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Research Article

On Sperner Γ-(semi)hypergroups

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Abstract. In this paper first we use the notion of Sperner family and we introduce some classes of Γ-(semi)hypergroups that we call them weak Sperner Γ-(semi)hypergroups and Sperner Γ-(semi)hypergroups. Then we introduce the class of complete Γ-(semi)hypergroups as a generalization of the class of complete semihypergroups and we show that every complete Γ-(semi)hypergroups is a Sperner Γ-(semi)hypergroups. Finally the class of complementable Γ-(semi)hypergroups are investigated.

Keywords. Sperner Γ-(semi)hypergroup; Complete Γ-(semi)hypergroup; Complentable Γ-(semi)hypergroups

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1. Introduction

The algebraic hyperstructure notion was introduced in 1934 by a French mathematician F. Marty [12] at the 8th Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure the composition of two elements is an element while in an algebraic hyperstructure the composition of two elements is a set. Around the 40's, the general aspects of the theory, connections with groups and various applications in geometry were studied. The theory knew an important progress starting with the 70's, when its research area enlarged. A recent book on hyperstructures [6] points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Many authors around the word studied different aspects of semihypergroups. In 1986, Sen and Saha [17] defined the notion of a Γ-semigroup as a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [2, 3], Chinram and Jirojkul [4], Chinram and Siammai [5], Hila [8, 9], Jafarpour and et al. [11], Saha [13], Sen and et al. [14–17, 19] and Seth [18]. In [1] and [7] Anvariyeh et al. introduced the notion of Γ-semihypergroup as a generalization of a semihypergroups, they extended classical notions of semigroups and semihypergroups to Γ-semihypergroup. Let $M = \{\alpha, \beta, \gamma, \ldots\}$ and $S = \{x, y, z, \ldots\}$ be two non-empty sets. Then (S,Γ) is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on S and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in S$, we have $x\alpha(y\beta z) = (x\alpha y)\beta z$. A Γ-semihypergroup (S,Γ) is called a Γ-hypergroup if for every $\gamma \in \Gamma$ and $x \in S$, we have $S\gamma x = x\gamma S = S$. Moreover in a Γ-(semi)hypergroup if every $\gamma \in \Gamma$ is an operation on S then it is a Γ-(semi)group.

In this research a generalization of the notion of Sperner semihypergroup introduced in [10] investigated and we introduce and study some classes of Γ-(semi)hypergroups. Using the notion of Sperner family of sets, first we introduce the classes weak Sperner Γ-(semi)hypergroups, row Sperner Γ-(semi)hypergroups, column Sperner Γ-(semi)hypergroups and Sperner Γ-(semi)hypergroups, respectively. Then we introduce the class of complete Γ-(semi)hypergroups as a generalization of the class of complete semihypergroups. We prove that the class of complete Γ-(semi)hypergroups is a subset of the class of Sperner Γ-(semi)hypergroups. Finally a complementable Γ-(semi)hypergroup is introduced and we show that a non-trivial Γ-group is a complementable Γ-hypergroup. A connection between complementable Γ-hypergroups and Sperner Γ-hypergroups are investigated.

2. Preliminaries

We recall here some basic notions of hypergroup theory and we refer the readers to the following fundamental books Corsini [12].

Let H be a non-empty set and $\mathcal{P}^*(H)$ denote the set of all non-empty subsets of H. Let $\circ$ be a *hyperoperation* (or *join operation*) on H that is a function from the chartezian product $H \times H$ into $\mathcal{P}^*(H)$. The image of the pair $(a,b) \in H \times H$ under the hyperoperation $\circ$ in $\mathcal{P}^*(H)$ is denoted by $a \circ b$. The join operation can be extended in a natural way to subsets of H as follows: for non-empty subsets A, B of H, define $A \circ B = \cup\{a \circ b \mid a \in A, b \in B\}$. The notation $a \circ A$ is used for $\{a\} \circ A$ and $A \circ a$ for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its element a. The hyperstructure $(H, \circ)$ is called a *semihypergroup* if $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in H$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

A semihypergroup $(H,\circ)$ is called a *hypergroup* if the reproduction law holds: $a\circ H = H\circ a = H$, for all $a\in H$.

A semihypergroup $(H,\circ)$ is called *complete* if for all natural numbers $n,m\geq 2$ and all tuples $(x_1,x_2,\dots,x_n)\in H^n$ and $(y_1,y_2,\dots,y_m)\in H^m$, we have the following implication:

$$\prod_{i=1}^{n} x_i \cap \prod_{j=1}^{m} y_j \neq \emptyset \quad \Rightarrow \quad \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j,$$

where $\prod\limits_{i=1}^{n} x_i = x_1\circ x_2\circ\dots\circ x_n$. In practice the next characterization is more useful.

