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Spaces of Series Summable by Absolute Cesàro and Matrix Operators

Research Article

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Abstract. In this paper giving some algebraic and topological properties of $|C_\alpha|_k$, we characterize the classes of all infinite matrices $(|C_\alpha|, |C_\delta|_k)$ and $(|C_\alpha|_k, |C_\delta|)$ for $\alpha, \delta > -1$ and $k \geq 1$, show that each element of this classes correspond to a continuous linear mapping, which also enables us to extend some well known results of Flett [7], Orhan and Sarıgöl [15], Bosanquet [2], Mehdi [13], Mazhar [11], and Sarıgöl [18], where $|C_\alpha|_k$ is the space of series summable by absolute Cesàro summability $|C,\alpha|_k$ in Flett's notation.

Keywords. Summability factors; Matrix transformations; Sequence spaces; Cesàro spaces

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1. Introduction

Let $\sum a_n$ be an infinite series with s_n as its n-th partial sum. Let (σ_n^α) and (t_n^α) be the n-th Cesàro means (C,α) of order $\alpha(\alpha > -1)$ of the sequences (s_n) and (na_n) respectively, i.e., $\sigma_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v$ and $t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v$, where $A_0^\alpha = 1$, $A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!}$ and $A_{-n}^\alpha = 0$, $(n \in N)$. The concept of absolute summability of order k was defined by Flett [7] as follows. A series $\sum a_n$ is summable $|C,\alpha|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty. \tag{1.1}$$

On the other hand, in view of the well known identity $t_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$, the condition (1.1) can be stated by

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^{\alpha}|^k < \infty, \qquad [8] \tag{1.2}$$

The summability $|C,\alpha|_k$ is one of ancestor summability methods and includes all Cesàro methods depending on α and k, for example, $|C,\alpha|_1$ is identical to $|C,\alpha|$. Now we denote by $|C_\alpha|_k$ the set of series summable by the summability method $|C,\alpha|_k$. Then a series $\sum a_v$ is summable by $|C,\alpha|_k$ iff $a=(a_v) \in |C_\alpha|_k$, where

$$|C_\alpha|_k = \left\{ a=(a_v) : \sum_{n=1}^{\infty} \left| \frac{1}{n^{1/k} A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \right|^k < \infty \right\}. \tag{1.3}$$

Let X and Y be any two sequence subsets and $A=(a_{nv})$ be an infinite matrix of complex numbers. Then we say that A defines a matrix transformation from X into Y, i.e., $A \in (X,Y)$ if $Ax=(A_n(x)) \in Y$ whenever $x \in X$, where

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v \tag{1.4}$$

provided that the series on the right side of (1.4) converge for each n.

Das [6] defined a matrix A to be absolutely k-th power conservative for $k \ge 1$ if $A \in B(A_k, A_k)$, where

$$\mathscr{A}_k = \left\{ s=(s_v) : \sum_{v=1}^{\infty} v^{k-1} |s_v - s_{v-1}|^k < \infty \right\}$$

and proved every consersative Hausdorff matrix $H \in B(\mathscr{A}_k, \mathscr{A}_k)$. Note that there exists a relation between A_k and $|C_0|_k$ obtained in the special case $\alpha=0$ if A lower triangular matrix. In fact, $a \in |C_0|_k$ if and only if $s \in A_k$, and so $A \in (A_k, A_k)$, iff $A \in (|C_0|_k, |C_0|_k)$, where

$$\hat{a}_{nv} = \begin{cases} \sum_{r=v}^{n} (a_{nr} - a_{n-1,r}), & 0 \le v \le n \\ 0, & v > n. \end{cases} \tag{1.5}$$

According to the terminology in [15], if A is a Riesz matrix, i.e., $a_{nv} = \frac{p_v}{P_n}$ for $0 \le v \le n$, and 0 otherwise, then $\sum a_v$ is summable $|R,p_n|_k$ iff $(R_n(a)) \in |C_0|_k$, where $R_0(a) = a_0, R_n(a) = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v$, $n \ge 1$, (p_n) is a sequence of positive constants such that $P_n = p_0 + p_1 + \ldots + p_n \to \infty$ as $n \to \infty$. So, if we say $|R_p|_k$ as the set of series summable by $|R,p_n|_k$, then we can write $|R_p|_k = \{a=(a_v) : (R_n(a)) \in |C_0|_k\}$. For real number α and nonnegative integers n we write $\Delta^{\alpha} x_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1} x_v$, whenever the series convergent, and

