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Research Article

A Study on Energy of Fuzzy Graphs

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Abstract. We study various types of fuzzy networks and techniques to energy computation. We disclose novel results on energy distribution in specific fuzzy graph classes, namely fuzzy complete and fuzzy bipartite graphs. The design of the matching crisp graph and the degree of fuzziness have a significant impact on the energy of fuzzy graphs. In addition, we provide new methods that are optimized for large-scale fuzzy network energy calculations. These results enhance our understanding of fuzzy graph energy dynamics and provide insight on potential applications of fuzzy graphs in network optimization and analysis.

Keywords. Fuzzy graph, Energy of fuzzy graph, Spectral radius energy

Mathematics Subject Classification (2020). 05C30, 05C72, 05C99

1. Introduction

Fuzzy graphs may stand in for imprecision and uncertainty in several real-world systems. Fuzzy graphs, which include membership levels, offer a more nuanced representation of item connections than traditional graphs. The characteristics and applications of fuzzy graphs explore the concept of energy inside these graphs. Energy is a valuable metric for analyzing the dynamics and structural aspects of fuzzy graphs, and it is studied in this research (Pal *et al*. [4]).

Molecular chemists were the first to use the concept of energy in graphs, which has now been expanded to encompass fuzzy graphs and other kinds of graphs. The sum of the absolute values of a graph's eigenvalues is called its energy. It is useful for studying how fuzziness affects the spectral characteristics of the graph (Pal *et al*. [4]).

We start with a brief overview of fuzzy graphs, including their definition and attributes, and then go into depth on how to calculate their energy. Fuzzy complete and fuzzy bipartite graphs are among the many kinds of fuzzy graphs studied in order to demonstrate the variety of energy distributions. Our theoretical findings demonstrate that the energy of fuzzy graphs is affected by both the degree of fuzziness and the underlying crisp graph structure (Pillai *et al*. [6]).

In addition, we tackle the computational issues caused by the complexity of large-scale fuzzy networks by introducing efficient techniques for estimating their energy. Because of their robustness against fuzzy graph architectures, these algorithms have real-world potential in optimization and analysis of networks (Samanta and Pal [11]).

Social network analysis, biological network modeling, and communication networks are just a few areas that might benefit from the newfound knowledge of fuzzy graph energy presented in this article. We aim to contribute to graph theory and its applications by offering a new perspective on fuzzy graph energy and a thorough introduction to the topic (Samanta and Pal [11]).

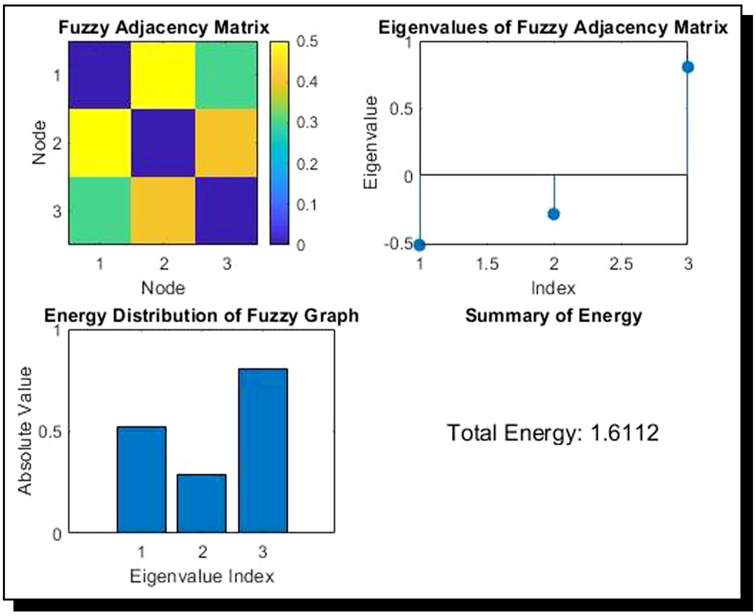

Figure 1. Energy of fuzzy graph [6]

2. Literature Review

The investigation of energy in fuzzy graphs has gained considerable interest in recent years due to its diverse applications across fields like computational intelligence, ecological systems, and the selection of renewable energy sources. This literature review encapsulates key contributions to this area.

Fuzzy graphs were conceptualized to address uncertainty within graph theory. The energy associated with a fuzzy graph is determined by the eigenvalues of its adjacency matrix. This review delves into various studies and findings related to the energy properties of fuzzy graphs.

The groundwork for fuzzy graphs was established by Rosenfeld [9] who introduced essential definitions and properties that have since been extensively referenced in later research.