Theorem 2.1 ([**?**]). *A (semi)hypergroup $(H,*)$ is complete if it can be written as the union $H=\bigcup\limits_{s\in G} A_s$ of its subsets, where G and A_s satisfy the conditions:*

(1) *$(G,\cdot)$ is a (semi)group;*

(2) *for all $(s,t)\in G^2$, where $s\neq t$, we have $A_s\bigcap A_t=\emptyset$;*

(3) *if $(a,b)\in A_s\times A_t$, then $a*b=A_{s\cdot t}$.*

We call G is the (semi)group related to H and is shown with G_*. Also we call the disjoint family $\{A_i\}_{i\in G}$ is related partition of H.

3. Sperner Γ-semihypergroups

Let $\mathcal{F}$ be a family of subsets of H. We call it a *Sperner family* if the following implication is valid:

$$\forall (X,Y)\in\mathcal{F}^2,\ \ [X\subseteq Y \ \text{ or } \ Y\subseteq X]\Rightarrow X=Y.$$

In this section we use the notion of Sperner family and we introduce and study some classes of Γ-(semi)hypergroups that we call ....

Definition 3.1. Let (S,Γ) be a Γ-semihypergroup then S is called weak Sperner Γ-semihypergroup if the following implication valid:

$$[x\alpha y\subseteq x\beta y\Rightarrow x\alpha y = x\beta y],$$

for every $(x,y)\in S^2$ and $(\alpha,\beta)\in\Gamma^2$.

Definition 3.2. Let (S,Γ) be a Γ-semihypergroup then a weak Sperner Γ-semihypergroup S is called:

(i) row Sperner Γ-semihypergroup if the following implication valid:
$[x\alpha y\subseteq x\alpha v\Rightarrow x\alpha y = x\alpha v]$, for every $(x,y,v)\in S^3$ and $\alpha\in\Gamma$;

(ii) column Sperner Γ-semihypergroup if the following implication valid:
$[x\alpha y\subseteq u\alpha y\Rightarrow x\alpha y = u\alpha y]$, for every $(x,y,u)\in S^3$ and $\alpha\in\Gamma$;

(iii) Sperner Γ-semihypergroup if it is row Sperner and column Sperner.

Example 3.3. Let $S = \{e,a,b,c\}$ and $\Gamma = \{\alpha,\beta\}$, where α and β are defined as follow:

α	e	a	b	c
e	e	a,b	a,b	c
a	a,b	c	c	e
b	a,b	c	c	e
c	c	e	e	a,b

β	e	a	b	c
e	c	e	e	a,b
a	e	a,b	a,b	c
b	e	a,b	a,b	c
c	a,b	c	c	e

In this case (S,Γ) is a Sperner Γ-semihypergroup.

Example 3.4. Let S be the set of integer numbers modulo n, where $n \in \mathbb{N}$ i.e. $S = \{\overline{0},\overline{1},\overline{2},\dots,\overline{n-1}\}$ and $\Gamma = \{\alpha_m | m \in \mathbb{N}\}$, where $\overline{x}\alpha_m\overline{y} = \overline{x+y+m}$, for every $(\overline{x},\overline{y}) \in S^2$ and $m \in \mathbb{N}$. Then (S,Γ) is a Γ-group and so it is a Sperner Γ-semihypergroup.

Example 3.5. Let (G,α) be a non-trivial group and (G,β) be a total hypergroup. Then (G,Γ), where $\Gamma = \{\alpha,\beta\}$, is a weak Sperner Γ-semihypergroup which is not Sperner Γ-semihypergroup.

Let α be a hyperoperation on semihypergroup S. For every $k \in \mathbf{N}$, $(x_1,x_2,\cdots,x_k) \in S^k$, we denote $\prod_{i=1}^{k_\alpha} x_i = x_1\alpha x_2\alpha\cdots\alpha x_k$.