$$X^{\beta} = \left\{ \varepsilon = (\varepsilon_v) : \sum_{v=0}^{\infty} \varepsilon_v x_v \text{ is convergent for every } x \in X \right\},$$

which is the β dual of X. Also we need the following notations for $v=1,2,\ldots$

$$\Gamma_\alpha = \left\{ \varepsilon : \Delta^{\alpha}\left(\frac{\varepsilon_v}{v}\right) \text{ exists, } \sup_{m,r} \left| r A_r^{\alpha} \sum_{v=r}^{m} \frac{\varepsilon_v}{v} A_{v-r}^{-\alpha-1} \right| < \infty \right\}$$

and

$$\Gamma_\alpha^{k^*} = \left\{ \varepsilon : \Delta^\alpha \left(\frac{\varepsilon_\nu}{\nu} \right) \text{ exists}, \sup_m \sum_{r=1}^m \left| r^{1/k} A_r^\alpha \sum_{\nu=r}^m \frac{\varepsilon_\nu}{\nu} A_{\nu-r}^{-\alpha-1} \right|^{k^*} < \infty \right\}$$

where $k > 1$, $1/k + 1/k^* = 1$.

2. Main Results

The problems of absolute summability factors and comparision of these methods goes to old rather and uptill now were widely examined by many authors, (see, [1–9], [10, 11, 23], [25, 26]). By other viewpoint we note that most of these results correspond to the special matrices $I, W \in (|C_\alpha|, |C_\delta|_k)$ or $I, W \in (|C_\alpha|_k, |C_\delta|)$ where I is an identity matrix and the matrix $W = (w_{n\nu})$ defined by $w_{n\nu} = \varepsilon_\nu$ for $\nu = n$, zero otherwise.

In the present paper giving some algebraic and topological properties of $|C_\alpha|_k$ we characterize the classes of all infinite matrices $(|C_\alpha|, |C_\delta|_k)$ and $(|C_\alpha|_k, |C_\delta|)$, show that each element of this classes corresponds to a continuous linear mapping, which also enables us to extends some well known results of Flett [7], Orhan and Sarıgöl [15], Bosanquet [2], Mehdi [13], Mazhar [11], and Sarıgöl [18], where $|C_\alpha|_k$ is the space of series summable by the summability $|C, \alpha|_k$. Our theorems read as follows.

Theorem 2.1. *Let $\alpha > -1$, $1 < k < \infty$ and $1/k + 1/k^* = 1$. Then,*

$$\{|C_\alpha|_k\}^\beta = \Gamma_\alpha^{k^*} \quad \text{and} \quad \{|C_\alpha|\}^\beta = \Gamma_\alpha.$$

Theorem 2.2. *Let $\alpha > -1$ and $k \geq 1$. Then $|C_\alpha|_k$ is a BK-space with respect to the norm*

$$\|a\|_{|C_\alpha|_k} = \left\{ |a_0|^k + \sum_{n=1}^\infty \left| \frac{1}{n^{1/k} A_n^\alpha} \sum_{\nu=1}^n A_{n-\nu}^{\alpha-1} \nu a_\nu \right|^k \right\}^{1/k}. \tag{2.1}$$

Theorem 2.3. *Assume that $1 \leq k < \infty$, $\alpha > -1$, $\delta > -1$. Then, $(|C_\alpha|, |C_\delta|_k) \subset B(|C_\alpha|, |C_\delta|_k)$ and $A \in (|C_\alpha|, |C_\delta|_k)$ if and only if*

$$\Delta^\alpha \left(\frac{1}{j} a_{\nu j} \right) \text{ exists for } j, \nu = 1, 2, \ldots, \tag{2.2}$$

$$\sup_{m,j} \left| j A_j^\alpha \sum_{r=j}^m \frac{1}{r} A_{r-j}^{-\alpha-1} a_{\nu r} \right| < \infty \text{ for } \nu = 0, 1, \ldots, \tag{2.3}$$

$$\sum_{n=1}^\infty \frac{1}{n} \left| \frac{1}{A_n^\delta} \sum_{\nu=1}^n \nu A_{n-\nu}^{\delta-1} a_{\nu 0} \right|^k < \infty \tag{2.4}$$

and

$$\sup_j \sum_{n=1}^\infty \left| \frac{j A_j^\alpha}{n^{1/k} A_n^\delta} \sum_{\nu=1}^n \nu A_{n-\nu}^{\delta-1} \Delta^\alpha \left(\frac{1}{j} a_{\nu j} \right) \right|^k < \infty. \tag{2.5}$$

