bucharest, 060042, Romania*
[4] *China Medical University, No. 91, Hsueh-Shih Road, Taichung, Taiwan & University Politehnica of Bucharest, 313 Splaiul Independentei, Bucharest, Romania*
***Corresponding author:** mihai@mathem.pub.ro

Abstract. The purpose of this paper is to extend Perov's fixed point theorem in the setting of modular generalized metric space, which is also established in this article. Further we discuss the Perov's result in the setting of two modular generalized metric spaces. As an application we prove the existence of solution for a system of integral equations.

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1. Introduction

It is not false to say that the metric fixed point theory has been started by the introduction of Banach contraction principle. This principle has many applications in nonlinear analysis. This theory frequently use to prove the existence of solution of differential and integral equations. Due to the applicability of Banach contraction principle in mathematical analysis, many researchers attracted towards it.

We have seen that in metric fixed point theory the Banach contraction principle has been extended mainly in two ways: By using weaker forms of contraction conditions; by using the structure which is more general than metric space.

Perov's fixed point theorem [10] is one of the earlier generalization of Banach contraction principle. In [10] Perov introduced the vector/matrix form of Banach contraction principle. For this purpose he first introduced the vector-valued/generalized metric space and then established the contraction condition regarding this space.

Before studying Perov's fixed point theorem, we need to discuss the following definition, notions and results:

Let X be a nonempty set and $\mathbb{R}^m$ is the set of all $m \times 1$ matrices with real entries. If $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)^T$, $\beta = (\beta_1, \beta_2, \dots, \beta_m)^T$ and $c \in \mathbb{R}$, then by $\alpha \leq \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \leq \beta_i$ (resp., $\alpha_i < \beta_i$) for each $i \in \{1, 2, \dots, m\}$ and by $\alpha \leq c$ we mean that $\alpha_i \leq c$ for each $i \in \{1, 2, \dots, m\}$. A mapping $d : X \times X \to \mathbb{R}^m$ is called a vector-valued/generalized metric on X if the following properties are satisfied:

(d_1) $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$, vice-versa;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A set X equipped with a vector-valued/generalized metric d is called a vector-valued/ generalized metric space and it is denoted by (X, d).

Throughout this paper we denote the set of all $m \times m$ matrices with non-negative elements by $M_{m,m}(\mathbb{R}_+)$, the zero $m \times m$ matrix by $\bar{0}$ and the identity $m \times m$ matrix by I, and note that $A^0 = I$.

A matrix A is said to be convergent to zero if and only if $A^n \to \bar{0}$ as $n \to \infty$ (see [13]). Following are some matrices which converges towards zero:

(a) Any matrix $A := \begin{pmatrix} b & b \\ a & a \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$.

(b) Any matrix $A := \begin{pmatrix} b & a \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$.

(c) Any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

Theorem 1.1 ([7]). *Let $A \in M_{m,m}(\mathbb{R}_+)$. The following statements are equivalent.*

(i) *A is convergent towards zero.*

(ii) *$A^n \to \bar{0}$ as $n \to \infty$.*

(iii) *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(A - \lambda I) = 0$.*

(iv) *The matrix $I - A$ is nonsingular and*

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots . \tag{1.1}$$

For a brief study of such matrices, we refer the reader to the following sources: Rus [11], Turinici [12], Bucur *et al*. [3] and O'Regan *et al*. [9].

Now, we are in position to recall the following Perov's fixed point theorem:

Theorem 1.2 ([10]). *Let (X,d) be a complete generalized metric space and $f: X \to X$ the mapping with the property that there exists a matrix $A \in M_{m,m}(\mathbb{R}_+)$ such that*

$$d(f(x),f(y)) \le Ad(x,y) \quad \text{for all } x,y \in X.$$

If A is a matrix convergent towards zero, then

(1) $\mathrm{Fix}(f) = \{x^*\}$;

(2) *the sequence of successive approximations $\{x_n\}$ such that, $x_n = f^n(x_0)$ is convergent and it has the limit x^*, for all $x_0 \in X$.*

Chistyakov [6] generalized the notion of metric space by introducing the notion of modular metric space. Under this notion we see that distance between two points depends upon another parameter. Chistyakov [6] defined modular metric space in the following way:

Definition 1.1 ([6]). A function $\omega: (0,\infty) \times X \times X \to [0,\infty]$ is known as a modular metric on X if the following axioms hold:

(i) $\omega(\lambda,x,y) = 0 \ \forall \ \lambda > 0$ if and only if $x = y$;

(ii) for each $x,y \in X$, $\omega(\lambda,x,y) = \omega(\lambda,y,x)$ for all $\lambda > 0$;

(iii) for each $x,y,z \in X$, $\omega(\lambda+\mu,x,z) \le \omega(\lambda,x,y) + \omega(\mu,y,z)$ for all $\lambda,\mu > 0$.

