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Research Article

Solution of Fuzzy Non-Homogeneous Differential Equation under Trapezoidal Fuzzy Number

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Abstract. This article proposes a result for *first order fuzzy non-homogeneous differential equation* under *trapezoidal fuzzy number* as preliminary value. We have used a method of interval arithmetic on α-cut interval of trapezoidal fuzzy number to obtained a general solution. We have presented a result of *non-homogeneous fuzzy differential equation* for four distinct circumstance of real valued functions tangled in differential equations. Also, an example of non-homogeneous first order linear fuzzy differential equation under *trapezoidal fuzzy number* as initial condition is being solved to verify the result at the end.

Keywords. Fuzzy differential equation, Trapezoidal fuzzy number, Support and core of fuzzy number, α-Cut Interval arithmetic

Mathematics Subject Classification (2020). 34A07, 3K36, 35R13

1. Introduction

Fuzzy differential equation models a fluctuating circumstance under ambiguous conditions plays an important role to several fields of science and engineering. The idea of fuzzy differential equation was first presented by Kandel and Byatt [13] in 1978. Followed by Dubois and Prade [6, 7] in 1982 discuss about derivative of crisp function and fuzzy function at fuzzy point and crisp point respectively. In 1985, Kaleva [12] presented an existence and uniqueness of solution of derivative and integration of fuzzy valued function. Seikkala [22] has conveyed a solution of initial value problems in fuzzy perimeter by means of an extension principle and provided a solution of fuzzy differential equation in 1987. In the year 1990, Baidosov [2] and Hullermeier [11] introduced a differential system with fuzzy constraint acknowledged as fuzzy differential inclusion. In year 1999, Oberguggenberger [17] used Zadeh's extension principle and provide a solution of fuzzy differential equation.

In year 2000, Buckley and Feuring [4] presented a new solution of first order fuzzy initial value problem which depends on various categories of derivatives of fuzzy function. In 2000, Park and Han [19] deliberated a unique result of differential equation in fuzzy circumference by using method of successive approximation whereas Chaco-Cano and Romain-Flores [5], projected a new explanation to the fuzzy differential equation based on generalised H-differentiability and π-derivative of function. In 2010, Bede and Gal [3] applied L-R type fuzzy number as preliminary condition of fuzzy differential equation and discussed wide-ranging results for fuzzy differential equation along with its existence. Duraisamy and Usha [9] solved a fuzzy initial value problem of first order by using third order Runge-Kutta method with trapezoidal fuzzy number as initial value. Salahshour [21] established the uniqueness and existence of solution of Nth-order fuzzy differential equation with generalised differentiability in Banach space. In 2012, Plotnikov with Skripnik [20] highlighted a new perspective of fuzzy function generalized derivative and presence of a fuzzy differential equation. In 2014, Mondal *et al.* [16] studied a solution of homogeneous first order linear fuzzy differential equation by means of method of Lagrange multiplier and intuitionistic Triangular fuzzy number and they have encouraged a solution of non-homogeneous Fuzzy differential equation using method of Lagrange multiplier. In 2020, Alamin *et al.* [1] discussed a solution of non-homogeneous difference equation and in 2023, Padmapriya and Kaliyappan [18] has deliberated a solution of system of non-homogeneous fuzzy fractional differential equations.

2. Preliminaries

Definition 2.1 (Fuzzy Set [23]). A set A state on universe of discourse X is called to be a fuzzy set, if $\forall\ a \in X$, a membership score of belongings is lies between the range $[0,1] \subset \mathbb{R}$. It is denoted and define as

$$A = \left\{ \frac{\mu_A\,(a)}{a} \,\middle|\, a \in X, \mu_A(a) \in [0,1] \right\},$$

where $\mu_A : X \to [0,1]$ is denote the degree of belongingness.