Theorem 2.4. *Assume that* $\alpha > -1$, $\delta > -1$, $1 < k < \infty$, $1/k + 1/k^* = 1$. *Then,* $(|C_\alpha|_k, |C_\delta|) \subset B(|C_\alpha|_k, |C_\delta|)$ *and* $A \in (|C_\alpha|_k, |C_\delta|)$ *if and only if* (2.2) *holds,*

$$\sup_m \sum_{j=1}^{m} \left| j^{1/k} A_j^{\alpha} \sum_{r=j}^{m} \frac{1}{r} A_{r-j}^{-\alpha-1} a_{vr} \right|^{k^*} < \infty, \quad v = 0, 1, \ldots, \tag{2.6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n A_n^{\delta}} \left| \sum_{v=1}^{n} v A_{n-v}^{\delta-1} a_{v0} \right| < \infty \tag{2.7}$$

and

$$\sum_{j=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{j^{1/k} A_j^{\alpha}}{n A_n^{\delta}} \left| \sum_{v=1}^{n} v A_{n-v}^{\delta-1} \Delta^{\alpha} \left(\frac{1}{j} a_{vj} \right) \right| \right\}^{k^*} < \infty. \tag{2.8}$$

3. Needed Lemmas

We need the following lemmas for the proof our theorems.

Lemma 3.1 ([17]). *Let* $1 < k < \infty$. *Then,* $A \in (l_k, l)$ *if and only if*

$$\sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} < \infty. \tag{3.1}$$

Lemma 3.2 ([10]). *Let* $1 \le k < \infty$. *Then,* $A \in (l, l_k)$ *if and only if*

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^k < \infty. \tag{3.2}$$

Lemma 3.3 ([24]). (a)

$$A \in (l, c) \Longleftrightarrow \begin{cases} \lim_n a_{nv} \text{ exists for each } v, \\ \sup_{n,v} |a_{nv}| < \infty. \end{cases} \tag{3.3}$$

(b) *Let* $1 < k < \infty$. *Then*

$$A \in (l_k, c) \Longleftrightarrow \begin{cases} \lim_n a_{nv} \text{ exists for each } v, \\ \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty. \end{cases} \tag{3.4}$$

Lemma 3.4 ([13]). *Let* $\beta > -1$, $1 \le k < \infty$ *and* $\sigma < \beta$. *Then, for* $k = 1$,

$$E_v = \begin{cases} O(v^{-\beta-1}), & \sigma \le -1 \\ O(v^{-\beta+\sigma}), & \sigma > -1 \end{cases}$$

and

$$E_v = \begin{cases} O(v^{-k\beta-1}), & \sigma < -1/k \\ O(v^{-k\beta-1} \log v), & \sigma = -1/k \\ O(v^{-k\beta+k\sigma}), & \sigma > -1/k \end{cases}$$

for $1 < k < \infty$, *where* $E_v = \sum_{n=v}^{\infty} \frac{|A_{n-v}^{\delta}|^k}{n (A_n^{\beta})^k}$ *for* $v \ge 1$.

Proof of Theorem 2.1. Let $\varepsilon \in \{|C_\alpha|_k\}^\beta$. Then, $\sum\limits_{\nu=0}^{\infty} \varepsilon_\nu x_\nu$ is convergent for every $x \in |C_\alpha|_k$. But $x \in |C_\alpha|_k$ if and only if $T \in l_k$, where

$$T_0^\alpha = x_0,\ T_n^\alpha = \frac{1}{n^{1/k} A_n^\alpha} \sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1} \nu x_\nu, \quad \text{for } n \ge 1. \tag{3.5}$$

By inversion of (3.5), we write, for $r \ge 1$,

$$x_r = \frac{1}{r} \sum_{\nu=1}^{r} A_{r-\nu}^{-\alpha-1} \nu^{1/k} A_\nu^\alpha T_\nu^\alpha \tag{3.6}$$

and so

$$\begin{aligned}\sum_{\nu=0}^{m} \varepsilon_\nu x_\nu &= \varepsilon_0 T_0^\alpha + \sum_{\nu=1}^{m} \frac{\varepsilon_\nu}{\nu} \sum_{r=1}^{\nu} A_{\nu-r}^{-\alpha-1} r^{1/k} A_r^\alpha T_r^\alpha \\ &= \varepsilon_0 T_0^\alpha + \sum_{r=1}^{m} \left(r^{1/k} A_r^\alpha \sum_{\nu=r}^{m} \frac{\varepsilon_\nu}{\nu} A_{\nu-r}^{-\alpha-1} \right) T_r^\alpha \\ &= \sum_{r=0}^{m} w_{mr} T_r^\alpha,\end{aligned}$$

where

$$w_{mr} = \begin{cases} \varepsilon_0, & r = 0 \\ r^{1/k} A_r^\alpha \sum\limits_{\nu=r}^{m} \dfrac{\varepsilon_\nu}{\nu} A_{\nu-r}^{-\alpha-1}, & 1 \le r \le m \\ 0, & r > m \end{cases} \tag{3.7}$$