A modular metric on X is said to be regular if (i) is replaced with the following axiom:

$$x = y \text{ if and only if } \omega(\lambda,x,y) = 0 \text{ for some } \lambda > 0.$$

In the literature, we have seen that fixed points of operators have been investigated by Abdou and Khamsi [1], Alfuraidan [2], Chaipunya *et al.* [4], Chistyakov [5], and Khamsi and Kozlowski [8] in modular metric spaces.

2. Main Result

In this section, we first introduce the notion of modular generalized metric space and then extend Perov's fixed point theorem in this new setting.

Let $\mathbb{R}_m$ is the set of all $m \times 1$ real matrices such that $\{(\alpha_1,\alpha_2,\cdots,\alpha_m)^T : \alpha_i \in \mathbb{R}\} \cup \{(\infty,\infty,\cdots,\infty)^T\}$. If $\alpha,\beta \in \mathbb{R}_m$, $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_m)^T$, $\beta = (\beta_1,\beta_2,\dots,\beta_m)^T$ and $c \in \mathbb{R} = (\infty,\infty)$, then by $\alpha \le \beta$ (resp., $\alpha < \beta$) we mean $\alpha_i \le \beta_i$ (resp., $\alpha_i < \beta_i$) for each $i \in \{1,2,\dots,m\}$ and by $\alpha \ge c$ we mean that $\alpha_i \ge c$ for each $i \in \{1,2,\dots,m\}$.

Let X be a nonempty set. A mapping $\omega: (0,\infty) \times X \times X \to \mathbb{R}_m$ is modular vector-valued/generalized metric on X, if it satisfies the following conditions, for all $x,y,z \in X$

(i) $\omega(\lambda,x,y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $\omega(\lambda,x,y) = \omega(\lambda,y,x)$ for all $\lambda > 0$;

(iii) $\omega(\lambda+\mu,x,y) \le \omega(\lambda,x,z)+\omega(\mu,z,y)$ for all $\lambda,\mu>0$.

If instead of (i) we have the following condition:

(i′): $\omega(\lambda,x,x)=0\ \forall\ \lambda>0$ and $x\in X$

then ω is pseudomodular generalized metric. A modular generalized metric ω on X is regular if the following weaker version of (i) is satisfied:

$$x=y \text{ if and only if } \omega(\lambda,x,y)=0 \text{ for some } \lambda>0.$$

Definition 2.1. Let ω be a pseudomodular generalized metric on X. For fix $x_0\in X$, the set

$$X_\omega=\{x\in X : \omega(\lambda,x,x_0)\to 0 \text{ as } \lambda\to\infty\}$$

is a modular space.

Following we discuss some important concepts regarding modular generalized metric space.

Definition 2.2. Let (X,ω) be a modular generalized metric space.

(i) The sequence $\{x_n\}$ in X_ω is ω-convergent to $x\in X_\omega$ if and only if $\omega(1,x_n,x)\to 0$, as $n\to\infty$.

(ii) The sequence $\{x_n\}$ in X_ω is ω-Cauchy if $\omega(1,x_m,x_n)\to 0$, as $m,n\to\infty$.

(iii) A subset D of X_ω is ω-complete if any ω-Cauchy sequence in D is a ω-convergent in D.

(iv) A subset D of X_ω is ω-closed if ω-limit of each ω-convergent sequence of D always belongs to D.

(v) A subset D of X_ω is ω-bounded if we have

$$\delta_\omega(D)=\sup\{\omega(1,x,y): x,y\in D\}<\infty.$$

(vi) A subset D of X_ω is ω-compact if for any $\{x_n\}$ in D there exists a subsequence $\{x_{n_k}\}$ and $x\in D$ such that $\omega(1,x_{n_k},x)\to 0$ as $k\to\infty$.