Definition 2.2 (α-cut of Fuzzy set [23]). A crisp set attained from a fuzzy set containing all elements whose membership grade is more than or equal to the value $\alpha \in [0,1]$ is recognized as α-cut of Fuzzy set and represented as

$$A^\alpha = \{a_k \mid a_k \in X \text{ and } \mu_A(a_k) \geq \alpha\}.$$

Definition 2.3 (Fuzzy Set Support [23]). A crisp set gained from a fuzzy set with all members whose membership grade is more than 0 is known support. It is represented as,

$$\operatorname{Supp}(A) = \{a \mid a \in X \text{ and } \mu_A(a) > 0\}.$$

Definition 2.4 (Fuzzy Set Core [23]). It is *Classical Set* of elements $a \in X$ whose membership value of belongingness in fuzzy set A is exactly equal to 1 and it is denoted as follows:

$$\operatorname{Core}(A) = \{a \mid t \in X \text{ and } \mu_A(a) = 1\}.$$

Definition 2.5 (Intervals Arithmetic's [10])**.** Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ two closed intervals defined on set of real numbers $\mathbb{R}$ in such a way that here $a_1 \le a_2$ and $b_1 \le b_2$, the arithmetic operations like subtraction and addition of intervals A and B are done as follows:

(a) $A - B = [a_1, a_2] - [b_1, b_2]$

$$= [\min((a_1-b_1),(a_1-b_2),(a_2-b_1),(a_2-b_2)), \max((a_1-b_1),(a_1-b_2),(a_2-b_1),(a_2-b_2))].$$

(b) $A + B = [a_1, a_2] + [b_1, b_2]$

$$= [\min((a_1+b_1),(a_1+b_2),(a_2+b_1),(a_2+b_2)), \max((a_1+b_1),(a_1+b_2),(a_2+b_1),(a_2+b_2))].$$

Definition 2.6 (Fuzzy Number [8])**.** A fuzzy set 'A' states on real number $\mathbb{R}$ is known as a fuzzy number provided it justify some conditions as below:

(i) Core of fuzzy set A must be non-empty, i.e., set A is normal fuzzy set.

(ii) Support of fuzzy set A must be bounded, i.e., for $\omega_1, \omega_2 \in \mathbb{R}$ there exist real number $a \in (-\infty, \omega_1)$ and (ω_2, ∞) such that $\mu_A(a) = 0$.

(iii) Fuzzy set A is monotonically continuous and convex, i.e., every α-cut set are closed; intervals, i.e., for every $\alpha \in [0,1]$, $A_\alpha = [a_\alpha, b_\alpha]$ is bounded interval.

Definition 2.7 (Trapezoidal Fuzzy Number (TrFN) [8])**.** A number A describe by four points such as $[a_1, a_2, a_3, a_4]$, where all $a_i \in \mathbb{R}$, $i = 1,2,3,4$ is known as *Trapezoidal Number* (TrFN) and its function of degree belongingness is defined as

$$\mu_A(t) = \begin{cases} 0, & \text{if } t < a_1 \text{ and } t > a_3, \\ \frac{t-a_1}{a_2-a_1}, & \text{if } a_1 \le t \le a_2, \\ 1, & \text{if } a_2 \le t \le a_3, \\ \frac{a_4-t}{a_4-a_3}, & \text{if } a_3 \le t \le a_4. \end{cases}$$

An α-cut interval of Trapezoidal Fuzzy Number is denoted and define as follows

$$A_\alpha = [\alpha(a_2 - a_1) + a_1, -\alpha(a_4 - a_3) + a_4] = [\alpha\ell + a_1, -\alpha r + a_4],$$

where $(a_2 - a_1) = \ell > 0$ and $(a_4 - a_3) = r > 0$.
If $r = \ell$, Trapezoidal Fuzzy Number $A = [a_1, a_2, a_3, a_4]$ is symmetric in nature.

Definition 2.8 (Fuzzy Homogeneous Differential Equation of First Order [12])**.** An equation $f\left(t, \mathcal{U}, \frac{d\mathcal{U}}{dt}\right) = 0$, $\mathcal{U}(t_0) = \mathcal{U}_0$ is said to be a homogeneous fuzzy differential equation of first order, if initial value $\mathcal{U}_0$ is fuzzy quantity and its solution $\mathcal{U}(t)$ is also fuzzy.

Definition 2.9 (Non-homogeneous Fuzzy Differential Equation of First Order [12])**.** An equation of the type $f\left(t, \mathcal{U}, \frac{d\mathcal{U}}{dt}\right) \ne 0$, $\mathcal{U}(t_0) = \mathcal{U}_0$ is known as Fuzzy Non-Homogeneous Differential Equation of first order, if preliminary value $\mathcal{U}_0$ is fuzzy value and its result $\mathcal{U}(t)$ is also fuzzy.