$\varepsilon \in \{|C_\alpha|_k\}^\beta \iff W \in (l_k, c)$. Therefore it follows from Lemma 3.3 that $\varepsilon \in \{|C_\alpha|_k\}^\beta$ iff

$$\sup_m \sum_{r=0}^{\infty} |w_{mr}|^{k^*} = \sup_m \left\{ |\varepsilon_0|^{k^*} + \sum_{r=1}^{m} \left| r^{1/k} A_r^\alpha \sum_{\nu=r}^{m} \frac{\varepsilon_\nu}{\nu} A_{\nu-r}^{-\alpha-1} \right|^{k^*} \right\} < \infty$$

and $\lim\limits_m w_{mr} = \Delta^\alpha \left(\dfrac{\varepsilon_r}{r} \right)$ exists for $r = 1, 2, \ldots$, that is to say, $\varepsilon \in \Gamma_\alpha^{k^*}$, completing the proof. □

The second part of the lemma is similarly proved by Lemma 3.3.

Proof of Theorem 2.2. It is easily seen from Minkowski inequality that $|C_\alpha|_k$ is a normed space with norm (2.1). Now, take a Cauchy sequence $\xi = (\xi^m)$ where $\xi^m = (a_\nu^m) \in |C_\alpha|_k$ $(m = 0, 1, \ldots)$. Given $\varepsilon > 0$. Then there exists at least a positive integer n_0 such that

$$\|\xi^{m_1} - \xi^{m_2}\|_{|C_\alpha|_k} < \varepsilon \tag{3.8}$$

for $m_1, m_2 > n_0$. This implies that $|a_\nu^{m_1} - a_\nu^{m_2}| \to 0$ as $m_1, m_2 \to \infty$. This means that $(a_\nu^{m_1})$ is a Cauchy sequence in C and so there exists limit $a_\nu^{m_1} \to x_\nu$ $(\nu = 0, 1, \ldots)$ as $m_1 \to \infty$, say. So it follows from (3.8) that $\|\xi^{m_1} - x\|_{|C_\alpha|_k} < \varepsilon$ for $m_1 > n_0$ and $x \in |C_\alpha|_k$. Therefore $|C_\alpha|_k$ is a Banach space. Finally, a coordinate functional $P_n : |C_\alpha|_k \to C$, $P_n(a) = a_n$ $(n = 0, 1, \ldots)$ is continuous since

$$\|P_n(a)\| = |a_n| \le \left(\frac{1}{n} \sum_{\nu=1}^{n} |A_{n-\nu}^{-\alpha-1}| \nu^{1/k} A_\nu^\alpha \right) \|a\|_{|C_\alpha|_k}$$

by (3.5) and (3.6). This completes the proof. □

Proof of Theorem 2.3. First part is seen by applying Banach Steinhause theorem in usual way since $|C_\alpha|$ is a BK-spaces by Theorem 2.2. Now take $A \in (|C_\alpha|, |C_\delta|_k)$. Then $A_n(x) = \sum a_{n\nu}x_\nu$ is convergent for every $x \in |C_\alpha|$ and $Ax = (A_n(x)) \in |C_\delta|_k$. This also means that $(a_{n0}, a_{n1}, \ldots) \in \{|C_\alpha|_k\}^\beta$ which is equivalent to (2.2) and (2.3) by Theorem 2.1. Now the statement (3.5) with $k = 1$ gives

$$T_0^\alpha = x_0,\ T_n^\alpha = \frac{1}{nA_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^{\alpha-1}\nu x_\nu, \quad \text{for } n \geq 1. \tag{3.9}$$

of which inversion implies $x_r = \frac{1}{r}\sum_{\nu=1}^{r} A_{r-\nu}^{-\alpha-1}\nu A_\nu^\alpha T_\nu^\alpha$, $r \geq 1$. Also, $x \in |C_\alpha| \Longleftrightarrow T^\alpha \in l$. Then we get $L_0^\delta = \sum_{\nu=0}^{\infty} a_{0\nu}x_\nu = T_0^\alpha = x_0$ and for $n \geq 1$