Definition 3.6. Let (S,Γ) be a Γ-semihypergroup and $\alpha \in \Gamma$. Then (S,Γ) is called

(i) α-complete if the following implication valid:

$$\prod_{i=1}^{n} {}_\alpha a_i \cap \prod_{i=1}^{m} {}_\alpha b_i \neq \emptyset \quad \Rightarrow \quad \prod_{i=1}^{n} {}_\alpha a_i = \prod_{i=1}^{m} {}_\alpha b_i,$$

where $n,m > 1$, $(a_1,a_2,\dots,a_n) \in S^n$ and $(b_1,b_2,\dots,b_n) \in S^m$;

(ii) Γ-complete if it is γ-complete, for every $\gamma \in \Gamma$;

(iii) complete if the following implication valid:

$$\prod_{i=1}^{n} {}_\alpha a_i \cap \prod_{i=1}^{m} {}_\beta b_i \neq \emptyset \quad \Rightarrow \quad \prod_{i=1}^{n} {}_\alpha a_i = \prod_{i=1}^{m} {}_\beta b_i,$$

for every $(\alpha,\beta) \in \Gamma^2$ and $n,m > 1$, $(a_1,a_2,\cdots,a_n) \in S^n$, $(b_1,b_2,\cdots,b_n) \in S^m$.

Remark 3.7. It is obvious if the Γ-semihypergroup introduced in the Example **??** is a Γ-complete Γ-semihypergroup which is not a complete Γ-semihypergroup.

Example 3.8. The Γ-semihypergroup introduced in the Example **??** is a complete Γ-semihypergroup.

Proposition 3.9. *Every complete Γ-semihypergroup (S,Γ) is a Sperner Γ-semihypergroup.*

Proof. The proof is straightforward. □

The following example shows that the converse of the previous proposition does not hold.

Example 3.10. Let $S = \{e, a, b\}$ and $\Gamma = \{\alpha\}$, where

α	e	a	b
e	$\{e,a\}$	$\{a,b\}$	$\{e,b\}$
a	$\{e,b\}$	$\{e,a\}$	$\{a,b\}$
b	$\{a,b\}$	$\{e,b\}$	$\{e,a\}$

In this case (S,Γ) is a Sperner Γ-semihypergroup which is not complete. Notice that $a^2 \cap ab \neq \emptyset$ but $a^2 \neq ab$.

Suppose that (S,Γ) is a Γ-semihypergroup and $\alpha \in \Gamma$, we mean by S_α is the semihypergroup (S,α), then we have the following:

Proposition 3.11. *Let (S,Γ) be a Γ-complete hypergroup and G_α and G_β are the related groups of S_α and S_β, respectively. Moreover suppose that $\{A_i^\alpha\}_{i\in G_\alpha}$ and $\{A_i^\beta\}_{i\in G_\beta}$ are the related partitions of S_α and S_β, respectively. If $\widehat{e} \in A_{e_{G_\alpha}} \cap A_{e_{G_\beta}}$($e_{G_\alpha}$ and e_{G_β} are the identity elements of G_α and G_β, respectively), for every $(\alpha,\beta) \in \Gamma^2$, then $|\Gamma| = 1$. (i.e. (S,Γ) is a complete hypergroup).*

Proof. Let $(\alpha,\beta) \in \Gamma^2$ and $(x,y) \in S^2$. Then we have

$$\begin{aligned} x\beta y &= \widehat{e}\alpha(x\beta y) \\ &= (\widehat{e}\alpha x)\beta y \\ &= (x\alpha\widehat{e})\beta y \\ &= x\alpha(\widehat{e}\beta y) \\ &= x\alpha y. \end{aligned}$$

Hence $\alpha = \beta$. Therefore $|\Gamma| = 1$. $\square$

Proposition 3.12. *Let $(G,\{\circ_i\}_{i\in I})$ be a $\{\circ_i\}_{i\in I}$-group and $\{A_i\}_{i\in G}$ be a disjoint family of sets. Then (S_G,Γ) is a complete Γ-hypergroup, where $S_G = \bigcup_{i\in G} A_i$ and $\Gamma = \{\alpha_i\}_{i\in I}$ in which $x\alpha_i y = A_{s\circ_i t}$, for every $(x,y) \in A_s \times A_t$.*

Proof. Let $(x,y,z) \in A_s \times A_t \times A_k$ and $\alpha_i, \alpha_j \in \Gamma$. Then we have $(x\alpha_i y)\alpha_j z = A_{s\circ_i t}\alpha_j z = A_{(s\circ_i t)\circ_j k}$. On the other hand $x\alpha_i(y\alpha_j z) = x\alpha_i A_{t\circ_j k} = A_{s\circ_i(t\circ_j k)}$. Because $(G,\{\circ_i\}_{i\in I})$ is a Γ-group and the family is disjoint we have $(s\circ_i t)\circ_j k = s\circ_i(t\circ_j k)$. Therefore $(x\alpha_i y)\alpha_j z = x\alpha_i(y\alpha_j z)$ for every $i,j \in I$. Moreover, $S_G\alpha_i x = x\alpha_i S_G = S_G$, for every $\alpha_i \in \Gamma$, and $x \in S_G$. $\square$

From now on we call (S_G,Γ) is the derived complete hypergroup from the $\{\circ_i\}_{i\in I}$-group, $(G,\{\circ_i\}_{i\in I})$.

