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# Ulam-Hyers Stability and Well-Posedness of the Fixed Point Problems for Contractive Multi-valued Operator in $b$-metric Spaces

Research Article

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**Abstract.** In this paper, we establish some fixed point results for new classes of contractive multi-valued mappings via $\alpha_*$-admissible mapping with respect to $\eta$ in the class of $b$-metric spaces. To illustrate the obtained results, we provide some example. We also study the generalized Ulam-Hyers stability and well-posedness of fixed point problems are given. The theorems presented will extend, generalize or unify several statements currently exist in the literature on those topics.

**Keywords.** $\alpha$-admissible mappings; $b$-metric spaces; Fixed points; Multi-valued operator; Ulam-Hyers stability; Well-posedness

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## 1. Introduction

The study of fixed points for multi-valued contraction mappings using the Hausdorff metric was initiated by Nadler [17] in 1969, who extended the Banach contraction principle to set-valued mappings. Since then many authors have studied fixed points for set-valued maps. The theory of set-valued maps has many applications in control theory, convex optimization, differential equations and economics. Many mathematician generalized Banach's contraction principle and Nadler's contraction principle in different spaces.

The notion of $b$-metric space, which is a metric space satisfying a relaxed form of triangle inequality; see Bakhtin [3] and Czerwik [10] to extend the concept of metric space. Since then, several papers discussed fixed point results for single-valued and multi-valued operators in $b$-metric spaces (see [3, 10–12, 21, 24] and references therein).

**Definition 1.1** (Czerwik [10])**.** Let $X$ be a nonempty set and the functional $d : X \times X \to [0,\infty)$ satisfies:

(b1) $d(x,y) = 0$ if and only if $x = y$;

(b2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(b3) there exists a real number $s \geq 1$ such that $d(x,z) \leq s[d(x,y)+d(y,z)]$ for all $x,y,z \in X$.

Then $d$ is called a $b$-metric on $X$ and a pair $(X,d)$ is called a $b$-metric space with coefficient $s$.

**Remark 1.2.** If we take $s = 1$ in above definition then $b$-metric spaces turns into ordinary metric spaces. Hence, the class of $b$-metric spaces is larger than the class of metric spaces.

For examples of $b$-metric spaces was given in [3, 4, 7, 10, 12].

**Example 1.3.** The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \left\{ \{x_n\} \subset \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$, together with the functional $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to [0,\infty)$,

$$d(x,y) := \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{\frac{1}{p}},$$

(where $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$) is a $b$-metric space with coefficient $s = 2^{\frac{1}{p}} > 1$. Notice that the above result holds for the general case $l_p(X)$ with $0 < p < 1$, where $X$ is a Banach space.

**Example 1.4.** Let $X$ be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of $X$ such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to [0,\infty)$ defined by:

$$d(x,y) := \begin{cases} 0, & x = y \\ 2s, & x,y \in X_1 \\ 1, & \text{otherwise}, \end{cases}$$

is a $b$-metric on $X$ with coefficient $s > 1$.

Throughout this work, we denotes the families of subset of a $b$-metric space $(X,d)$ listed below:

$$\mathcal{S}(X) := \{Y \,|\, Y \subset X\};$$

$$\mathcal{P}(X) := \{Y \in \mathcal{S}(X) \,|\, Y \neq \emptyset\};$$

$$\mathcal{B}(X) := \{Y \in \mathcal{P}(X) | Y \text{ is bounded}\};$$
$$\mathcal{W}(X) := \{Y \in \mathcal{P}(X) | Y \text{ is compact}\};$$
$$\mathcal{CL}(X) := \{Y \in \mathcal{P}(X) | Y \text{ is closed}\};$$
$$\mathcal{CB}(X) := \{Y \in \mathcal{P}(X) | Y \text{ is closed and bounded}\}.$$

Additionally, we will also need the gap, excess generalized, Pompeiu-Hausdorff and $\delta$-functionals which will be used to define certain $b$-metric spaces to study various multi-valued operators. The definition of those functionals are given as follows:

(I) The gap functional: $D : \mathcal{S}(X) \times \mathcal{S}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$D(A,B) := \begin{cases} \inf\{d(a,b) \mid a \in A, b \in B\}, & A \neq \emptyset \neq B; \\ 0, & A = \emptyset = B; \\ +\infty, & \text{otherwise.} \end{cases}$$

(II) The excess generalized functional: $\rho : \mathcal{S}(X) \times \mathcal{S}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$\rho(A,B) := \begin{cases} \sup\{D(a,B) \mid a \in A\}, & A \neq \emptyset \neq B; \\ 0, & A = \emptyset; \\ +\infty, & B = \emptyset \neq A. \end{cases}$$

(III) Pompeiu-Hausdorff functional: $H : \mathcal{S}(X) \times \mathcal{S}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$H(A,B) := \begin{cases} \max\{\rho(A,B), \rho(B,A)\}, & A \neq \emptyset \neq B; \\ 0, & A = \emptyset = B; \\ +\infty, & \text{otherwise.} \end{cases}$$

(IV) $\delta$ functional: $\delta : \mathcal{S}(X) \times \mathcal{S}(X) \to \mathbb{R}_+ \cup \{+\infty\}$

$$\delta(A,B) := \begin{cases} \sup\{d(a,b) \mid a \in A, b \in B\}, & A \neq \emptyset \neq B; \\ 0, & A = \emptyset = B; \\ +\infty, & \text{otherwise.} \end{cases}$$

From the definition above we can see that if $x_0 \in X$ in the gap functional then $D(x_0,B) := D(\{x_0\},B)$. If $A = B$ in $\delta$ functional then $\delta(A,A) := \delta(A)$. It well known that $(\mathcal{CB}(X),H)$ is a complete $b$-metric space provided that $(X,d)$ is a complete $b$-metric space (see [10]).

Let $(X,d)$ be a $b$-metric space. We cite the following lemmas from Czerwik [10–12] and Singh *et al.* [32].

**Lemma 1.5.** *Let $(X,d)$ be a $b$-metric space. For any $A,B,C \in \mathcal{P}(X)$ and any $x,y \in X$, we have the following:*

(i) *$d(x,B) \leq d(x,b)$ for all $b \in B$;*

(ii) $d(x,B) \leq H(A,B)$ *for all* $x \in A$*;*

(iii) $\delta(A,B) \leq H(A,B)$*;*

(iv) $H(A,A) = 0$*;*

(v) $H(A,B) = H(B,A)$*;*

(vi) $H(A,C) \leq s(H(A,B) + H(B,C))$*;*

(vii) $d(x,A) \leq s(d(x,y) + d(y,A))$.

**Remark 1.6.** The function $H : \mathcal{CL}(X) \times \mathcal{CL}(X) \to [0,+\infty)$ is a generalized Pompeiu-Hausdorff $b$-metric, that is, $H(A,B) = +\infty$ if $\max\{\delta(A,B), \delta(B,A)\}$.