Definition 2.10 (Strong solution and Weak Solution [12])**.** Let $f\left(t, \mathcal{U}, \frac{d\mathcal{U}}{dt}\right) = 0$ or $f\left(t, \mathcal{U}, \frac{d\mathcal{U}}{dt}\right) \ne 0$ with preliminary condition $\mathcal{U}(t_0) = \mathcal{U}_0$ is the first order fuzzy differential equation, where preliminary condition $\mathcal{U}_0$ is fuzzy number. A solution $\mathcal{U}(t)$ is said to be a strong solution if and only if it's α-cut interval, $\mathcal{U}(t, \alpha) = [\mathcal{U}_1(t, \alpha)\mathcal{U}_2(t, \alpha)]$ satisfy following conditions:
(I) $\frac{\partial \mathcal{U}_1(t,\alpha)}{\partial \alpha} > 0$ and (II) $\frac{\partial \mathcal{U}_2(t,\alpha)}{\partial \alpha} < 0$, otherwise it is a weak solution.

3. Case Study

Consider a non-homogeneous fuzzy differential equation of first order :

$$\frac{d\mathcal{U}(t)}{dt}+\mathcal{P}(t)\mathcal{U}(t)=\mathcal{Q}(t) \tag{1}$$

with $\widetilde{\mathcal{U}}(t)=\mathcal{U}(t_0)=[a_1,a_2,a_3,a_4]$ is *Trapezoidal Fuzzy Number* (TFN). Here, $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are any functions of real valued variable t. The solution of equation (1) is value of dependent variable $\mathcal{U}(t)$ whose α-cut interval is denoted as $\mathcal{U}(t)=[\inf\mathcal{U}(t),\sup\mathcal{U}(t)]=[\mathcal{U}_1(t,\alpha),\mathcal{U}_2(t,\alpha)]$ and the α-cut interval of given initial condition is denoted as

$$\mathcal{U}(t_0)=[a_1+\alpha\ell,a_4-\alpha r]=[\mathcal{U}_1(t_0,\alpha),\mathcal{U}_2(t_0,\alpha)],$$

where $\ell=(a_2-a_1)>0$ and $r=(a_4-a_3)>0$.

Case 3.1: If $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are any real crisp functions.
An α-cut of differential equation (1) is

$$\left[\frac{d\mathcal{U}_1(t,\propto)}{dt},\frac{d\mathcal{U}_2(t,\propto)}{dt}\right]+\mathcal{P}(t)[\mathcal{U}_1(t,\propto),\mathcal{U}_2(t,\propto)]=\mathcal{Q}(t),$$

i.e.,

$$\frac{d\mathcal{U}_1(t,\propto)}{dt}+\mathcal{P}(t)\ \mathcal{U}_1(t,\propto)=\mathcal{Q}(t), \tag{2}$$

$$\frac{d\mathcal{U}_2(t,\propto)}{dt}+\mathcal{P}(t)\mathcal{U}_2(t,\propto)=\mathcal{Q}(t). \tag{3}$$

Since $\mathcal{P}(t)$ is real valued function, the integrating factor is $\mathcal{I}(t)=e^{\int\mathcal{P}(t)dt}$ and solutions are

$$\mathcal{U}_1(t,\propto)\mathcal{I}(t)=\int\mathcal{Q}(t)\mathcal{I}(t)dt+C_1, \tag{4}$$

$$\mathcal{U}_2(t,\propto)\mathcal{I}(t)=\int\mathcal{Q}(t)\mathcal{I}(t)dt+C_2, \tag{5}$$

at $t=t_0$, $C_1=\mathcal{U}_1(t_0,\propto)\mathcal{I}(t_0)-\left(\int\mathcal{Q}(t)\mathcal{I}(t)dt\right)_{t=t_0}$ and $C_2=\mathcal{U}_2(t_0,\propto)\mathcal{I}(t_0)-\left(\int\mathcal{Q}(t)\ I(t)dt\right)_{t=t_0}$ and from equations (4) and (5), we get

$$\mathcal{U}_1(t,\propto)=(\mathcal{I}(t))^{-1}\left\{\int\mathcal{Q}(t)\mathcal{I}(t)dt+(a_1+\alpha\ell)\mathcal{I}(t_0)-\left(\int\mathcal{Q}(t)\mathcal{I}(t)dt\right)_{t=t_0}\right\}, \tag{6}$$

$$\mathcal{U}_2(t,\propto)=(\mathcal{I}(t))^{-1}\left\{\int\mathcal{Q}(t)\mathcal{I}(t)dt+(a_4-\alpha r)\mathcal{I}(t_0)-\left(\int\mathcal{Q}(t)\mathcal{I}(t)dt\right)_{t=t_0}\right\} \tag{7}$$

are satisfying a condition of strong solution, i.e.,

$$\frac{\partial\mathcal{U}_1(t,\alpha)}{\partial\alpha}=\ell\mathcal{I}(t_0)(\mathcal{I}(t))^{-1}>0\quad\text{and}\quad\frac{\partial\mathcal{U}_2(t,\alpha)}{\partial\alpha}=-r\mathcal{I}(t_0)(\mathcal{I}(t))^{-1}<0.$$