$$\begin{aligned}
L_n^\delta &= \frac{1}{n^{1/k}A_n^\delta}\sum_{\nu=1}^{n} A_{n-\nu}^{\delta-1}\nu A_\nu(x) \\
&= \frac{1}{n^{1/k}A_n^\delta}\sum_{\nu=1}^{n} A_{n-\nu}^{\delta-1}\nu \sum_{r=0}^{\infty} a_{\nu r}x_r \\
&= \frac{1}{n^{1/k}A_n^\delta}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\delta-1}\left(a_{\nu 0}T_0^\alpha + \sum_{r=1}^{\infty} a_{\nu r}\frac{1}{r}\sum_{j=1}^{r} A_{r-j}^{-\alpha-1} jA_j^\alpha T_j^\alpha\right) \\
&= \frac{1}{n^{1/k}A_n^\delta}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\delta-1}(a_{\nu 0}T_0^\alpha + U_\nu), \text{ say.}
\end{aligned}$$

On the other hand, it follows from (2.3) that the series

$$\sum_{j=1}^{\infty} jA_j^\alpha\left(\sum_{r=j}^{m}\frac{1}{r}a_{\nu r}A_{r-j}^{-\alpha-1}\right)T_j^\alpha$$

convergent in uniformly in m, and so we get

$$\begin{aligned}
U_\nu &= \lim_m \sum_{r=1}^{m} a_{\nu r}\frac{1}{r}\sum_{j=1}^{r} A_{r-j}^{-\alpha-1} jA_j^\alpha T_j^\alpha \\
&= \lim_m \sum_{j=1}^{m} jA_j^\alpha\left(\sum_{r=j}^{m}\frac{1}{r}a_{\nu r}A_{r-j}^{-\alpha-1}\right)T_j^\alpha \\
&= \sum_{j=1}^{\infty} jA_j^\alpha \Delta^\alpha\left(\frac{a_{\nu j}}{j}\right)T_j^\alpha.
\end{aligned}$$

Therefore

$$\begin{aligned}
L_n^\delta &= \frac{1}{n^{1/k}A_n^\delta}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\delta-1}\left\{a_{\nu 0}T_0^\alpha + \sum_{j=1}^{\infty} jA_j^\alpha \Delta^\alpha\left(\frac{a_{\nu j}}{j}\right)T_j^\alpha\right\} \\
&= \frac{1}{n^{1/k}A_n^\delta}\left\{\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\delta-1}a_{\nu 0}T_0^\alpha + \sum_{j=1}^{\infty}\sum_{\nu=1}^{n} \nu A_{n-\nu}^{\delta-1} jA_j^\alpha \Delta^\alpha\left(\frac{a_{\nu j}}{j}\right)T_j^\alpha\right\} \\
&= \sum_{j=0}^{\infty} b_{nj}T_j^\alpha,
\end{aligned}$$

where

$$b_{nj}=\begin{cases}\dfrac{1}{n^{1/k}A_n^\delta}\sum\limits_{v=1}^{n} vA_{n-v}^{\delta-1}a_{v0}, & j=0,\ n\ge 1\\ \dfrac{jA_j^\alpha}{n^{1/k}A_n^\delta}\sum\limits_{v=1}^{n} vA_{n-v}^{\delta-1}\Delta^\alpha\left(\dfrac{a_{vj}}{j}\right), & n\ge 1,\ j\ge 1\end{cases}$$

Now, $A\in(|C_\alpha|,|C_\delta|_k)\Longleftrightarrow B\in(l,l_k)$, i.e., equivalently, $\sup\limits_j\sum\limits_{n=0}^{\infty}|b_{nj}|^k<\infty$ by Lemma 3.2. Thus, it follows from the definition of the matrix B that

$$\sup_j\sum_{n=0}^{\infty}|b_{nj}|^k=\sup_{j\ge 1}\left\{\sum_{n=1}^{\infty}|b_{n0}|^k+\sum_{n=1}^{\infty}|b_{nj}|^k\right\}<\infty$$

which is satisfied if and only if the conditions (2.4) and (2.5) hold, completing the proof. □