**Lemma 1.7.** *Let* $(X,d)$ *be a* $b$*-metric space. For* $A \in \mathcal{P}(X)$ *and* $x \in X$*, then we have*

$$d(x,A) = 0 \Longleftrightarrow x \in \bar{A} = A,$$

*where* $\bar{A}$ *denotes the closure of the set* $A$.

**Lemma 1.8.** *Let* $(X,d)$ *be a* $b$*-metric space. For* $A,B \in \mathcal{P}(X)$ *and* $q > 1$*. Then, for all* $a \in A$*, there exists* $b \in B$ *such that* $d(a,b) \leq qH(A,B)$ *if* $H(A,B) > 0$.

**Definition 1.9** (Boriceanu *et al.* [7])**.** Let $(X,d)$ be a $b$-metric space. Then a sequence $\{x_n\}$ in $X$ is called:

(a) Convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$.

(b) Cauchy if and only if $d(x_n,x_m) \to 0$ as $m,n \to \infty$.

(c) Complete if and only if every Cauchy sequence is convergent.

**Lemma 1.10** (Czerwik [10])**.** *Let* $(X,d)$ *be a* $b$*-metric space and let* $\{x_k\}_{k=0}^{n} \subset X$*. Then:*

$$d(x_0,x_n) \leq sd(x_0,x_1) + \ldots + s^{n-1}d(x_{n-2},x_{n-1}) + s^{n-1}d(x_{n-1},x_n).$$

Let $(X,d)$ is a $b$ metric space, if $T : X \to \mathcal{P}(X)$ is a multivalued operator, then $x \in X$ is called a fixed point for $T$ if and only if $x \in T(x)$. The set $Fix(T) := \{x \in X : x \in T(x)\}$ is called the fixed point set of $T$, while $SFix(T) := \{x \in X : T(x) = \{x\}\}$ is called the strict fixed point set of $T$. $Graph(T) := \{(x,y) \in X \times X : y \in T(x)\}$, denotes the graph of $T$.

For the following notions see Rus *et al.* [28] and A. Petruşel [18].

**Definition 1.11.** Let $(X,d)$ be a $b$-metric space and $T : X \to \mathcal{CL}(X)$ be a multivalued operator. By definition, $T$ is multivalued weakly Picard (briefly MWP) operator if for each $(x,y) \in Graph(T)$ there exists a sequence $\{x_n\}$ for all $n \in \mathbb{N}$ such that

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$, for each $n \in \mathbb{N}$;

(iii) the sequence $\{x_n\}$ for all $n \in \mathbb{N}$ is convergent and its limit is a fixed point of $T$.

**Remark 1.12.** A sequence $\{x_n\}$ for all $n \in \mathbb{N}$ satisfying the condition (i) and (ii), in the Definition above is called a sequence of successive approximations of $T$ starting from $(x,y) \in Graph(T)$.

Finally, to prove our results we need the following class of functions.

**Definition 1.13** (Rus [25])**.** A mapping $\psi : [0,\infty) \to [0,\infty)$ is called a comparison function if it is increasing and $\psi^n(t) \to 0$ as $n \to \infty$ for any $t \in [0,\infty)$, where $\psi^n$ is $n$-th iterate of $\psi$.

**Lemma 1.14** (Rus [25], Berinde [6])**.** *If $\psi : [0,\infty) \to [0,\infty)$ is a comparison function, then:*

(1) *$\psi^n$ is also a comparison function;*

(2) *$\psi$ is continuous at $0$;*

(3) *$\psi(t) < t$, for any $t > 0$.*

The concept of $(c)$-comparison function was introduced by Berinde [6] in the following definition.

**Definition 1.15** (Berinde [6])**.** A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a $(c)$-comparison function if

(1) $\psi$ is increasing;

(2) there exists $n_0 \in \mathbb{N}$, $k \in (0,1)$ and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} \epsilon_n$ such that $\psi^{n+1}(t) \le k\psi^n(t) + \epsilon_n$, for $n \ge n_0$ and any $t \in [0,\infty)$.

Here we recall the definitions of the following class of $(b)$-comparison function as given by Berinde [5] in order to extend some fixed point results to the class of a $b$-metric spaces :

**Definition 1.16** (Berinde [5])**.** Let $s \ge 1$ be a real number. A mapping $\psi : [0,\infty) \to [0,\infty)$ is called a $(b)$-comparison function if the following conditions are fulfilled:

(1) $\psi$ is increasing;

(2) there exist $n_0 \in \mathbb{N}$, $k \in (0,1)$ and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} \epsilon_n$ such that $s^{n+1}\psi^{n+1}(t) \le ks^n\psi^n(t) + \epsilon_n$, for $n \ge n_0$ and any $t \in [0,\infty)$.

In this work, we use $\Psi_b$ stands for the class of all $(b)$-comparison functions $\psi : [0,\infty) \to [0,\infty)$ unless and until it is stated. It is evident that the concept of $(b)$-comparison function reduces to that of $(c)$-comparison function when $s = 1$.

**Lemma 1.17** (Berinde [4])**.** *If $\psi : [0,\infty) \to [0,\infty)$ is a $(b)$-comparison function, then the following assertions hold:*

(i) *the series $\sum_{n=0}^{\infty} s^n\psi^n(t)$ converges for any $t \in [0,\infty)$;*

(ii) *the function $S : [0,\infty) \to [0,\infty)$ defined by $S(t) = \sum_{n=0}^{\infty} s^n\psi^n(t)$ for $t \in [0,\infty)$, is increasing and continuous at $0$.*

### 1.1 An $\alpha$-Admissible Mappings

Samet *et al.* [31] introduced the concept of $\alpha$-admissible mapping, where $\alpha$ is mapping from nonempty set $X$ to $[0,\infty)$, and established fixed point theorems via this concept and also showed that these results can be utilized to derive fixed point theorems in partially ordered spaces. Moreover, they applied the main results to ordinary differential equations. Afterward, Asl *et al.* [2] extended the concept of $\alpha$-admissible for single valued mappings to multivalued mappings called $\alpha_*$-admissible.

Later on, Salimi *et al.* [29], established fixed point theorems for $\alpha_*$-admissible contractions mapping with respect to $\eta_*$ on metric space for multifunction. There are many researchers improved and generalized fixed point results by using the concept of an $\alpha$-admissible mapping for single valued and multivalued mappings (see [1, 16, 21–24, 30]).