Hence, based on α-cut interval arithmetic, the solution of differential equation (1) as follows

$$\begin{aligned}\mathcal{U}(t)&=[\mathcal{U}_1(t,0),\mathcal{U}_1(t,1),\mathcal{U}_2(t,0),\mathcal{U}_2(t,1)]\\&=(\mathcal{I}(t))^{-1}\left\{\int\mathcal{Q}(t)\ \mathcal{I}(t)dt+\mathcal{U}(t_0)\mathcal{I}(t_0)-\left(\int\mathcal{Q}(t)\mathcal{I}(t)dt\right)_{t=t_0}\right\}.\end{aligned} \tag{8}$$

Case 3.2: If $\mathcal{P}(t)$ is real valued crisp function and $\mathcal{Q}(t)$ is real valued fuzzy function.
Let $\mathcal{Q}(t)$ is real valued fuzzy function such that $\inf\mathcal{Q}(t)=\mathcal{Q}_1(t)$ and $\sup\mathcal{Q}(t)=\mathcal{Q}_2(t)$ then, α-cut of differential equation (1) is

$$\left[\frac{d\mathcal{U}_1(t,\propto)}{dt},\frac{d\mathcal{U}_2(t,\propto)}{dt}\right]+\mathcal{P}(t)[\mathcal{U}_1(t,\propto),\mathcal{U}_2(t,\propto)]=[\mathcal{Q}_1(t,\propto),\mathcal{Q}_2(t,\propto)]$$

i.e.,

$$\frac{d\mathcal{U}_1(t,\propto)}{dt}+\mathcal{P}(t)\mathcal{U}_1(t,\propto)=\mathcal{Q}_1(t,\propto), \tag{9}$$

$$\frac{d\mathcal{U}_2(t,\propto)}{dt}+\mathcal{P}(t)\mathcal{U}_2(t,\propto)=\mathcal{Q}_2(t,\propto). \tag{10}$$

Since $\mathcal{P}(t)$ is real valued function, the integrating factor is $\mathcal{I}(t)=e^{\int\mathcal{P}(t)dt}$ and solutions are

$$\mathcal{U}_1(t,\propto)\mathcal{I}(t)=\int\mathcal{I}(t)\mathcal{Q}_1(t,\propto)dt+C_1, \tag{11}$$

$$\mathcal{U}_2(t,\propto)\mathcal{I}(t)=\int\mathcal{I}(t)\mathcal{Q}_2(t,\propto)dt+C_2, \tag{12}$$

at $t=t_0$, $C_1=\mathcal{U}_1(t_0,\propto)\mathcal{I}(t_0)-\left(\int\mathcal{Q}_1(t,\propto)\mathcal{I}(t)dt\right)_{t=t_0}$ and $C_2=\mathcal{U}_2(t_0,\propto)\mathcal{I}(t_0)-\left(\int\mathcal{Q}_2(t,\propto)\mathcal{I}(t)dt\right)_{t=t_0}$ and from equations (11) and (12), we get

$$\mathcal{U}_1(t,\propto)=(\mathcal{I}(t))^{-1}\left\{\int\mathcal{I}(t)\mathcal{Q}_1(t,\propto)dt+\mathcal{I}(t_0)(a_1+\alpha\ell)-\left(\int\mathcal{I}(t)\mathcal{Q}_1(t,\propto)dt\right)_{t=t_0}\right\}, \tag{13}$$

$$\mathcal{U}_2(t,\propto)=(\mathcal{I}(t))^{-1}\left\{\int\mathcal{I}(t)\mathcal{Q}_2(t,\propto)dt+\mathcal{I}(t_0)(a_4-\alpha r)-\left(\int\mathcal{I}(t)\mathcal{Q}_2(t,\propto)dt\right)_{t=t_0}\right\} \tag{14}$$