A Γ-hypergroup (S,Γ) is called commutative if for every $x,y \in S$ and for every $\gamma \in \Gamma$ we have $x\gamma y = y\gamma x$.

Example 3.13. Consider the Γ-group $(G,\{\circ_1,\circ_2\})$ where $\circ_1$ and $\circ_2$ are defined as follow:

$\circ_1$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\circ_2$	e	a	b
e	b	e	a
a	e	a	b
b	a	b	e

Now let $A_e=\{x\}, A_a=\{y,z\}, A_b=\{t\}$ and $S_G=\{x,y,z,t\}$. Then derived complete hypergroup from $(G,\{\circ_1,\circ_2\})$ is as follows:

α_1	x	y	z	t
x	x	y,z	y,z	t
y	y,z	t	t	x
z	y,z	t	t	x
t	t	x	x	y,z

α_2	x	y	z	t
x	t	x	x	y,z
x	x	y,z	y,z	t
y	x	y,z	y,z	t
t	y,z	t	t	x

Proposition 3.14. *Let $(G,\{\circ_i\}_{i\in I})$ be a $\{\circ_i\}_{i\in I}$-group. Then the derived complete hypergroup from (S_G,Γ) is commutative if and only if $(G,\{\circ_i\}_{i\in I})$ is commuttive.*

Proof. The proof is striaghtforward. □

Let α be a hyperoperation on S such that $x\alpha y\neq S$, for all $(x,y)\in S^2$. Then hyperoperation α^c is called complement of hyperoperation α, where $x\alpha^c y=S-x\alpha y$, for all $(x,y)\in S^2$.

Definition 3.15. Let (S,Γ) be a Γ-semihypergroup such that $x\alpha y\neq S$, for all $(x,y)\in S^2$ and $\alpha\in\Gamma$. If (S,Γ^c) is a Γ^c-semihypergroup, where $\Gamma^c=\{\alpha^c|\alpha\in\Gamma\}$, then (S,Γ) is called a complementable Γ-semihypergroup.

Example 3.16. The Γ-semihypergroup in Example 3.13 is a complementable Γ-semihypergroup that we can show it as following:

α_1^c	x	y	z	t
x	y,z,t	x,t	x,t	x,y,z
y	x,t	x,y,z	x,y,z	y,z,t
z	x,t	x,y,z	x,y,z	y,z,t
t	x,y,z	y,z,t	y,z,t	x,t

α_2^c	x	y	z	t
x	x,y,z	y,z,t	y,z,t	x,t
x	y,z,t	x,t	x,t	x,y,z
y	y,z,t	x,t	x,t	x,y,z
t	x,t	x,y,z	x,y,z	y,z,t

Lemma 3.17. *Every non-trivial Γ-group (i.e. it is not a trivial group) is a complementable Γ-group.*

Proof. Let (G,Γ) be a Γ-group. If $|G|=2$, then $|\Gamma|=1$ or $|\Gamma|=2$. If $|\Gamma|=1$ then (G,Γ) is a group and hence it is complementable. It is easy to see that if $|\Gamma|=2$ then $\Gamma^c=\Gamma$. Hence (G,Γ) is a complementable Γ-group. Now suppose that $|G|\geq 3$. In this case we have

$$(x\alpha^c y)\beta^c z=\bigcup_{u\in x\alpha^c y} u\beta^c z\supseteq u_1\beta^c z\cup u_2\beta^c z=G,$$

where $u_1\neq u_2$ and $u_1,u_2\in x\alpha^c y$, for every $(\alpha,\beta)\in\Gamma^2$ and $(x,y,z)\in G^3$. On the other hand

$$x\alpha^c(y\beta^c z)=\bigcup_{v\in y\beta^c z} x\alpha^c v\supseteq x\alpha^c v_1\cup x\alpha^c v_2=G$$

where $v_1 \neq v_2$ and $v_1, v_2 \in x\beta^c y$, for every $(\alpha,\beta) \in \Gamma^2$ and $(x,y,z) \in G^3$. Thus $(x\alpha^c y)\beta^c z = x\alpha^c(y\beta^c z)$, for every $(\alpha,\beta) \in \Gamma^2$ and $(x,y,z) \in G^3$. □