**Definition 1.18** (Samet *et al.* [31])**.** Let $X$ be a nonempty set, $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$. We say that $f$ is an $\alpha$-admissible mapping if satisfies the following condition:

$$\text{for } x, y \in X \text{ for which } \alpha(x,y) \geq 1 \Longrightarrow \alpha(T(x),T(y)) \geq 1.$$

**Definition 1.19** (Asl *et al.* [2])**.** Let $X$ be a nonempty set, $T : X \to \mathcal{P}(X)$ and $\alpha : X \times X \to [0,\infty)$. We say that $T$ is $\alpha_*$-admissible if

$$\text{for } x, y \in X, \quad \alpha(x,y) \geq 1 \Longrightarrow \alpha_*(T(x),T(y)) \geq 1,$$

where

$$\alpha_*(A,B) := \inf\{\alpha(a,b) : a \in A \text{ and } b \in B\}.$$

**Definition 1.20** (Salimi *et al.* [29])**.** $T : X \to \mathcal{P}(X)$ on a metric space $(X,d)$ and let $\alpha, \eta : X \times X \to [0,\infty)$ be two functions, where $\eta$ is bounded. We say that $T$ is an $\alpha_*$-admissible mapping with respect to $\eta$ mapping if

$$x, y \in X, \quad \alpha(x,y) \geq \eta(x,y) \Longrightarrow \alpha_*(T(x),T(y)) \geq \eta_*(T(x),T(y))$$

where $\alpha_*(A,B) := \inf\{\alpha(a,b) : a \in A \text{ and } b \in B\}$ and $\eta_*(A,B) := \sup\{\eta(a,b) : a \in A \text{ and } b \in B\}$.

If we take $\eta(a,b) = 1$ for all $a, b \in X$, then this definition reduces to Definition 1.19 of Asl *et al.* In case $\alpha(a,b) = 1$ for all $a, b \in X$, then $T$ is called an $\eta$-subadmissible mapping.

Recently Bota *et al.* [8] proved some existence and uniqueness theorems for $\alpha_*$-contractive type operators defined over $b$-metric spaces. In particular, they also provide results related to Ulam-Hyers stability, well-posedness and limit shadowing.

The purpose of this work is to establish the existence and the uniqueness of fixed point theorems for some class of multivalued contraction mappings via $\alpha_*$-admissible mapping with respect to $\eta$ mappings. We also give some example shows that the our fixed point theorems for new types of contractive mappings are independent. The generalized Ulam-Hyers stability and well-posedness of fixed point problems for these classes in the framework of $b$-metric spaces are proved.

## 2. Fixed Point Results in $b$-Metric Spaces

In this section, we shall state and prove Pompeiu-Hausdorff generalized functional, gap functional, $\delta$-functional for some class of multivalued contraction mappings via $\alpha_*$-admissible mapping with respect to $\eta$ mappings in $b$-metric spaces. First we start with $b$-metric version of multivalued contraction mappings via the Pompeiu-Hausdorff generalized functional and $\alpha_*$-admissible mapping with respect to $\eta$ mappings:

**Theorem 2.1.** *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible with respect to $\eta$ on $X$ and $\psi \in \Psi_b$. Suppose that the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0,x_1) \ge \eta(x_0,x_1)$;*

(b) *for all $x,y \in X$, we have*

$$\alpha_*(T(x),T(y)) \ge \eta_*(T(x),T(y)) \Longrightarrow H(T(x),T(y)) \le \psi(d(x,y)); \tag{2.1}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n,x) \ge \eta(x_n,x)$ for all $n \in \mathbb{N}$.*

*Then*

(i) *$Fix(T) \ne \emptyset$;*

(ii) *$T$ is MWP operator.*

*Proof.* Let $x_0 \in X$ and $x_1 \in T(x_0)$, we have $\alpha(x_0,x_1) \ge \eta(x_0,x_1)$ (by condition (a)). Since $T$ is an $\alpha_*$-admissible mapping with respect to $\eta_*$, then $\alpha_*(T(x_0),T(x_1)) \ge \eta_*(T(x_0),T(x_1))$. Therefore by (2.1), we have $H(T(x_0),T(x_1)) \le \psi(d(x_0,x_1))$.

If $x_0 = x_1$ we obtain the desired conclusion. Also,if $x \in T(x)$, then $x_1$ is a fixed point of $T$. Hence we assume that $x_0 \ne x_1$ and $x_1 \notin T(x_1)$ and hence $d(x_1,T(x_1)) > 0$. By Lemma 1.8 and (2.1), there exist $\tau > 1$ such that

$$0 < d(x_1,Tx_1) \le H(T(x_0),T(x_1)) \le \psi(d(x_0,x_1)) < \psi(\tau d(x_0,x_1)).$$

Since $\psi$ is increasing function. This implies that there exists $x_2 \in T(x_1)$ (obviously, $x_2 \ne x_1$) such that

$$0 < d(x_1,x_2) \le H(T(x_0),T(x_1)) < \psi(\tau d(x_0,x_1)).$$

From condition $\alpha_*(T(x_0),T(x_1)) \ge \eta_*(T(x_0),T(x_1))$ with $x_1 \in T(x_0)$ and $x_2 \in T(x_1)$, we have $\alpha(x_1,x_2) \ge \eta(x_1,x_2)$. Since $T$ is an $\alpha_*$-admissible mapping with respect to $\eta_*$, then $\alpha_*(T(x_1),T(x_2)) \ge \eta_*(T(x_1),T(x_2))$. Assume that $x_2 \notin T(x_2)$, that is, $d(x_2,T(x_2)) > 0$. By (2.1), we get

$$H(T(x_1),T(x_2)) \le \psi(d(x_1,x_2)).$$

Since $\psi$ is increasing function, we have that

$$0 < d(x_2,T(x_2)) \le H(T(x_1),T(x_2)) \le \psi(d(x_1,x_2)) < \psi^2(\tau d(x_0,x_1)).$$

This implies that there exists $x_3 \in T(x_2)$ (obviously, $x_3 \neq x_2$) such that

$$0 < d(x_2, x_3) < \psi^2(\tau d(x_0, x_1)).$$

By continuing this process, we define the sequence $\{x_n\}$ in $X$ such that $x_n \notin Tx_n$, $x_{n+1} \in T(x_n)$, $x_n \neq x_{n+1}$ $x_{n+1} \in T(x_n)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since $T$ is an $\alpha_*$-admissible mapping with respect to $\eta_*$, we get

$$\alpha_*(T(x_n), T(x_{n+1})) \geq \eta_*(T(x_n), T(x_{n+1})) \quad \text{for all} \quad n \in \mathbb{N}.$$

Also, we have

$$0 < d(x_n, T(x_n)) \leq d(x_n, x_{n+1}) < \psi^n(\tau d(x_0, x_1)) \quad \text{for all} \quad n \in \mathbb{N}.$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} s^{k-n+1} d(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} s^{k-n+1} \psi^n(\tau d(x_0, x_1)).$$

By Lemma (1.17) we know that the series $\sum_{i=1}^{\infty} s^i \psi^i(\tau d(x_0, x_1))$ converges. Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. By the completeness of $X$, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$. By condition (c), we have $\alpha(x_n, x^*) \geq \eta(x_n, x^*)$ then $\alpha_*(T(x_n), T(x^*)) \geq \eta_*(T(x_n), T(x^*))$. As a consequence, we derive that

$$\begin{aligned} D(x^*, T(x^*)) &\leq s[d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + H(T(x_n), T(x^*))] \\ &\leq s[d(x^*, x_{n+1}) + \psi(d(x_n, x^*))]. \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, since $\psi$ is continuous at 0, we obtain

$$D(x^*, T(x^*)) = 0.$$

Since, $T(x^*)$ is closed we obtain that $x^* \in T(x^*)$, that is, $x^*$ is a fixed point of $T$. This completes the proof. □

Next, we propose a $b$-metric version of multivalued contraction mappings via the gap functional.