are satisfying a condition of strong solution, i.e., $\frac{\partial\mathcal{U}_1(t,\alpha)}{\partial\alpha}>0$ and $\frac{\partial\mathcal{U}_2(t,\alpha)}{\partial\alpha}<0$.
Hence, based on α-cut interval arithmetic, the solution of the differential equation (1) as follows

$$\begin{aligned}\mathcal{U}(t)&=[\mathcal{U}_1(t,0),\mathcal{U}_1(t,1),\mathcal{U}_2(t,0),\mathcal{U}_2(t,1)]\\&=(\mathcal{I}(t))^{-1}\left\{\int[\mathcal{Q}_1(t),\mathcal{Q}_2(t)]\mathcal{I}(t)dt+\mathcal{U}(t_0)\mathcal{I}(t_0)-\left(\int[\mathcal{Q}_1(t),\mathcal{Q}_2(t)]\mathcal{I}(t)dt\right)_{t=t_0}\right\},\end{aligned} \tag{15}$$

where $\mathcal{Q}_1(t)=\min\{\mathcal{Q}_1(t,0),\mathcal{Q}_1(t,1),\mathcal{Q}_2(t,0),\mathcal{Q}_2(t,1)\}$ and $\mathcal{Q}_2(t)=\max\{\mathcal{Q}_1(t,0),\mathcal{Q}_1(t,1),\mathcal{Q}_2(t,0),\mathcal{Q}_2(t,1)\}$.

Case 3.3. If $\mathcal{P}(t)$ is real valued fuzzy function and $\mathcal{Q}(t)$ is real valued crisp function.
Let $\mathcal{P}(t)$ is fuzzy valued function such that $\inf\mathcal{P}(t)=\mathcal{P}_1(t)$ and $\sup\mathcal{P}(t)=\mathcal{P}_2(t)$.
Then, α-cut of differential equation (1) is

$$\left[\frac{d\mathcal{U}_1(t,\propto)}{dt},\frac{d\mathcal{U}(t,\propto)}{dt}\right]+[\mathcal{P}_1(t,\propto),\mathcal{P}_2(t,\propto)][\mathcal{U}_1(t,\propto),\mathcal{U}_2(t,\propto)]=\mathcal{Q}(t),$$

i.e.,

$$\frac{d\mathcal{U}_1(t,\propto)}{dt}+\mathcal{P}_1(t,\propto)\mathcal{U}_1(t,\propto)=\mathcal{Q}(t), \tag{16}$$

$$\frac{d\mathcal{U}_2(t,\propto)}{dt}+\mathcal{P}_2(t,\propto)\mathcal{U}_2(t,\propto)=\mathcal{Q}(t). \tag{17}$$

Since $P(t)$ is fuzzy valued function, the integrating factors are $\mathcal{I}_1(t,\propto)=e^{\int\mathcal{P}_1(t,\propto)dt}$ and $\mathcal{I}_2(t,\propto)=e^{\int\mathcal{P}_2(t,\propto)dt}$ and hence solutions are

$$\mathcal{U}_1(t,\propto)\mathcal{I}_1(t,\propto)=\int\mathcal{I}_1(t,\propto)\mathcal{Q}(t)dt+C_1, \tag{18}$$

$$\mathcal{U}_2(t,\propto)\mathcal{I}_2(t,\propto)=\int\mathcal{I}_2(t,\propto)\mathcal{Q}(t)dt+C_2 \tag{19}$$

at $t=t_0$, $C_1=\mathcal{U}_1(t_0,\propto)\mathcal{I}_1(t_0,\propto)-\left(\int\mathcal{I}_1(t,\propto)\mathcal{Q}(t)dt\right)_{t=t_0}$ and
$C_2=\mathcal{U}_2(t_0,\propto)\mathcal{I}_2(t_0,\propto)-\left(\int\mathcal{I}_2(t,\propto)\mathcal{Q}(t)dt\right)_{t=t_0}$ and from equations (18) and (19), we get

$$\mathcal{U}_1(t,\propto)=(\mathcal{I}_1(t,\propto))^{-1}\left\{\int\mathcal{I}_1(t,\propto)\mathcal{Q}(t)dt+\mathcal{I}_1(t_0,\propto)(a_1+\alpha\ell)-\left(\int\mathcal{I}_1(t,\propto)\mathcal{Q}(t)dt\right)_{t=t_0}\right\}, \tag{20}$$