Definition 2.3. Let (X,ω) be a modular generalized metric space and $\{x_n\}$ be a sequence in X_ω. Then:

- ω satisfies the Δ_M-condition if $\lim\limits_{m,n\to\infty}\omega(m-n,x_n,x_m)=0$ for $m,n\in\mathbb{N}$ with $m>n$ implies $\lim\limits_{m,n\to\infty}\omega(\lambda,x_n,x_m)=0$ for all $\lambda>0$.
- ω satisfies Fatou property if for any $\{x_n\}$ ω-convergent to x and $\{y_n\}$ ω-convergent to y, we have $\omega(1,x,y)\le \liminf\limits_{n\to\infty}\omega(1,x_n,y_n)$.

Definition 2.4. A modular generalized metric ω on X is strongly regular if the following conditions hold:

(a) condition (i) of modular generalized metric ω is replaced with

(i′): $x=y$ if and only if $\omega(1,x,y)=0$.

(b) $\lim\limits_{n\to\infty}\omega(1,x_n,x)=0$ and $\lim\limits_{n\to\infty}\omega(1,x_n,y)=0$ implies $\omega(1,x,y)=0$.

Throughout this section, $N(W)$ denotes the set of all nonempty subsets of W. Now, we state and prove the first theorem of this section.

Theorem 2.1. *Let X be a nonempty set endowed with ω as a strongly regular modular generalized metric satisfying Δ_M-condition and Fatou property. Let W is ω-complete and ω-bounded subset of X_ω. Let $F: W \to N(W)$ be a multivalued mapping with $A,B \in M_{m,m}(\mathbb{R}_+)$ such that for each $x,y \in W$ and $u \in Fx$, there exists $v \in Fy$ satisfying*

$$\omega(1,u,v) \le A\omega(1,x,y) + B\omega(1,y,u). \tag{2.1}$$

Then F has a fixed point, provided that the matrix A converges towards zero.

Proof. Consider $x_0 \in W$ and $x_1 \in Fx_0$. From (2.1), for $x_0,x_1 \in W$ with $x_1 \in Fx_0$, we have $x_2 \in Fx_1$ such that

$$\begin{aligned}\omega(1,x_1,x_2) &\le A\omega(1,x_0,x_1) + B\omega(1,x_1,x_1)\\ &= A\omega(1,x_0,x_1).\end{aligned} \tag{2.2}$$

Again from (2.1), for $x_1,x_2 \in W$ with $x_2 \in Fx_1$, we have $x_3 \in Fx_2$ such that

$$\begin{aligned}\omega(1,x_2,x_3) &\le A\omega(1,x_1,x_2) + B\omega(1,x_2,x_2)\\ &\le A^2\omega(1,x_0,x_1) \qquad \text{(by using (2.2))}.\end{aligned}$$

Continuing in the same way, we get a sequence $\{x_n\}$ in W such that $x_n \in Fx_{n-1}$ and

$$\omega(1,x_n,x_{n+1}) \le A^n\omega(1,x_0,x_1), \text{ for all } n \in \mathbb{N}.$$

By using the triangular inequality, for each $n,m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\omega(m-n,x_n,x_m) &\le \omega(1,x_n,x_{n+1}) + \omega(1,x_{n+1},x_{n+2}) + \cdots + \omega(1,x_{m-1},x_m)\\ &\le \sum_{i=n}^{m-1} A^i\omega(1,x_0,x_1)\\ &\le A^n\left(\sum_{i=0}^{\infty} A^i\right)\omega(1,x_0,x_1)\\ &= A^n(I-A)^{-1}\omega(1,x_0,x_1).\end{aligned}$$

Since the matrix A converges towards 0. Thus the sequence $\{x_n\}$ is ω-Cauchy sequence in W. As W is ω-complete. Then there exists $x^* \in W$ such that $\{x_n\}$ is ω-convergent to x^*, that is, $\lim\limits_{n\to\infty} \omega(1,x_n,x^*) = 0$. From (2.1), for $x_n,x^* \in W$ and $x_{n+1} \in Fx_n$ we have $w^* \in Fx^*$ such that

$$\omega(1,x_{n+1},w^*) \le A\omega(1,x_n,x^*) + B\omega(1,x^*,x_{n+1}).$$

Letting $n \to \infty$ in the above inequality and by using Fatou property, we get $\omega(1,x^*,w^*) = 0$, that is, $x^* = w^*$. Thus $x^* \in Fx^*$. □

Remark 2.1. If $\omega(1,x,y)$ is finite for each $x,y \in W$, then we can leave the boundedness of W from the above theorem.