**Theorem 2.2.** *Let $(X, d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible with respect to $\eta$ on $X$ and $\psi \in \Psi_b$. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) := D(x, T(x))$ is a lower semicontinuous mapping. Suppose that the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*

(b) *for all $x \in X$ and $y \in T(x)$, we have*

$$\alpha_*(T(x), T(y)) \geq \eta_*(T(x), T(y)) \Longrightarrow D(y, T(y)) \leq \psi(d(x, y)); \tag{2.2}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.*

*Then there exists an orbit $\{x_n\}$ of $T$ at $x_0$ and $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $x^* \in Fix(T)$.*

*Proof.* Let $x_0 \in X$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$ (from condition (a)). We define the sequence $\{x_n\}$ in $X$ such that

$$x_{n+1} \in T(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is an $\alpha_*$-admissible, with respect to $\eta$ then $\alpha_*(T(x_0),T(x_1)) \geq \eta_*(T(x_0),T(x_1))$. Therefore by (2.2), we have

$$D(x_1,T(x_1)) \leq \psi(d(x_0,x_1)).$$

By induction, we get $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Next, we will show that $\{x_n\}$ is Cauchy sequence in $X$. For each $n \in \mathbb{N}$, we have Also, we have

$$0 < D(x_n,T(x_n)) \leq D(x_n,x_{n+1}) < \psi^n(d(x_0,x_1)) \quad \text{for all} \quad n \in \mathbb{N}.$$

Following the same lines of argument given in the proof of Theorem 2.1 we know that $\{x_n\}$ is a Cauchy sequence in $X$ which converges to $x^*$ as $n \to \infty$. Since $(X,d)$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (c), we have $\alpha(x_n,x^*) \geq \eta(x_n,x^*)$ then $\alpha_*(T(x_n),T(x^*)) \geq \eta_*(T(x_n),T(x^*))$. Now, by the lower semicontinuity of the function $f$, we have

$$0 \leq f(x^*) = D(x^*,T(x^*)) \leq \liminf_{n\to\infty} f(x_n) = 0.$$

Hence $f(x^*) = 0$, which means that $D(x^*,T(x^*)) = 0$. Thus $x^* \in Fix(T)$. This completes the proof. □

**Theorem 2.3.** *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible with respect to $\eta$ on $X$ and $\psi \in \Psi_b$. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) := \delta(x,T(x))$ is a lower semicontinuous mapping, satisfy the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$;*

(b) *for all $x \in X$ and $y \in T(x)$, we have*
$$\alpha_*(T(x),T(y)) \geq \eta_*(T(x),T(y)) \Longrightarrow \delta(y,T(y)) \leq \psi(d(x,y)); \tag{2.3}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(x_n,x) \geq \eta(x_n,x)$ for all $n \in \mathbb{N}$.*

*Then there exists an orbit $\{x_n\}$ of $T$ at $x_0$ and $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $x^* \in SFix(T)$.*

*Proof.* Let $x_0 \in X$ such that $\alpha(x_0,x_1) \geq \eta(x_0,x_1)$ (from condition (a)). We define the sequence $\{x_n\}$ in $X$ such that

$$x_{n+1} \in T(x_n) \quad \text{for all } n \in \mathbb{N}.$$

Since $T$ is an $\alpha_*$-admissible, with respect to $\eta$ then $\alpha_*(T(x_0),T(x_1)) \geq \eta_*(T(x_0),T(x_1))$. Therefore by (2.3), we have

$$\delta(x_1,T(x_1)) \leq \psi(d(x_0,x_1)).$$

Then

$$d(x_1,x_2) \le \delta(x_1,T(x_1)) \le \psi(d(x_0,x_1)).$$

By induction, we get $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Next, we will show that $\{x_n\}$ is Cauchy sequence in $X$. For each $n \in \mathbb{N}$, we have Also, we have

$$0 < d(x_n,x_{n+1}) \le \delta(x_n,T(x_n)) \le \psi^n(d(x_0,x_1)) \quad \text{for all} \quad n \in \mathbb{N}.$$

Following the same lines of argument given in the proof of Theorem 2.1 we know that $\{x_n\}$ is a Cauchy sequence in $X$ which converges to $x^*$ as $n \to \infty$. Since $(X,d)$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (c), we have $\alpha(x_n,x^*) \ge \eta(x_n,x^*)$ then $\alpha_*(T(x_n),T(x^*)) \ge \eta_*(T(x_n),T(x^*))$. Now, by the lower semicontinuity of the function $f$, we have

$$0 \le f(x^*) = \delta(x^*,T(x^*)) \le \liminf_{n\to\infty} f(x_n) = 0.$$

Hence $f(x^*) = 0$, which means that $\delta(x^*,T(x^*)) = 0$. Next, we will prove that $x^* \in SFix(T)$. Let $x^* \in Fix(T)$ and using (2.3) with $x = y = x^*$, we obtain

$$\delta(T(x^*)) = \delta(x^*,T(x^*)) \le \psi(d(x^*,x^*)).$$

Thus, $\delta(T(x^*)) = 0$ and $T(x^*) = \{x^*\}$. This completes the proof. □

**Theorem 2.4.** *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible with respect to $\eta$ on $X$ and $\psi \in \Psi_b$. Suppose that all the hypotheses of Theorem 2.1 hold. Then $Fix(T) = SFix(T) = \{x^*\}$ such that $\alpha(x^*,y^*) \ge \eta(x^*,y^*)$ for all $x^*,y^* \in X$ which is $x^* \in SFix(T)$ and $y^* \in Fix(T)$.*

*Proof.* We shall prove now that $Fix(T) = SFix(T)$. Because $SFix(T) \subset Fix(T)$, we need to show that $Fix(T) \subset SFix(T)$. Let $x^* \in SFix(T)$ and $y^* \in Fix(T)$ with $y^* \neq x^*$. By hypothesis $\alpha(x^*,y^*) \ge \eta(x^*,y^*)$. Since, $T$ is an $\alpha_*$-admissible with respect to $\eta$ on $X$, we get $\alpha(Tx^*,Ty^*) \ge \eta(Tx^*,Ty^*)$. By (2.1), we have

$$\begin{aligned} d(x^*,y^*) = D(T(x^*),y^*) &\le H(T(x^*),T(y^*)) \\ &\le \psi(d(x^*,y^*)) < d(x^*,y^*). \end{aligned}$$

This is a contradiction. So, we have $x^* = y^*$. Hence, $Fix(T) = SFix(T) = \{x^*\}$. □

If we set $\eta(x,y) = 1$ for all $x,y \in X$ in Theorems 2.1 or 2.2 or 2.3, we get the following results.