$$\mathcal{U}_2(t,\propto) = (\mathcal{I}_2(t,\propto))^{-1}\left\{\int \mathcal{I}_2(t,\propto)\mathcal{Q}(t)dt + \mathcal{I}_2(t_0,\propto)(a_4-\alpha r) - \left(\int \mathcal{I}_2(t,\propto)\mathcal{Q}(t)dt\right)_{t=t_0}\right\} \tag{21}$$

are satisfying a condition of strong solution, i.e., $\frac{\partial \mathcal{U}_1(t,\alpha)}{\partial \alpha} > 0$ and $\frac{\partial \mathcal{U}_2(t,\alpha)}{\partial \alpha} < 0$.
Hence, based on α-cut interval arithmetic, the solution of differential equation (1) as follows

$$\begin{aligned}\mathcal{U}(t) &= [\mathcal{U}_1(t,0),\mathcal{U}_1(t,1),\mathcal{U}_2(t,0),\mathcal{U}_2(t,1)]\\ &= (\mathcal{I}(t,\propto))^{-1}\left\{\int \mathcal{Q}(t)\mathcal{I}(t,\propto)dt + \mathcal{U}(t_0)\mathcal{I}(t_0,\propto) - \left(\int \mathcal{Q}(t)\mathcal{I}(t,\propto)dt\right)_{t=t_0}\right\},\end{aligned} \tag{22}$$

where $(\mathcal{I}(t,\propto)) = [(\mathcal{I}_1(t,\propto)),(\mathcal{I}_2(t,\propto))]$.
Here, $\mathcal{I}_1(t) = \min\{\mathcal{I}_1(t,0),\mathcal{I}_1(t,1),\mathcal{I}_2(t,0),\mathcal{I}_2(t,1)\}$ and $\mathcal{I}_2(t) = \max\{\mathcal{I}_1(t,0),\mathcal{I}_1(t,1),\mathcal{I}_2(t,0),\mathcal{I}_2(t,1)\}$.

Case 3.4. If $\mathcal{P}(t)$ and $\mathcal{Q}(t)$ are real valued fuzzy function.
Let $\mathcal{Q}(t)$ is fuzzy valued function such that $\inf \mathcal{Q}(t) = \mathcal{Q}_1(t)$ and $\sup \mathcal{Q}(t) = \mathcal{Q}_2(t)$.
Let $\mathcal{P}(t)$ is fuzzy valued function such that $\inf \mathcal{P}(t) = \mathcal{P}_1(t)$ and $\sup \mathcal{P}(t) = \mathcal{P}_2(t)$.
Then, α-cut of differential equation (1) is

$$\left[\frac{d\mathcal{U}_1(t,\propto)}{dt},\frac{d\mathcal{U}_2(t,\propto)}{dt}\right] + [\mathcal{P}_1(t,\propto),\mathcal{P}_2(t,\propto)][\mathcal{U}_1(t,\propto),\mathcal{U}_2(t,\propto)] = [\mathcal{Q}_1(t,\propto),\mathcal{Q}_2(t,\propto)],$$

i.e.,

$$\frac{d\mathcal{U}_1(t,\propto)}{dt} + \mathcal{P}_1(t,\propto)x_1(t,\propto) = \mathcal{Q}_1(t,\propto), \tag{23}$$

$$\frac{d\mathcal{U}_2(t,\propto)}{dt} + \mathcal{P}_2(t,\propto)x(t,\propto) = \mathcal{Q}_2(t,\propto). \tag{24}$$

Since $\mathcal{P}(t)$ is fuzzy valued function, the integrating factors are $\mathcal{I}_1(t,\propto) = e^{\int \mathcal{P}_1(t,\propto)dt}$ and $\mathcal{I}_2(t,\propto) = e^{\int \mathcal{P}_2(t,\propto)dt}$ and hence solutions are

$$\mathcal{U}_1(t,\propto)\mathcal{I}_1(t,\propto) = \int \mathcal{I}_1(t,\propto)\mathcal{Q}_1(t,\propto)dt + C_1, \tag{25}$$

$$\mathcal{U}_2(t,\propto)\mathcal{I}_2(t,\propto) = \int \mathcal{I}_2(t,\propto)\mathcal{Q}_2(t,\propto)dt + C_2 \tag{26}$$