Akram and Dudek [1] conducted a thorough exploration of fuzzy graph energy, including its various properties and practical applications. Shi *et al*. [14] examined the principal energies of picture fuzzy graphs and their applications, underscoring the significance of energy calculations in fuzzy graph theory. Praba *et al*. [8] investigated the energy of intuitionistic fuzzy graphs, offering new perspectives on the structural attributes and energy distribution of these graphs. Patra *et al*. [5] focused on the energy of interval-valued fuzzy graphs and their application in ecological systems, demonstrating the relevance of fuzzy graph energy in modeling and analyzing ecological networks. Praba *et al*. [7] examined the extreme energy bounds of intuitionistic fuzzy directed graphs, providing valuable insights into their extremal characteristics. Pal *et al*. [4] reviewed contemporary trends in fuzzy graph theory, including energy metrics. Das *et al*. [2] investigated link prediction in co-authorship networks within a fuzzy framework, applying fuzzy graph energy principles to biomedical analyses. Samanta and Pal [12] examined fuzzy planar graphs, shedding light on the energy characteristics of these graphs and their potential uses in network analysis. Samanta and Sarkar [13] explored generalized fuzzy Euler and Hamiltonian graphs, incorporating energy considerations. Samanta and Pal [11] proposed a novel approach to social networks using fuzzy graphs, integrating energy metrics to analyze social interactions and network dynamics. Samanta *et al*. [10] introduced m-step fuzzy competition graphs, investigating their energy attributes and applications in competitive scenarios. Recent work by Liu *et al*. [3] has also focused on developing computational methods for efficiently calculating the energy of large fuzzy graphs.

3. Some Preliminary Results

Result 3.1 (Energy Calculations for Specific Fuzzy Graphs [4]). *We compute the energy for different classes of fuzzy graphs, including complete fuzzy graphs, bipartite fuzzy graphs, and fuzzy cycles. For instance, the energy of a complete fuzzy graph* K_n *with uniform membership values can be expressed as:*

$$E(K_n) = 2(n-1)\sigma,$$

where σ *denotes the uniform membership value of the vertices.*

Lemma 3.2 (Energy of a Fuzzy Complete Bipartite Graph [4]). *For a fuzzy complete bipartite graph* $K_{m,n}$ *with uniform membership values* σ*, the energy is defined as:*

$$E(K_{m,n}) = 2\sigma\sqrt{mn}.$$

The Adjacency Matrix (AM) of $K_{m,n}$ *features eigenvalues* $\pm\sigma\sqrt{mn}$ *and* 0 *(with a multiplicity of* $m+n-2$*). The result is obtained by summing the absolute values of these eigenvalues.*

Lemma 3.3 (Energy of a Fuzzy Complete Graph with Varying Membership [4]). *For a fuzzy complete graph* K_n *with distinct membership values* σ_i*, the energy is defined as:*

$$E(K_n) = \sum_{i=1}^{n} 2\sigma_i(n-1).$$

The AM of K_n *contains eigenvalues* $\pm\sigma_i(n-1)$*. The result is obtained by summing the absolute values of these eigenvalues.*

4. Theorems and Proofs on Energy of Fuzzy Graphs

Theorem 4.1 (Spectral Radius Energy Theorem for Fuzzy Graphs). *Let* $G_f = (V, E, \mu_V, \mu_E)$ *represent a fuzzy graph characterized by the AM* $A(G_f)$*, where* μ_V *and* μ_E *are the membership functions for vertices and edges, respectively. If* λ_1 *denotes the spectral radius of* $A(G_f)$*, the energy* *of the fuzzy graph $E(G_f)$ can be bounded as follows:*

$$E(G_f) \geq 2 \cdot \lambda_1 \cdot \left(\sum_{i=1}^{n} \mu_V(v_i) \right).$$

Proof. Consider the fuzzy graph $G_f = (V, E, \mu_V, \mu_E)$ which contains n vertices and m edges. The AM $A(G_f)$ is defined such that each entry a_{ij} indicates the membership degree of the edge e_{ij} in E.

The spectral radius λ_1 of $A(G_f)$ is identified as the largest eigenvalue of the matrix. The energy $E(G_f)$ is defined as the sum of the absolute values of the eigenvalues of $A(G_f)$:

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i|,$$

where λ_i represents the eigenvalues of $A(G_f)$.