**Corollary 2.5.** *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible on $X$ and $\psi \in \Psi_b$. Suppose that the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0,x_1) \ge 1$;*

(b) *for all $x,y \in X$, we have*

$$\alpha_*(T(x),T(y)) \ge 1 \Longrightarrow H(T(x),T(y)) \le \psi(d(x,y)); \tag{2.4}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n,x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n,x) \ge 1$ for all $n \in \mathbb{N}$.*

*Then*

(i) $Fix(T) \neq \emptyset$;

(ii) $T$ *is MWP operator.*

**Corollary 2.6.** *Let* $(X,d)$ *be a complete* $b$*-metric space,* $T : X \to \mathcal{CL}(X)$ *be an* $\alpha_*$*-admissible on* $X$ *and* $\psi \in \Psi_b$*. Suppose that* $f : X \to \mathbb{R}_+$ *defined* $f(x) := D(x,T(x))$ *is a lower semicontinuous mapping, satisfy the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0,x_1) \geq 1$;

(b) *for all* $x \in X$ *and* $y \in T(x)$*, we have*

$$\alpha_*(T(x),T(y)) \geq 1 \Longrightarrow D(y,T(y)) \leq \psi(d(x,y)); \tag{2.5}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, then* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then there exists an orbit* $\{x_n\}$ *of* $T$ *at* $x_0$ *and* $x^* \in X$ *such that* $x_n \to x^*$ *as* $n \to \infty$*. Moreover,* $x^* \in Fix(T)$.

**Corollary 2.7.** *Let* $(X,d)$ *be a complete* $b$*-metric space,* $T : X \to \mathcal{CL}(X)$ *be an* $\alpha_*$*-admissible on* $X$ *and* $\psi \in \Psi_b$*. Suppose that* $f : X \to \mathbb{R}_+$ *defined* $f(x) := \delta(x,T(x))$ *is a lower semicontinuous mapping, satisfy the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0,x_1) \geq 1$;

(b) *for all* $x \in X$ *and* $y \in T(x)$*, we have*

$$\alpha_*(T(x),T(y)) \geq 1 \Longrightarrow \delta(y,T(y)) \leq \psi(d(x,y)); \tag{2.6}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, then* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then there exists an orbit* $\{x_n\}$ *of* $T$ *at* $x_0$ *and* $x^* \in X$ *such that* $x_n \to x^*$ *as* $n \to \infty$*. Moreover,* $x^* \in SFix(T)$.

**Corollary 2.8.** *Let* $(X,d)$ *be a complete* $b$*-metric space,* $T : X \to \mathcal{CL}(X)$ *be an* $\alpha_*$*-admissible on* $X$ *and* $\psi \in \Psi_b$*. Suppose that the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0,x_1) \geq 1$;

(b) *for all* $x,y \in X$*, we have*

$$\alpha_*(T(x),T(y))H(T(x),T(y)) \leq \psi(d(x,y)); \tag{2.7}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$*, then* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then*

(i) $Fix(T) \neq \emptyset$;

(ii) $T$ *is MWP operator.*

**Corollary 2.9.** *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible on $X$ and $\psi \in \Psi_b$. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) := D(x, T(x))$ is a lower semicontinuous mapping, satisfy the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;*

(b) *for all $x \in X$ and $y \in T(x)$, we have*
$$\alpha_*(T(x), T(y))D(y, T(y)) \leq \psi(d(x, y)); \tag{2.8}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists an orbit $\{x_n\}$ of $T$ at $x_0$ and $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $x^* \in Fix(T)$.*

**Corollary 2.10** ( [8]). *Let $(X,d)$ be a complete $b$-metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible on $X$ and $\psi \in \Psi_b$. Suppose that $f : X \to \mathbb{R}_+$ defined $f(x) := \delta(x, T(x))$ is a lower semicontinuous mapping, satisfy the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq 1$;*

(b) *for all $x \in X$ and $y \in T(x)$, we have*
$$\alpha_*(T(x), T(y))\delta(y, T(y)) \leq \psi(d(x, y)); \tag{2.9}$$

(c) *if $\{x_n\}$ is a sequence in $X$ converging to $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

*Then there exists an orbit $\{x_n\}$ of $T$ at $x_0$ and $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, $x^* \in SFix(T)$.*

If the coefficient $s = 1$ in Corollary 2.1, we obtain immediately the following fixed point theorems in metric spaces.

**Corollary 2.11** ( [29]). *Let $(X,d)$ be a complete metric space, $T : X \to \mathcal{CL}(X)$ be an $\alpha_*$-admissible with respect to $\eta$ on $X$ and $\psi$ is $(c)$-comparison function. Suppose that the following assertions hold:*

(a) *there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;*

(b) *for all $x, y \in X$, we have*
$$\alpha_*(T(x), T(y)) \geq \eta_*(T(x), T(y)) \Longrightarrow H(T(x), T(y)) \leq \psi(d(x, y)); \tag{2.10}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point.*

If $\eta = 1$ in Corollary 2.11, we obtain that

**Corollary 2.12.** *Let* $(X,d)$ *be a complete metric space,* $T : X \to \mathcal{CL}(X)$ *be an* $\alpha_*$*-admissible on* $X$ *and* $\psi$ *is* $(c)$*-comparison function. Suppose that the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0, x_1) \geq 1$;

(b) *for all* $x, y \in X$, *we have*

$$\alpha_*(T(x), T(y)) \geq 1 \Longrightarrow H(T(x), T(y)) \leq \psi(d(x,y)); \tag{2.11}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point.*

**Corollary 2.13** ( [2]). *Let* $(X,d)$ *be a complete metric space,* $T : X \to \mathcal{CL}(X)$ *be an* $\alpha_*$*-admissible on* $X$ *and* $\psi$ *is* $(c)$*-comparison function. Suppose that the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0, x_1) \geq 1$;

(b) *for all* $x, y \in X$, *we have*

$$\alpha_*(T(x), T(y)) H(T(x), T(y)) \leq \psi(d(x,y)); \tag{2.12}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point.*

**Corollary 2.14.** *Let* $(X,d)$ *be a complete metric space,* $T : X \to \mathcal{CL}(X)$ *and* $\psi$ *is* $(c)$*-comparison function. Suppose that the following assertions hold:*

(a) *there exists* $x_0 \in X$ *and* $x_1 \in T(x_0)$ *such that* $\alpha(x_0, x_1) \geq 1$;

(b) *for all* $x, y \in X$, *we have*

$$H(T(x), T(y)) \leq \psi(d(x,y)); \tag{2.13}$$

(c) *if* $\{x_n\}$ *is a sequence in* $X$ *converging to* $x \in X$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

*Then* $T$ *has a fixed point.*

**Remark 2.15.** If $\psi(t) = kt$, where $k \in (0,1)$ in Corollary 2.14, we obtain the Nadler's contraction principle [17].