at $t = t_0$, $C_1 = \mathcal{U}_1(t_0,\propto)\mathcal{I}_1(t_0,\propto) - \left(\int \mathcal{I}_1(t,\propto)\mathcal{Q}_1(t,\propto)dt\right)_{t=t_0}$ and
$C_2 = \mathcal{U}_2(t_0,\propto)\mathcal{I}_2(t_0,\propto) - \left(\int \mathcal{I}_2(t,\propto)\mathcal{Q}_2(t,\propto)dt\right)_{t=t_0}$ and from equations (18) and (19), we get

$$\mathcal{U}_1(t,\propto) = (\mathcal{I}_1(t,\propto))^{-1}\left\{\int \mathcal{Q}_1(t,\propto)\mathcal{I}_1(t,\propto)dt + (a_1+\alpha\ell)\mathcal{I}_1(t_0,\propto) - \left(\int \mathcal{Q}_1(t,\propto)\mathcal{I}_1(t,\propto)dt\right)_{t=t_0}\right\}, \tag{27}$$

$$\mathcal{U}_2(t,\propto) = (\mathcal{I}_2(t,\propto))^{-1}\left\{\int \mathcal{Q}_2(t,\propto)\mathcal{I}_2(t,\propto)dt + (a_4-\alpha r)\mathcal{I}_2(t_0,\propto) - \left(\int \mathcal{Q}_2(t,\propto)\mathcal{I}_2(t,\propto)dt\right)_{t=t_0}\right\} \tag{28}$$

are satisfying a condition of strong solution, i.e., $\frac{\partial \mathcal{U}_1(t,\alpha)}{\partial \alpha} > 0$ and $\frac{\partial \mathcal{U}_2(t,\alpha)}{\partial \alpha} < 0$.
Hence, based on α-cut interval arithmetic, a solution of differential equation (1) is as follows

$$\begin{aligned}\mathcal{U}(t) &= [\mathcal{U}_1(t,0),\mathcal{U}_1(t,1),\mathcal{U}_2(t,0),\mathcal{U}_2(t,1)]\\ &= (\mathcal{I}(t,\propto))^{-1}\left\{\int \mathcal{Q}(t)\mathcal{I}(t,\propto)dt + \mathcal{U}(t_0)\mathcal{I}(t_0,\propto) - \left(\int \mathcal{Q}(t)\mathcal{I}(t,\propto)dt\right)_{t=t_0}\right\},\end{aligned} \tag{29}$$

where $(\mathcal{I}(t,\propto)) = [(\mathcal{I}_1(t,\propto)),(\mathcal{I}_2(t,\propto))]$ and $\mathcal{Q}(t) = [\mathcal{Q}_1(t),\mathcal{Q}_2(t)]$.
Here, $\mathcal{I}_1(t) = \min\{\mathcal{I}_1(t,0),\mathcal{I}_1(t,1),\mathcal{I}_2(t,0),\mathcal{I}_2(t,1)\}$, $\mathcal{I}_2(t) = \max\{\mathcal{I}_1(t,0),\mathcal{I}_1(t,1),\mathcal{I}_2(t,0),\mathcal{I}_2(t,1)\}$, $\mathcal{Q}_1(t) = \min\{\mathcal{Q}_1(t,0),\mathcal{Q}_1(t,1),\mathcal{Q}_2(t,0),\mathcal{Q}_2(t,1)\}$ and $\mathcal{Q}_2(t) = \max\{\mathcal{Q}_1(t,0),\mathcal{Q}_1(t,1),\mathcal{Q}_2(t,0),\mathcal{Q}_2(t,1)\}$.

4. Solved Example

Example 4.1. Consider non-homogeneous differential equation $\frac{d\mathcal{U}(t)}{dt}+2\mathcal{U}(t)=t$ with starting condition $\mathcal{U}_0=\mathcal{U}(t_0)=\mathcal{U}(0)=[0,1,2,3]$ is trapezoidal fuzzy number.
Here, $\mathcal{P}(t)=2$ and $\mathcal{Q}(t)=t$ are real valued functions
Then, an integrating factor is $\mathcal{I}(t)=e^{\int 2dt}=e^{2t}$ and at $t_0=0$, $\mathcal{I}(t_0)=e^0=1$ and therefore, its solution (8) is