Example 2.1. Let $X = [0,10] \times [0,10]$ be endowed with $w(\lambda,(x_1,x_2),(y_1,y_2)) = \frac{1}{\lambda}\begin{pmatrix}|x_1-y_1|\\|x_2-y_2|\end{pmatrix}$. Define the mapping

$$F: X \to N(X), \quad T(x_1,x_2) = \left\{\left(\frac{x_1}{2},\frac{x_2}{2}\right),\left(\frac{x_1+1}{2},\frac{x_2+1}{2}\right)\right\}$$

It is easy to see that F satisfies (2.1) with $A = \begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}$ and all the other conditions of Theorem 2.1 hold. Thus we conclude that F has a fixed point.

In the following theorem we discuss the Perov's fixed point theorem for two modular generalized metric spaces.

Theorem 2.2. *Let X be a nonempty set endowed with ω as a modular generalized metric space and with ρ as a strongly regular modular generalized metric satisfying Δ_M-condition. Let W is ω-complete and ρ-bounded subset of X_ω. Let $F: W \to N(W)$ be a multivalued mapping with $A,B \in M_{m,m}(\mathbb{R}_+)$, such that for each $x,y \in W$ and $u \in Fx$ there exists $v \in Fy$ satisfying*

$$\rho(1,u,v) \le A\rho(1,x,y) + B\rho(1,y,u). \tag{2.3}$$

Further, assume that the following conditions hold:

(i) *the matrix A converges towards zero;*

(ii) *for each $x,y \in W$, there exists $C \in M_{m,m}(\mathbb{R}_+)$ such that $\omega(1,u,v) \le C\cdot\rho(1,x,y)$, for each $u \in Fx$ and $v \in Fy$;*

(iii) *$Graph(F) = \{(x,y): x \in W, y \in Fx\}$ is ω-closed.*

Then F has a fixed point.

Proof. Consider $x_0 \in W$ and $x_1 \in Fx_0$. From (2.3), for $x_0,x_1 \in W$ with $x_1 \in Fx_0$, we have $x_2 \in Fx_1$ such that

$$\begin{aligned}\rho(1,x_1,x_2) &\le A\rho(1,x_0,x_1) + B\rho(1,x_1,x_1)\\ &= A\rho(1,x_0,x_1).\end{aligned}$$

Again from (2.3), for $x_1,x_2 \in W$ with $x_2 \in Fx_1$, we have $x_3 \in Fx_2$ such that

$$\begin{aligned}\rho(1,x_2,x_3) &\le A\rho(1,x_1,x_2) + B\rho(1,x_2,x_2)\\ &\le A^2\rho(1,x_0,x_1).\end{aligned}$$

Continuing in the same way, we get a sequence $\{x_n\}$ in X such that $x_n \in Fx_{n-1}$ and

$$\rho(1,x_n,x_{n+1}) \le A^n\rho(1,x_0,x_1) \ \text{ for each } \ n \in \mathbb{N}.$$

Now, we will show that $\{x_n\}$ is ρ-Cauchy sequence in (W,ρ). Let $n,m \in \mathbb{N}$, then by using the triangular inequality we get

$$\begin{aligned}\rho(m-n,x_n,x_m) &\le \sum_{i=n}^{m-1}\rho(1,x_i,x_{i+1})\\ &\le \sum_{i=n}^{m-1}A^i\rho(1,x_0,x_1)\end{aligned}$$

$$\leq A^n\left(\sum_{i=0}^{\infty}A^i\right)\rho(1,x_0,x_1)$$
$$= A^n(I-A)^{-1}\rho(1,x_0,x_1).$$

Since the matrix A converges towards zero. Thus $\{x_n\}$ is ρ-Cauchy sequence in (X,ρ). By hypothesis, we have $C\in M_{m\times m}$ such that

$$\omega(1,x_{n+1},x_{m+1})\leq C\rho(1,x_n,x_m).$$

Since $\lim_{n,m\to\infty}\rho(m-n,x_n,x_m)=0$ and ρ satisfies Δ_M-condition. Hence, $\{x_n\}$ is also a ω-Cauchy sequence in W. As W is ω-complete. Then there exists $x^*\in W$ such that $\{x_n\}$ is ω-convergent to x^*, that is, $\lim_{n\to\infty}\omega(1,x_n,x^*)=0$. As $x_{n+1}\in Fx_n$ and $\{x_n\}$ is ω-convergent to x^*, then by using hypothesis (iii), we have $x^*\in Fx^*$. □