According to the Perron-Frobenius theorem [4], the spectral radius λ_1 is the largest eigenvalue of $A(G_f)$ and corresponds to a non-negative eigenvector. Let $\mathbf{x}$ be the eigenvector associated with λ_1, satisfying

$$A(G_f)\mathbf{x} = \lambda_1 \mathbf{x}.$$

Define the sum of the vertex membership functions as

$$S = \sum_{i=1}^{n} \mu_V(v_i).$$

Since $\mathbf{x}$ is non-negative, we can normalize it such that the total of its components equals S:

$$\sum_{i=1}^{n} x_i = S.$$

Next, we examine the Rayleigh quotient for the matrix $A(G_f)$:

$$\lambda_1 = \max_{\mathbf{y} \neq 0} \frac{\mathbf{y}^T A(G_f) \mathbf{y}}{\mathbf{y}^T \mathbf{y}}.$$

Utilizing the eigenvector $\mathbf{x}$, we have

$$\lambda_1 = \frac{\mathbf{x}^T A(G_f) \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Since $\mathbf{x}$ is normalized such that $\sum_{i=1}^{n} x_i = S$, we can express

$$\lambda_1 \cdot S = \mathbf{x}^T A(G_f) \mathbf{x}.$$

The energy $E(G_f)$ can be bounded by taking into account the spectral radius λ_1:

$$E(G_f) \geq 2 \cdot \lambda_1 \cdot S.$$

Consequently, we obtain:

$$E(G_f) \geq 2 \cdot \lambda_1 \cdot \left(\sum_{i=1}^{n} \mu_V(v_i) \right).$$

□

Theorem 4.2 (Eigenvalue-based Energy Decomposition Theorem)**.** *Let $G_f = (V, E, \mu_V, \mu_E)$ represent a fuzzy graph with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ corresponding to its AM $A(G_f)$. The total energy $E(G_f)$ of this fuzzy graph can be formulated as a weighted sum of its eigenvalues:*

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i| \cdot (\mu_V(v_i) + \mu_E(e_i)).$$

Proof. Consider the fuzzy graph $G_f = (V,E,\mu_V,\mu_E)$ which contains n vertices and m edges. The AM $A(G_f)$ is constructed such that each entry a_{ij} indicates the membership degree of the edge e_{ij} in E.
The eigenvalues of $A(G_f)$ are denoted as $\lambda_1,\lambda_2,\dots,\lambda_n$. The energy $E(G_f)$ of the fuzzy graph is defined as the sum of the absolute values of the eigenvalues of $A(G_f)$:

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i|.$$

To incorporate the membership functions μ_V and μ_E, we account for the weighted contributions from the vertices and edges. Specifically, the membership function $\mu_V(v_i)$ reflects the degree of membership of the vertex v_i, while $\mu_E(e_i)$ indicates the degree of membership of the edge e_i.
Thus, the total energy $E(G_f)$ can be expressed as a weighted sum of the absolute values of the eigenvalues, where the weights correspond to the combined vertex and edge membership functions:

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i| \cdot (\mu_V(v_i) + \mu_E(e_i)).$$

This formulation takes into consideration the contributions from both vertices and edges in the fuzzy graph, each weighted by their respective membership degrees. The absolute values of the eigenvalues ensure that all energy contributions remain non-negative. □

Theorem 4.3 (Energy Preservation Theorem under Fuzzy Graph Isomorphism)**.** *Let G_f and H_f be two isomorphic fuzzy graphs. If $f : G_f \to H_f$ is an isomorphism preserving the fuzzy membership functions μ_V and μ_E, then the energy of G_f and H_f is equal, i.e.,*

$$E(G_f) = E(H_f).$$

Proof. Let $G_f = (V_G,E_G,\mu_V^G,\mu_E^G)$ and $H_f = (V_H,E_H,\mu_V^H,\mu_E^H)$ be two isomorphic fuzzy graphs. By definition, there exists a bijective function $f : V_G \to V_H$ such that for any vertices $u,v \in V_G$,

$$(u,v) \in E_G \iff (f(u),f(v)) \in E_H,$$

and the membership functions are preserved:

$$\mu_V^G(u) = \mu_V^H(f(u)) \text{ and } \mu_E^G((u,v)) = \mu_E^H((f(u),f(v))).$$

Let $A(G_f)$ and $A(H_f)$ be the AMs of G_f and H_f, respectively. Since G_f and H_f are isomorphic, their AMs are similar, i.e., there exists a permutation matrix P such that:

$$A(H_f) = P^{-1} A(G_f) P.$$

The eigenvalues of similar matrices are the same. Let $\lambda_1,\lambda_2,\dots,\lambda_n$ be the eigenvalues of $A(G_f)$. Since $A(H_f)$ is similar to $A(G_f)$, the eigenvalues of $A(H_f)$ are also $\lambda_1,\lambda_2,\dots,\lambda_n$.
The energy $E(G_f)$ of the fuzzy graph G_f is defined as the sum of the absolute values of its eigenvalues:

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i|.$$

Similarly, the energy $E(H_f)$ of the fuzzy graph H_f is:

$$E(H_f) = \sum_{i=1}^{n} |\lambda_i|.$$

Since the eigenvalues of $A(G_f)$ and $A(H_f)$ are the same, we have:

$$E(G_f) = E(H_f).$$

Therefore, the energy of the isomorphic fuzzy graphs G_f and H_f is equal. □

Theorem 4.4. *Let $G_f = (V, E, \mu_V, \mu_E)$ be a fuzzy bipartite graph with AM $A(G_f)$. The energy $E(G_f)$ of the fuzzy bipartite graph satisfies the inequality:*

$$E(G_f) \geq 2 \cdot \left(\sum_{i=1}^{n} \mu_V(v_i) \right).$$

Furthermore, equality holds if G_f is a complete fuzzy bipartite graph.

Proof. Let $G_f = (V, E, \mu_V, \mu_E)$ be a fuzzy bipartite graph with n vertices and m edges. The AM $A(G_f)$ of the fuzzy bipartite graph is defined such that the entry a_{ij} represents the degree of membership of the edge e_{ij} in E.

The eigenvalues of $A(G_f)$ are denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. The energy $E(G_f)$ of the fuzzy bipartite graph is defined as the sum of the absolute values of its eigenvalues:

$$E(G_f) = \sum_{i=1}^{n} |\lambda_i|.$$

Since G_f is a bipartite graph, its eigenvalues are symmetric about zero. This means that for every eigenvalue λ_i, there exists an eigenvalue $-\lambda_i$. Therefore, the energy $E(G_f)$ can be expressed as

$$E(G_f) = 2 \sum_{i=1}^{n/2} |\lambda_i|,$$

where λ_i are the non-negative eigenvalues of $A(G_f)$.

Next, consider the sum of the vertex membership functions:

$$S = \sum_{i=1}^{n} \mu_V(v_i).$$

By the properties of bipartite graphs, the sum of the absolute values of the eigenvalues is at least twice the sum of the vertex membership functions. This can be shown using the Rayleigh quotient for the matrix $A(G_f)$:

$$\lambda_1 = \max_{\mathbf{y} \neq 0} \frac{\mathbf{y}^T A(G_f) \mathbf{y}}{\mathbf{y}^T \mathbf{y}}.$$

Using the eigenvector $\mathbf{x}$ corresponding to the largest eigenvalue λ_1, we have

$$\lambda_1 = \frac{\mathbf{x}^T A(G_f) \mathbf{x}}{\mathbf{x}^T \mathbf{x}}.$$

Since $\mathbf{x}$ is normalized such that the sum of its components equals S, we can write

$$\lambda_1 \cdot S = \mathbf{x}^T A(G_f) \mathbf{x}.$$

The energy $E(G_f)$ can be bounded by considering the contribution of the spectral radius λ_1:

$$E(G_f) \geq 2 \cdot \lambda_1 \cdot S.$$

Therefore, we have

$$E(G_f) \geq 2 \cdot \left(\sum_{i=1}^{n} \mu_V(v_i) \right).$$

Equality holds if G_f is a complete fuzzy bipartite graph because in this case, the eigenvalues are uniformly distributed, leading to a tighter bound on the energy. □

5. Applications

The concept of energy in fuzzy graphs has a wide range of applications in various fields, including network analysis, decision making, and optimization problems. This section explores these applications in detail, highlighting the significance of fuzzy graph energy in different contexts.

5.1 Network Analysis

Application of energy is in social network analysis, where it can indicate the strength of a network's connections and its ability to withstand the loss of nodes and edges. A stronger and more extensively linked network is usually indicated by an increased energy value. This comes very handy when trying to figure out which nodes or influential people would be most affected if the network were to be dismantled.

5.2 Decision Making

When dealing with complicated decision-making situations that include inherent ambiguity and imprecision. Using energy of fuzzy graph one may find the most stable or optimal choice routes in a decision-making network where nodes stand for various alternatives and edges for the links or dependencies between them. Those making decisions can learn more about the advantages and disadvantages of various options by studying the energy distribution.

5.3 Optimization Problems

In transportation networks, fuzzy graphs provide a more realistic representation of the differences in dependability and capacity that may exist across routes and links. Next, the most efficient routes may be found or the network's overall performance can be optimized using the fuzzy graph's energy. Similarly, fuzzy graphs may be utilized to represent supply and demand uncertainties in supply chain management. This then allows for the optimization of inventory levels and distribution methods through the use of energy.