Next, we give some example shows that the contractive conditions in our results are independent. Also, our results are real generalizations of Banach contraction principle and several results in literature.

**Example 2.16.** Let $X = \mathbb{R}$ and defined $d : X \times X \to [0,\infty)$ as

$$d(x,y) = |x-y|^2 \quad \text{for all} \quad x,y \in X.$$

Then $(X,d)$ is a complete $b$-metric space with coefficient $s = 2 > 1$, but it is not usual metric space. Let us define $T : \mathbb{R} \to \mathcal{CL}(X)$ by

$$T(x) = \begin{cases} \left\{0, \dfrac{x+1}{2}\right\}, & 0 \le x < 1, \\ \{4,5\}, & \text{otherwise.} \end{cases}$$

Also, define $\alpha : X \times X \to [0,\infty)$, $\eta : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & 0 \le x,y < 1, \\ 2, & \text{otherwise} \end{cases}$$

$$\eta(x,y) = \begin{cases} 2, & 0 \le x,y < 1, \\ 4, & \text{otherwise} \end{cases}$$

and $\psi(t) = \dfrac{1}{2}t$ for all $t \ge 0$. Clearly $T$ is an $\alpha_*$-admissible with respect to $\eta$ on $X$. There exists $x_0 = 0 \in X$ assume that $x_0 \ne x_1$. So, $T(x_0) = T(0) = \frac{1}{2}$ such that

$$\alpha\left(0, \frac{1}{2}\right) = 3 > 2 = \eta\left(0, \frac{1}{2}\right).$$

Since, $T$ is an $\alpha_*$-admissible with respect to $\eta$ on $X$, we get

$$\alpha_*(Tx(_0), T(x_1)) = \alpha_*\left(\frac{1}{2}, 0\right) = 3 > 2 = \eta_*\left(\frac{1}{2}, \frac{3}{4}\right) = \eta_*(Tx(0), T(x_1)).$$

For all $x,y \in X$, with

$$\alpha_*(T(x),T(y)) \ge \eta_*(T(x),T(y)).$$

Then we have

$$\begin{aligned} H(Tx,Ty) &= \max\left\{\sup_{x \in T(x)} D(x,T(y)), \sup_{y \in T(y)} D(y,T(x))\right\} \\ &= \max\left\{\frac{1}{4}|x-y|^2, \frac{1}{4}|y-x|^2\right\} \\ &= \frac{1}{4}|x-y|^2 < \frac{1}{2}|x-y|^2 = \psi(d(x,y)). \end{aligned}$$

Hence, we can see that condition (a) and (b) of Theorem 2.1 hold. Also, we can easily to prove that condition (c) in Theorem 2.1 holds. Therefore, all of conditions in Theorem 2.1 hold. Moreover, we have that $Fix(T) = SFix(T) = \{0\}$.

Next, we show that the contractive condition in Corollary 2.14 and Nadler's contraction principle cannot be applied to this example For $x = 0$ and $y = 1$, we obtain that

$$H(Tx,Ty) = 25 > 1 = d(x,y).$$

## 3. The Generalized Ulam-Hyers Stability of Fixed Point Inclusion in $b$-Metric Spaces

In this section, we prove the generalized Ulam-Hyers stability of fixed point inclusion in $b$-metric spaces which correspondence to Theorems 2.1, 2.2 and 2.3.

Stability problems of functional analysis is the another one which play the most important in mathematics analysis. It was introduced by Ulam [34], he was concern the stability of group homomorphisms. Afterward, Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach space, this type of stability is called Ulam-Hyers stability. Several authors consider Ulam-Hyers stability results in fixed point theory and remarkable result on the stability of certain classes of functional equations via fixed point approach (see [9, 13, 14, 21, 24, 26, 27, 33] and references therein).

We recall the following definitions of generalized Ulam-Hyer stability for multivalued operator in the class of $b$-metric spaces:

**Definition 3.1.** Let $(X,d)$ be a $b$-metric space with coefficient $s$ and $T : X \to \mathcal{P}(X)$ be a multivalued operator. By definition, the fixed point inclusion

$$u \in T(u), \quad u \in X \tag{3.1}$$

is said to be generalized Ulam-Hyers stable in the framework of a $b$-metric space if there exists an increasing operator $\varphi : [0,\infty) \to [0,\infty)$, continuous at 0 and $\varphi(0) = 0$ such that for each $\varepsilon > 0$ and an $\varepsilon$-solution $v^* \in X$, of inequality,

$$D(v^*, T(v^*)) \le \varepsilon, \tag{3.2}$$

there exists a solution $u^* \in X$ of the fixed point inclusion (3.1) such that

$$d(v^*, u^*) \le \varphi(s\varepsilon). \tag{3.3}$$

If there exists $c > 0$ such that $\varphi(t) := ct$ for all $t \in [0,\infty)$, then the fixed point inclusion (3.1) is said to be Ulam-Hyers stable in the framework of a $b$-metric space.

**Remark 3.2.** If $s = 1$, then Definition 3.1 reduce to generalized Ulam-Hyers stability in metric spaces. Also, if $\varphi(t) := ct$, for all $t \in [0,\infty)$, where $c > 0$, then it reduces to classical Ulam-Hyers stability.

**Theorem 3.3.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s$. Suppose that all the hypotheses of Theorem 2.1 hold and also that the function $\xi : [0,\infty) \to [0,\infty)$ defined by $\xi(r) := r - s\psi(r)$ is strictly increasing and onto. If $\alpha(u^*, v^*) \ge \eta(u^*, v^*)$ for all $u^* \in X$ which is an $\varepsilon$-solution, then the fixed point inclusion (3.1) is generalized Ulam-Hyers stable.*

*Proof.* By Theorem 2.1, we have $x^* \in T(x^*)$, that is, $x^* \in X$ is a solution of fixed point inclusion (3.1). Let $\varepsilon > 0$ and $y^* \in X$ is an $\varepsilon$-solution, that is

$$d(y^*, T(y^*)) \le \varepsilon.$$

Since $x^*, y^* \in X$ are $\varepsilon$-solution, we have

$$\alpha(x^*, y^*) \geq \eta(x^*, y^*).$$

Since, $T$ is $\alpha_*$-admissible with respect to $\eta$. So, we have

$$\alpha(T(x^*), T(y^*)) \geq \eta(T(x^*), T(y^*)).$$

Now, we obtain

$$\begin{aligned} d(x^*, y^*) &= D(T(x^*), y^*) \\ &\leq s[H(T(x^*), T(y^*)) + D(T(y^*), y^*)] \\ &\leq s[\psi(d(x^*, y^*)) + \varepsilon]. \end{aligned}$$