Proof of Theorem 2.4. First part is seen by applying Banach Steinhause theorem in usual way since $|C_\alpha|_k$ is a BK-spaces by Theorem 2.2. Now take $A\in(|C_\alpha|_k,|C_\delta|)$. Then $A_n(x)=\sum\limits_{v=0}^{\infty}a_{nv}x_v$ is convergent for every $x\in|C_\alpha|_k$ and $A(x)=(A_n(x))\in|C_\delta|$. This also gives us $(a_{n0},a_{n1},\dots)\in\{|C_\alpha|_k\}^\beta$ which is the same as (2.2) and (2.6). Now by considering (3.5) we write that $x\in|C_\alpha|_k\Longleftrightarrow T^\alpha\in l_k$ and

$$\begin{aligned}A_n(x)&=\sum_{r=0}^{\infty}a_{nr}x_r\\ &=a_{n0}T_0^\alpha+\sum_{r=1}^{\infty}a_nr\frac{1}{r}\sum_{v=1}^{r}A_{r-v}^{-\alpha-1}v^{1/k}A_v^\alpha T_v^\alpha\\ &=a_{n0}T_0^\alpha+\lim_m\sum_{v=1}^{m}v^{1/k}A_v^\alpha\left(\sum_{r=v}^{m}\frac{1}{r}A_{r-v}^{-\alpha-1}a_{nr}\right)T_v^\alpha.\end{aligned}$$

As in proof of Theorem 2.3,

$$A_n(x)=a_{n0}T_0^\alpha+\sum_{v=1}^{\infty}v^{1/k}A_v^\alpha\left(\sum_{r=v}^{\infty}\frac{1}{r}A_{r-v}^{-\alpha-1}a_{nr}\right)T_v^\alpha.$$

Now, $A(x)=A_n(x)\in|C_\delta|$ means that $L^\delta=(L_n^\delta)\in l$, where $L_0^\delta=A_0(x)=x_0$ and $L_n^\delta=\dfrac{1}{nA_n^\delta}\sum\limits_{v=1}^{n}A_{n-v}^{\delta-1}vA_v(x)$ for $n\ge 1$. On the other hand, we can write

$$\begin{aligned}L_n^\delta&=\frac{1}{nA_n^\delta}\sum_{v=1}^{n}A_{n-v}^{\delta-1}vA_v(x)\\ &=\frac{1}{nA_n^\delta}\sum_{v=1}^{n}vA_{n-v}^{\delta-1}\sum_{r=0}^{\infty}a_{vr}x_r\\ &=\frac{1}{nA_n^\delta}\left\{\sum_{v=1}^{n}vA_{n-v}^{\delta-1}a_{v0}T_0^\alpha+\sum_{j=1}^{\infty}j^{1/k}A_j^\alpha\sum_{v=1}^{n}vA_{n-v}^{\delta-1}\Delta^\alpha\left(\frac{1}{j}a_{vj}\right)\right\}T_j^\alpha\\ &=\sum_{j=0}^{\infty}d_{nj}T_j^\alpha\end{aligned}$$

where

$$d_{nj} = \begin{cases} \dfrac{1}{nA_n^\delta} \sum\limits_{v=1}^{n} vA_{n-v}^{\delta-1} a_{v0}, & j=0,\ n\geq 1 \\ \dfrac{j^{1/k} A_j^\alpha}{nA_n^\delta} \sum\limits_{v=1}^{n} vA_{n-v}^{\delta-1} \Delta^\alpha \left(\dfrac{1}{j} a_{vj}\right), & j\geq 1,\ n\geq 1 \end{cases}$$

By Lemma 3.1, $A \in (|C_\alpha|_k, |C_\delta|)$ iff $D \in (l_k, l)$, i.e., equivalently,

$$\sum_{j=0}^{\infty} \left(\sum_{n=0}^{\infty} |d_{nj}| \right)^{k^*} < \infty. \tag{3.10}$$

But, (3.10) holds if and only if

$$\sum_{n=1}^{\infty} |d_{n0}| = \sum_{n=1}^{\infty} \frac{1}{nA_n^\delta} \left| \sum_{v=1}^{n} vA_{n-v}^{\delta-1} a_{v0} \right| < \infty$$

and

$$\sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} |d_{nj}| \right)^{k^*} = \sum_{j=1}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{j^{1/k} A_j^\alpha}{nA_n^\delta} \left| \sum_{v=1}^{n} vA_{n-v}^{\delta-1} \Delta^\alpha \left(\frac{1}{j} a_{vj} \right) \right| \right\}^{k^*} < \infty.$$

Therefore the proof is completed. □

4. Applications

Our theorems include some well known results. Now we list them with our notations.