3. Application

In this section, we prove the existence theorem for the following system of integral equations:

$$x(t)=f(t)+\int_a^b g_1(t,s,x(s),y(s))ds,$$
$$y(t)=f(t)+\int_a^b g_2(t,s,x(s),y(s))ds, \tag{3.1}$$

for each $t,s\in I=[a,b]$, where $g_i: I\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function for $i=1,2$. We denote by $(C[a,b],\mathbb{R})$ the space of all continuous real valued function defined on $[a,b]$.

Theorem 3.1. *Let $X=(C[a,b],\mathbb{R})$. Consider the operator $T_i: X\times X\to X$ given by the formula*

$$T_i(x(t),y(t))=f(t)+\int_a^b g_i(t,s,x(s),y(s))ds,$$

where $g_i: I\times I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function for $i=1,2$. Also assume that for each $t,s\in[a,b]$ and $x,y,u,v\in X$, we have

$$|g_i(t,s,x(s),y(s))-g_i(t,s,u(s),v(s))|\leq a_{i1}|x(s)-u(s)|+a_{i2}|y(s)-v(s)| \quad \text{for } i=1,2,$$

where $A=(b-a)\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$ converges to zero. Then the system of integral equations (3.1) *has at least one solution.*

Proof. By the hypothesis of this theorem we observe that for each $t,s\in[a,b]$ and $x,y,u,v\in X$, we have the following:

$$\begin{aligned}|T_i(x(t),y(t))-T_i(u(t),v(t))| &\leq \int_a^b |g_i(t,s,x(s),y(s))-g_i(t,s,u(s),v(s))|ds\\ &\leq \int_a^b [a_{i1}|x(s)-u(s)|+a_{i2}|y(s)-v(s)|]ds\\ &\leq (b-a)[a_{i1}\max_{s\in I}|x(s)-u(s)|+a_{i2}\max_{s\in I}|y(s)-v(s)|,\end{aligned}$$

for $i=1,2$.

Define the operator

$$T : W = X \times X \to W = X \times X, \quad T(\overline{x}) = T(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2))$$

for each $\overline{x} = (x_1, x_2) \in X \times X$, and consider the modular generalized metric

$$\omega : (0,\infty) \times W \times W \to \mathbb{R}_m, \qquad \omega(\lambda, \overline{x}(t), \overline{y}(t)) = \frac{1}{\lceil \lambda \rceil} \begin{pmatrix} \max_{t \in I} |x_1(t) - y_1(t)| \\ \max_{t \in I} |x_2(t) - y_2(t)| \end{pmatrix}.$$

It is easy to see that W is ω-complete and ω satisfies both Δ_M-condition and Fatou property. Thus we conclude that

$$\omega(1, T\overline{x}, T\overline{y}) \leq A\omega(1, \overline{x}, \overline{y})$$

for each $\overline{x}, \overline{y} \in W$. Therefore, by Theorem 2.1, we have at least one $\overline{v} \in W$ such that $T\overline{v} = T(v_1, v_2) = (T_1(v_1, v_2), T_2(v_1, v_2)) = (v_1, v_2)$. This implies that $v_1 = T_1(v_1, v_2)$ and $v_2 = T_2(v_1, v_2)$, that is, the system of integral equations (3.1) has at least one solution. □

4. Conclusion

In regular modular metric space, $x = y$ if and only if $\omega(\lambda, x, y) = 0$ for some $\lambda > 0$. It means that if $x = y$, then $\omega(\lambda, x, y) = 0$ for some $\lambda > 0$. Similarly, if $\omega(\lambda, x, y) = 0$ for some $\lambda > 0$, then $x = y$. Thus, if $\omega(1, x, y) = 0$, then we may have $x \neq y$. To overcome this ambiguity from some existing results of the literature, the notion of strongly regular modular metric space has been introduced in this paper. Since there are many generalizations of Banach contraction principle, one may use our technique to further generalize some interesting results.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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