It follows that

$$d(x^*, y^*) - s\psi(d(x^*, y^*)) \leq s\varepsilon.$$

Since $\xi(r) := r - s\psi(r)$, we have

$$\xi(d(x^*, y^*)) = d(x^*, y^*) - s\psi(d(x^*, y^*)).$$

This implies that

$$d(x^*, y^*) \leq \xi^{-1}(s\varepsilon).$$

Notice that $\xi^{-1} : [0, \infty) \to [0, \infty)$ exists, is increasing, continuous at 0 and $\varphi^{-1}(0) = 0$. Therefore, the fixed point inclusion (3.1) is generalized Ulam-Hyers stable. □

## 4. Well-Posedness of the Fixed Point Problems in $b$-Metric Spaces

In this section we present some well-posedness results for the fixed point problem. We consider both the well-posedness and the well-posedness in the generalized sense for a multivalued operator $T$ with respect to $H$ and $D$ in the class of $b$-metric spaces We begin by recalling the definition of these notions from [19] and [20].

**Definition 4.1.** Let $(X, d)$ be a $b$-metric spaces with coefficient $s$ and $T : X \to \mathcal{P}(X)$ be a multivalued operator. By definition, the fixed point problem is well posed for $T$ with respect to $H$ if:

(i) $SFix(T) = \{x^*\}$;

(ii) If $x_n$ is a sequence in $X$ such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.

**Definition 4.2.** Let $(X, d)$ be a $b$-metric spaces with coefficient $s$ and $T : X \to \mathcal{P}(X)$ be a multivalued operator. By definition, the fixed point problem is well posed in the generalized sense for $T$ with respect to $H$ if:

(i) $SFix(T) \neq \emptyset$;

(ii) If $x_n$ is a sequence in $X$ such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$, then there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x^*$, as $n \to \infty$.

**Definition 4.3.** Let $(X,d)$ be a $b$-metric spaces with coefficient $s$ and $T : X \to \mathcal{P}(X)$ be a multivalued operator. By definition, the fixed point problem is well posed for $T$ with respect to $D$ if:

(i) $SFix(T) = \{x^*\}$;

(ii) If $x_n$ is a sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \to x^*$, as $n \to \infty$.

**Definition 4.4.** Let $(X,d)$ be a $b$-metric spaces with coefficient $s$ and $T : X \to \mathcal{P}(X)$ be a multivalued operator. By definition, the fixed point problem is well posed in the generalized sense for $T$ with respect to $D$ if:

(i) $SFix(T) \neq \emptyset$;

(ii) If $x_n$ is a sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$, then there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x^*$, as $n \to \infty$.

In the following next theorems, we add a new condition to assure the well-posedness of the fixed point problems with respect to $H$ and $D$ in $b$-metric spaces.

(H1) If $\{x_n\}$ is sequence in $X$ such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$ and $x \in SFix(T)$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.

(H2) If $\{x_n\}$ is sequence in $X$ such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$ and $x \in SFix(T)$, then there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x$, as $n \to \infty$ and $\alpha(x_{k_n}, x) \geq \eta(x_{k_n}, x)$ for all $n \in \mathbb{N}$.

(D1) If $\{x_n\}$ is sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$ and $x \in SFix(T)$, then $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$.

(D2) If $\{x_n\}$ is sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$ and $x \in SFix(T)$, then there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x$, as $n \to \infty$ and $\alpha(x_{k_n}, x) \geq \eta(x_{k_n}, x)$ for all $n \in \mathbb{N}$.

**Theorem 4.5.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s$, $T : X \to \mathcal{P}(X)$ and $\alpha, \eta : X \times X \to [0,\infty)$ be three mappings and $\psi \in \Psi_b$. Suppose that*

(i) *The hypotheses of Theorem 2.4, Theorem 3.3 and condition $(H1)$ hold.*

(ii) *For any $x_n$ is a sequence in $X$ with $H(x_n, T(x_n)) \to 0$, as $n \to \infty$, we have $x_n \to x^*$, as $n \to \infty$.*

*Then, the fixed point problem is well-posed for $T$ with respect to $H$.*

*Proof.* By Theorem 2.1 and Theorem 2.4, there is a point $x^* \in X$ such that $SFix(T) = x^*$. Let $\{x_n\}$ be sequence in $X$ such that $H(x_n, T(x_n)) \to 0$, as $n \to \infty$. By condition $(H1)$, we get

$$\alpha(x_n, x^*) \geq \eta(x_n, x^*) \text{ for all } n \in \mathbb{N}.$$

Since, $T$ is $\alpha_*$-admissible with respect to $\eta$. So, we have

$$\alpha(T(x_n),T(x^*)) \geq \eta(T(x_n),T(x^*)) \text{ for all } n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned} d(x_n,x^*) &= d(x_n,T(x^*)) \\ &\leq s[d(x_n,T(x_n)) + H(T(x_n),T(x^*))] \\ &\leq s[H(x_n,T(x_n)) + H(T(x_n),T(x^*))] \\ &\leq s[H(x_n,T(x_n)) + \psi(x_n,x^*)]. \end{aligned}$$

Thus we get $d(x_n,x^*) - s\psi(x_n,x^*) \leq sH(x_n,T(x_n))$. Since $\xi(r) := r - s\psi(r)$, we have

$$\xi(d(x_n,x^*)) = d(x_n,x^*) - s\psi(x_n,x^*).$$

This implies that

$$d(x_n,x^*) \leq \xi^{-1}(sH(x_n,T(x_n))).$$

Since $H(x_n,T(x_n)) \to 0$, as $n \to \infty$. It implies that $x_n \to x^*$, as $n \to \infty$. Therefore, the fixed point problem is well-posed for $T$ with respect to $H$. □

**Theorem 4.6.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s$, $T : X \to \mathcal{P}(X)$ and $\alpha,\eta : X \times X \to [0,\infty)$ be three mappings and $\psi \in \Psi_b$. Suppose that*

(i) *The hypotheses of Theorem 2.4, Theorem 3.3 and condition $(H2)$ hold.*

(ii) *For $x_n$ is a sequence in $X$ with $H(x_n,T(x_n)) \to 0$, as $n \to \infty$, there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x^*$, as $n \to \infty$ and $H(x_{k_n},T(x_{k_n})) \to 0$, as $n \to \infty$.*

*Then, the fixed point problem is well-posed in the generalized sense for $T$ with respect to $H$.*

*Proof.* By Theorem 2.1 and Theorem 2.4, we know that $SFix(T) \neq \emptyset$. Let $\{x_n\}$ be sequence in $X$ which satisfies (ii). By condition $(H2)$, we get

$$\alpha(x_{k_n},x^*) \geq \eta(x_{k_n},x^*) \text{ for all } n \in \mathbb{N}.$$