Corollary 4.1 ([7]). *If $\alpha > -1$, $\beta > \alpha + \frac{1}{k^*}$ and $k \geq 1$, then $I \in (|C_\alpha|, |C_\beta|_k)$.*

Proof. Consider the special case $A = I$ in Theorem 2.3. Then, it is clear that (2.2), (2.3) and (2.4) hold. On the other hand, since

$$\Delta^\alpha \left(\frac{1}{j} a_{vj} \right) = \sum_{m=j}^{\infty} A_{m-j}^{-\alpha-1} \frac{1}{m} a_{vm} = \begin{cases} \dfrac{1}{v} A_{v-j}^{-\alpha-1}, & 1 \leq j \leq v \\ 0, & j > 0 \end{cases} \tag{4.1}$$

we have, by Lemma 3.4,

$$\sup_j \sum_{n=1}^{\infty} \left| \frac{jA_j^\alpha}{n^{1/k} A_n^\beta} \sum_{v=1}^{n} vA_{n-v}^{\beta-1} \Delta^\alpha \left(\frac{1}{j} a_{vj} \right) \right|^k = \sup_j (jA_j^\alpha)^k \sum_{n=j}^{\infty} \frac{|A_{n-v}^{\beta-\alpha-1}|^k}{n(A_n^\beta)^k} < \infty$$

for $\beta > \alpha + \dfrac{1}{k^*}$. So (2.5) holds, which completes the proof. □

It is well known that the case $k = 1$ and $\beta > \alpha$ of this result was given by Kogbetliantz [8].

Corollary 4.2 ([7]). *If $\alpha,\beta > -1$ and $k > 1$ then $I \notin (|C_(\beta)|_k, |C_\alpha|)$.*

Proof. Take the special case $A = I$ in Theorem 2.4. Then, using (4.1) we have, by Lemma 3.4,

$$\sum_{j=1}^{\infty}\left\{\sum_{n=1}^{\infty}\frac{j^{1/k}A_j^{\beta}}{nA_n^{\alpha}}\left|\sum_{v=1}^{n} vA_{n-v}^{\alpha-1}\Delta^{\beta}\left(\frac{1}{j}a_{vj}\right)\right|\right\}^{k^*} \geq \sum_{j=1}^{\infty}(j^{1/k}A_j^{\alpha})^{k^*}\left|\sum_{n=j}^{\infty}\frac{A_{n-j}^{\alpha-\beta-1}}{n(A_n^{\alpha})}\right|^{k^*} = \infty,$$

i.e., (2.8) does not hold. This completes the proof. □

Corollary 4.3 ([13]). *If $k > 1$, $\beta \geq 0$ and α is nonnegative integer, then $W \in (|C_\alpha|, |C_\beta|_k)$ if and only if*

(i) $\Delta^{\alpha}\varepsilon_v = O(v^{-\alpha})$

(ii.a) $\varepsilon_v = O(v^{\beta-\alpha-1+1/k})$ $(\beta < \alpha + 1/k^*)$,

(ii.b) $\varepsilon_v = O((\log v)^{-1/k})$ $(\beta = \alpha + 1/k^*)$,

(ii.c) $\varepsilon_v = O(1)$, $(\beta > \alpha + 1/k^*)$

When $\beta = \frac{1}{k^*}$ (i) has to be strengthened by factor $(\log v)^{-1/k}$. Conditions for the case $k = 1$ were obtained by Bosanquet [2], *Chow* [5] *and Peyerimhoff* [16]*; cf. also Bosanquet and Chow* [4].

Proof. Put $A = W$ in Theorem 2.3. Then the conditions (2.2), (2.3) and (2.4) are satisfied and the condition (2.5) is reduced to

$$\sup_j (jA_j^{\alpha})^k \sum_{n=j}^{\infty}\frac{1}{n(A_n^{\beta})^k}\left|\sum_{v=j}^{n}A_{n-v}^{\beta-1}A_{n-j}^{-\alpha-1}\varepsilon_v\right|^k < \infty \tag{4.2}$$

which are equivalent to the conditions of Corollary 4.3, see, for detail, in [13]. □

Corollary 4.4 ([11]). *Let $\alpha \geq 0$, $k > 1$. Then, $W \in (|C_\alpha|_k, |C_1|)$ if and only if*

(i) $\sum_{v=1}^{\infty} v^{(\alpha+1)k^*-1}\left|\Delta^{\alpha}\left(\frac{\varepsilon_v}{v}\right)\right|^{k^*} < \infty$

(iia) $\sum_{v=1}^{\infty}\frac{1}{v}|\varepsilon_v|^{k^*} < \infty$, $\alpha \leq 1$,

(iib) $\sum_{v=1}^{\infty} v^{\alpha k^*-k^*-1}|\varepsilon_v|^{k^*} < \infty$, $\alpha > 1$.