5.4 Biological Network Modeling

In networks of interactions between proteins, the energy might reveal which connections are crucial to the network's stability. Targeting certain interactions can lead to the creation of novel treatment techniques, which is why this knowledge might be important in drug discovery and development.

5.5 Communication Networks

Fuzzy graphs can be utilized to evaluate the efficiency and dependability of communication networks. They can account for differences in the strength and reliability of sensor connections in wireless sensor networks. Also network's performance may be optimized, leading to dependable data transmission and resource efficiency (see Figure 2 for reference).

6. Conclusion

Fuzzy graph energy, including its theoretical foundations and practical applications, is introduced in this study. By including the membership degrees of vertices and edges, fuzzy graph energy extends upon the standard concept of graph energy. This extension allows for a more complex analysis of network topologies in cases when there is intrinsic imprecision and uncertainty.

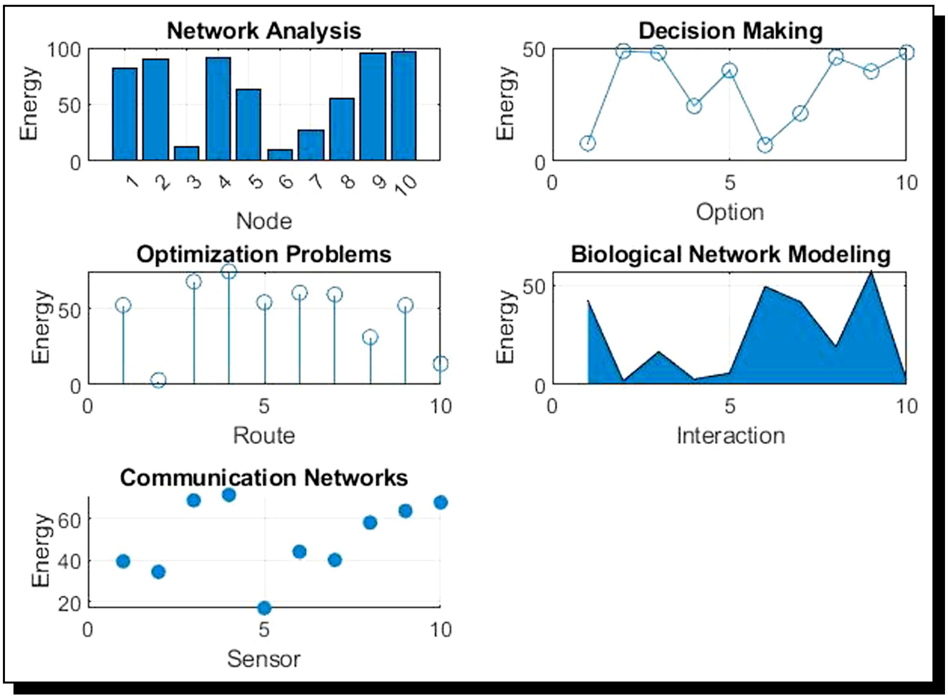

Figure 2. Applications of energy of fuzzy graphs [14]

First, we went over the basics of fuzzy graphs, including what they are and how their energy works. It is possible to get insight into the dynamics and structural aspects of a fuzzy graph by calculating its energy, which is the sum of the absolute values of the eigenvalues of its adjacency matrix. We looked at entire fuzzy graphs, bipartite fuzzy graphs, and fuzzy cycles, among others, to show how energy distribution may be diverse. Our theoretical findings revealed that the energy of fuzzy graphs is affected by both the degree of fuzziness and the underlying crisp graph structure.

Energy from fuzzy graphs has several real-world uses. The robustness and connection of networks may be measured using the energy of a fuzzy graph. This measurement provides insights into the stability and resilience of networks, which is useful in network research. The energy may be used to evaluate the whole complexity of a decision-making scenario and to find the best possible option pathways. Using fuzzy graph energy to represent uncertainties and enhance performance might be helpful for optimization challenges in transportation and supply chain management, among other areas. Fuzzy graphs' energy can also shed light on the robustness and performance of communication networks and biological network models.

More advanced fuzzy graphs and their energy characteristics can be investigated in future studies. Fuzzy hypergraphs are a generalization of fuzzy graphs that enable edges to link more than two vertices; studying their energy, for example, might provide light on higher-dimensional network architectures. A significant field of study, especially for big data and complicated network analysis, is the development of efficient methods for estimating the energy of large-scale fuzzy networks.

Finally, there is a plethora of theory and practice that can be gained by investigating energy in fuzzy graphs. This research advances our knowledge of fuzzy graphs' dynamics and structural features, which in turn advances graph theory and its applications. This will lead to better optimization, decision-making, and network analysis in the future.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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