Theorem 3.18. *The complement of every Γ-group is a Sperner Γ-hypergroup.*

Proof. If (G,Γ) is a Γ-group then by previous Lemma (G,Γ^c) is a Γ-hypergroup. Moreover, we have

$$x\alpha y = x\beta y \Longleftrightarrow x\alpha^c y = x\beta^c y,$$

for every $(\alpha,\beta) \in \Gamma^2$ and $x, y \in G$. Hence (G,Γ^c) is a weak Sperner Γ-hypergroup. Also

$$x\alpha^c y = x\alpha^c v \Longleftrightarrow y = v,$$

for every $(x,y,v) \in G^3$. Therefore (G,Γ^c) is a row Sperner Γ-hypergroup. Similarly it is a column Sperner Γ-hypergroup, so it is a Sperner Γ-hypergroup. □

4. conclusion

In this paper, a generalization of the notion of Sperner semihypergroup introduced in [10] investigated and studied some classes of Γ-(semi)hypergroups. Using the notion of Sperner family of sets, we introduce the classes weak Sperner Γ-(semi)hypergroups, row Sperner Γ-(semi)hypergroups, column Sperner Γ-(semi)hypergroups and Sperner Γ-(semi)hypergroups, respectively. Moreover, we intend to continue this study in order to obtain fuzzy weak Sperner Γ-(semi)hypergroups, interval-valued fuzzy weak Sperner Γ-(semi) hypergroups.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S.M. Anvariyeh, S. Mirvakili and B. Davvaz, On Γ-hyperideals in Γ-semihypergroups, *Carpathian Journal of Mathematics* **26** (1) (2010), 11 – 23.

[2] S. Chattopadhyay, Right inverse Γ-semigroup, *Bull. Cal. Math. Soc.* **93** (2001), 435 – 442.

[3] S. Chattopadhyay, Right orthodox Γ-semigroup, *Southeast Asian Bull. Math.* **29** (2005), 23 – 30.

[4] R. Chinram and C. Jirojkul, On bi-Γ-ideals in Γ-semigroups, *Songklanakarin J. Sci. Technol.* **29** (2007), 231 – 234.

[5] R. Chinram and P. Siammai, On Green's relations for Γ-semigroups and reductive Γ-semigroups, *Int. J. Algebra* **2** (2008), 187 – 195.

[6] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory, Advances in Mathematics*, Kluwer Academic Publishers (2003).

[7] D. Heidari, S.O. Dehkordi and B. Davvaz, Γ-semihypergroups and their properties, *University Politehnica of Bucharest Scientific Bulletin, Series A* **72** (1) (2010), 195 – 208.

[8] K. Hila, On regular, semiprime and quasi-reflexive Γ-semigroup and minimal quasi-ideals, *Lobachevski J. Math.* **29** (2008), 141 – 152.

[9] K. Hila, On some classes of le-Γ-semigroups, *Algebras Groups Geom.* **24** (2007), 485 – 495.

[10] M. Jafarpour, H. Aghabozorgi and V. Leoreanu-Fotea, On Sperner (semi)hypergroups, *European Journal of Combinatorics* **44** (Part B) (2015), 231 – 235.

[11] M. Jafarpour, S.S. Mousavi and V. Leoreanu-Fotea, A class of semihypergroups connected to preordered weak Γ-semigroups, *Computers & Mathematics with Applications* **62** (I8) (2011), 2944 – 2949.

[12] F. Marty, Sur une generalization de la notion de groupe, *8iem Congres Math. Scandinaves*, Stockholm (1934), 45 – 49.

[13] N.K. Saha, On Γ-semigroup II, *Bull. Cal. Math. Soc.* **79** (1987), 331 – 335.

[14] M.K. Sen and S. Chattopadhyay, Semidirect product of a monoid and a Γ-semigroup, *East-West J. Math.* **6** (2004), 131 – 138.

[15] M.K. Sen and A. Seth, On po-Γ-semigroups, *Bull. Calcutta Math. Soc.* **85** (1993), 445 – 450.

[16] M.K. Sen and N.K. Saha, Orthodox Γ-semigroups, *Internat. J. Math. Math. Sci.* **13** (1990), 527 – 534.

[17] M.K. Sen and N.K. Saha, On Γ-semigroup I, *Bull. Cal. Math. Soc.* **78** (1986), 180 – 186.

[18] A. Seth, Γ-group congruences on regular Γ-semigroups, *Internat. J. Math. Math. Sci.* **15** (1992), 103 – 106.

[19] Z.X. Zhong and M.K. Sen, On several classes of orthodox Γ-semigroups, *J. Pure Math.* **14** (1997), 18 – 25.