Proof. In Theorem 2.4, take $A = W$ and $\delta = 1$. Then it is clear that the conditions (2.2), (2.6) and (2.7) are satisfied, and also (2.8) is reduced to the condition

$$\sum_{j=1}^{\infty} j^{k^*-1}(A_j^{\alpha})^{k^*}\left\{\sum_{n=j}^{\infty}\frac{1}{n(n+1)}\left|\sum_{v=j}^{n}A_{v-j}^{-\alpha-1}\varepsilon_v\right|\right\}^{k^*} < \infty \tag{4.3}$$

which is the same as the above conditions. In fact, for $\alpha \ge 0$, since $|C_0|_k \subset |C_\alpha|_k$ by Kogbetliantz [8], we get $W \in (|C_0|_k, |C_1|)$ whenever $W \in (|C_\alpha|_k, |C_1|)$. Now, it follows from (4.3) that $\sum_{\nu=1}^{\infty} \frac{1}{\nu}|\varepsilon_\nu|^{k^*} < \infty$, and so $|\varepsilon_\nu| = O(\nu^{\frac{1}{k^*}})$. Using (4.3), we have the condition (i) since

$$\begin{aligned}\sum_{n=j}^{\infty} \frac{1}{n(n+1)} \sum_{\nu=j}^{n} |A_{\nu-j}^{-\alpha-1}\varepsilon_\nu| &= \sum_{\nu=j}^{\infty} |A_{\nu-j}^{-\alpha-1}\varepsilon_\nu| \sum_{n=j}^{\infty} \frac{1}{n(n+1)} \\ &= \sum_{\nu=j}^{\infty} \left|A_{\nu-j}^{-\alpha-1}\frac{\varepsilon_\nu}{\nu}\right| \\ &= O(1)\sum_{\nu=j}^{\infty} |A_{\nu-j}^{-\alpha-1}| < \infty.\end{aligned}$$

Similarly, (4.3) directly implies (iib). Therefore the conditions (i), (iia) and (iib) are necessary. Sufficiency is shown as in [11]. □

Corollary 4.5 ([15]). *Let $k \ge 1$. Then, $I \in (|R_p|, |R_q|_k)$ if and only if*

(i) $\dfrac{P_\nu q_\nu}{Q_\nu p_\nu} = O(\nu^{(1/k)} - 1)$,

(ii) $q_\nu K_\nu = O\left(\frac{p_\nu}{P_\nu}\right)$,

(iii) $Q_\nu K_\nu = O(1)$,

where

$$K_\nu = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k \right\}^{\frac{1}{k}}.$$

Proof. In Theorem 2.3, take we take the matrix A as

$$a_{n\nu} = \begin{cases} \dfrac{q_n P_\nu P_{\nu-1}}{Q_n Q_{n-1} p_\nu} \Delta\left(\dfrac{Q_{\nu-1}}{P_{\upsilon-1}}\right), & 1 \le \nu \le n-1 \\ \dfrac{q_n P_n}{Q_n p_n}, & \nu = n \\ 0, & \nu > n. \end{cases}$$

Then, it is easy to see that, $I \in (|R_p|, |R_q|_k)$ iff $A \in (|C_\alpha|_1, |C_0|_k)$ and moreover the condition (2.5) is reduced to

$$\sup_\nu \left\{ \left(\nu^{k-1} \frac{q_\nu P_\nu}{Q_\nu p_\nu}\right)^k + \left| \frac{P_\nu P_{\nu-1}}{p_\nu} \Delta\left(\frac{Q_{\nu-1}}{P_{\nu-1}}\right)\right|^k \{K_\nu\}^k \right\} < \infty \tag{4.4}$$

which is equivalent to the conditions (i), (ii) and (iii) of Corollary 4.5.

The case $k = 1$ of this result was given by Bosanquet [2] and Sunouchi [26]. □

Corollary 4.6 ([18]). *If A is a lower triangular infinite matrix then $A \in (\mathscr{A}_1, \mathscr{A}_k)$, $k \ge 1$, if and only if*

$$\sup_{v} \sum_{n=v}^{\infty} n^{k-1} |\hat{a}_{nv}|^k < \infty.$$

Proof. If A is defined by (1.5), then, $A \in (\mathscr{A}_1, \mathscr{A}_k)$ iff $\hat{A} \in (|C_0|, |C_0|_k)$. Thus, the result is obtained from Theorem 2.3. □

5. Conclusion

In the present paper new series spaces have been introduced making use of absolute Cesàro summability $|C, \alpha|_k$, and some algebraic, topological properties and matrix operators on that space have been investigated. And so it has been brought a different perspective and studying field.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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