Since, $T$ is $\alpha_*$-admissible with respect to $\eta$. So, we have

$$\alpha(T(x_{k_n}),T(x^*)) \geq \eta(T(x_{k_n}),T(x^*)) \text{ for all } n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned} d(x_{k_n},x^*) &= d(x_{k_n},T(x^*)) \\ &\leq s[d(x_{k_n},T(x_{k_n})) + H(T(x_{k_n}),T(x^*))] \\ &\leq s[H(x_{k_n},T(x_{k_n})) + H(T(x_{k_n}),T(x^*))] \\ &\leq s[H(x_{k_n},T(x_{k_n})) + \psi(x_{k_n},x^*)]. \end{aligned}$$

Thus we get $d(x_{k_n},x^*) - s\psi(x_{k_n},x^*) \leq sH(x_{k_n},T(x_{k_n}))$. Since $\xi(r) := r - s\psi(r)$, we have

$$\xi(d(x_{k_n},x^*)) = d(x_{k_n},x^*) - s\psi(x_{k_n},x^*).$$

This implies that

$$d(x_{k_n},x^*) \leq \xi^{-1}(sH(x_{k_n},T(x_{k_n}))).$$

Since $H(x_{k_n}, T(x_{k_n})) \to 0$, as $n \to \infty$. It implies that $x_n \to x^*$, as $n \to \infty$. Therefore, the fixed point problem is well-posed in the generalized sense for $T$ with respect to $H$. □

**Theorem 4.7.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s$, $T : X \to \mathcal{P}(X)$ and $\alpha, \eta : X \times X \to [0,\infty)$ be three mappings and $\psi \in \Psi_b$. Suppose that*

(i) *The hypotheses of Theorem 2.4, Theorem 3.3 and condition $(D1)$ hold.*

(ii) *For any $x_n$ is a sequence in $X$ with $D(x_n, T(x_n)) \to 0$, as $n \to \infty$, we have $x_n \to x^*$, as $n \to \infty$.*

*Then, the fixed point problem is well-posed for $T$ with respect to $D$.*

*Proof.* By Theorem 2.1 and Theorem 2.4, there is a point $x^* \in X$ such that $x^* \in SFix(T)$. Let $\{x_n\}$ be sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. Then for $z_n \in T(x_n)$ such that $d(x_n, z_n) = D(x_n, T(x_n))$ for all $n \in \mathbb{N}$ and by condition $(D1)$, we get

$$\alpha(x_n, x^*) \geq \eta(x_n, x^*) \text{ for all } n \in \mathbb{N}.$$

Since, $T$ is $\alpha_*$-admissible with respect to $\eta$. So, we have

$$\alpha(T(x_n), T(x^*)) \geq \eta(T(x_n), T(x^*)) \text{ for all } n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned} d(x_n, x^*) &= d(x_n, T(x^*)) \\ &\leq s[d(x_n, T(x_n)) + D(T(x_n), T(x^*))] \\ &\leq s[d(x_n, z_n) + D(z_n, T(x^*))] \\ &\leq s[d(x_n, z_n) + H(T(x_n), T(x^*))] \\ &\leq s[D(x_n, T(x_n)) + \psi(x_n, x^*)]. \end{aligned}$$

Thus we get $d(x_n, x^*) - s\psi(x_n, x^*) \leq sD(x_n, T(x_n))$. Since $\xi(r) := r - s\psi(r)$, we have

$$\xi(d(x_n, x^*)) = d(x_n, x^*) - s\psi(x_n, x^*).$$

This implies that

$$d(x_n, x^*) \leq \xi^{-1}(sD(x_n, T(x_n))).$$

Since $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. It implies that $x_n \to x^*$, as $n \to \infty$. Therefore, the fixed point problem is well-posed for $T$ with respect to $D$. □

**Theorem 4.8.** *Let $(X,d)$ be a complete $b$-metric space with coefficient $s$, $T : X \to \mathcal{P}(X)$ and $\alpha, \eta : X \times X \to [0,\infty)$ be three mappings and $\psi \in \Psi_b$. Suppose that*

(i) *The hypotheses of Theorem 2.4, Theorem 3.3 and condition $(D2)$ hold.*

(ii) *For $x_n$ is a sequence in $X$ with $D(x_n, T(x_n)) \to 0$, as $n \to \infty$, there exists a subsequence $\{x_{k_n}\}$ of $x_n$ such that $x_{k_n} \to x^*$, as $n \to \infty$ and $D(x_{k_n}, T(x_{k_n})) \to 0$, as $n \to \infty$.*

*Then, the fixed point problem is well-posed in the generalized sense for $T$ with respect to $D$.*

*Proof.* By Theorem 2.1 and Theorem 2.4, we know that $SFix(T) \neq \emptyset$. Let $\{x_n\}$ be sequence in $X$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$. Let $\{x_{k_n}\}$ be a subsequence of $x_n$. Then for $w_{k_n} \in T(x_{k_n})$ such that $d(x_{k_n}, w_{k_n}) = D(x_{k_n}, T(x_{k_n}))$ for all $n \in \mathbb{N}$ and by condition $(D2)$, we get

$$\alpha(x_{k_n}, x^*) \geq \eta(x_{k_n}, x^*) \text{ for all } n \in \mathbb{N}.$$

Since, $T$ is $\alpha_*$-admissible with respect to $\eta$. So, we have

$$\alpha(T(x_{k_n}), T(x^*)) \geq \eta(T(x_{k_n}), T(x^*)) \text{ for all } n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned}
d(x_{k_n}, x^*) &= d(x_{k_n}, T(x^*)) \\
&\leq s[d(x_{k_n}, T(x_{k_n})) + D(T(x_{k_n}), T(x^*))] \\
&\leq s[d(x_{k_n}, w_{k_n}) + D(w_{k_n}, T(x^*))] \\
&\leq s[d(x_{k_n}, w_{k_n}) + H(T(x_{k_n}), T(x^*))] \\
&\leq s[D(x_{k_n}, T(x_{k_n})) + \psi(x_n, x^*)].
\end{aligned}$$

Thus we get $d(x_{k_n}, x^*) - s\psi(x_{k_n}, x^*) \leq sD(x_{k_n}, T(x_{k_n}))$. Since $\xi(r) := r - s\psi(r)$, we have

$$\xi(d(x_{k_n}, x^*)) = d(x_{k_n}, x^*) - s\psi(x_{k_n}, x^*).$$

This implies that

$$d(x_{k_n}, x^*) \leq \xi^{-1}(sD(x_{k_n}, T(x_{k_n}))).$$

Since $D(x_{k_n}, T(x_{k_n})) \to 0$, as $n \to \infty$. It implies that $x_n \to x^*$, as $n \to \infty$. Therefore, the fixed point problem is well-posed in the generalized sense for $T$ with respect to $D$. □

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### Competing Interests

The author declare that he/she has no competing interests.

### Authors' Contributions

The author wrote, read and approved the final manuscript.

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