$$\begin{aligned}
\mathcal{U}(t)&=(\mathcal{I}(t))^{-1}\left\{\int \mathcal{Q}(t)\mathcal{I}(t)dt+\mathcal{U}(t_0)\mathcal{I}(t_0)-\left(\int \mathcal{Q}(t)\mathcal{I}(t)dt\right)_{t=t_0}\right\}\\
&=e^{-2t}\left\{\int te^{2t}dt+[0,1,2,3]-\left(\int te^{2t}dt\right)_{t=0}\right\}\\
&=e^{-2t}\left\{\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t}+[0,1,2,3]-\left(\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t}\right)_{t=0}\right\}\\
&=e^{-2t}\left\{\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t}+[0,1,2,3]+\frac{1}{4}\right\}\\
&=\ e^{-2t}\left\{\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t}+\left[\frac{1}{4},\frac{5}{4},\frac{9}{4},\frac{13}{4}\right]\right\}
\end{aligned}$$

is the required solution

Example 4.2. Consider non-homogeneous differential equation $\frac{d\mathcal{U}(t)}{dt}+2\mathcal{U}(t)=[t-\delta,t+\delta]$ with starting condition $\mathcal{U}_0=\mathcal{U}(t_0)=\mathcal{U}(0)=[0,1,2,3]$ is Trapezoidal fuzzy number.
Here, $\mathcal{P}(t)=2$ is a real valued crisp function and $\mathcal{Q}(t)=[\mathcal{Q}_1(t),\mathcal{Q}_2(t)]=[t-\delta,t+\delta]$ is a real valued fuzzy function.
An integrating factor is $\mathcal{I}(t)=e^{\int 2dt}=e^{2t}$ and at $t=t_0=0$, $I(t_0)=e^0=1$ and hence its solution from (15) is

$$\begin{aligned}
\mathcal{U}(t)&=(\mathcal{I}(t))^{-1}\left\{\int[\mathcal{Q}_1(t),\mathcal{Q}_2(t)]\mathcal{I}(t)dt+\mathcal{U}(t_0)\mathcal{I}(t_0)-\left(\int[\mathcal{Q}_1(t),\mathcal{Q}_2(t)]\mathcal{I}(t)dt\right)_{t=t_0}\right\}\\
&=e^{-2t}\left\{\int[t-\delta,t+\delta]e^{2t}dt+[0,1,2,3]-\left(\int[t-\delta,t+\delta]e^{2t}dt\right)_{t=0}\right\}\\
&=e^{-2t}\left\{\left[\left(\frac{t}{2}-\frac{1}{4}-\frac{\delta}{2}\right),\left(\frac{t}{2}-\frac{1}{4}+\frac{\delta}{2}\right)\right]e^{2t}+[0,1,2,3]-\left(\left[\left(\frac{t}{2}-\frac{1}{4}-\frac{\delta}{2}\right),\left(\frac{t}{2}-\frac{1}{4}+\frac{\delta}{2}\right)\right]e^{2t}\right)_{t=0}\right\}\\
&=e^{-2t}\left\{\left[\left(\frac{t}{2}-\frac{1}{4}-\frac{\delta}{2}\right),\left(\frac{t}{2}-\frac{1}{4}+\frac{\delta}{2}\right)\right]e^{2t}+[0,1,2,3]+\left[\left(\frac{1}{4}-\frac{\delta}{2}\right),\left(\frac{1}{4}+\frac{\delta}{2}\right)\right]\right\}\\
&=e^{-2t}\left\{\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t}+\left[\frac{1}{4},\frac{5}{4},\frac{9}{4},\frac{13}{4}\right]+\left[-\frac{\delta}{2},\frac{\delta}{2}\right]\right\}
\end{aligned}$$

is the required solution.

Example 4.3. Consider non-homogeneous differential equation $\frac{d\mathcal{U}(t)}{dt}+\left[\frac{1}{t}-\delta,\frac{1}{t}+\delta\right]\mathcal{U}(t)=2$ with starting condition $\mathcal{U}_0=\mathcal{U}(t_0)=\mathcal{U}(1)=[0,1,2,3]$ is trapezoidal fuzzy number.
Here, $\mathcal{P}(t)=[\mathcal{P}_1(t,\propto),\mathcal{P}_2(t,\propto)]=\left[\frac{1}{t}-\delta,\frac{1}{t}+\delta\right]$ is a fuzzy function and $\mathcal{Q}(t)=2$ is a real valued crisp function then, an integrating factor is

$$\mathcal{I}(t)=e^{\int[\frac{1}{t}-\delta,\frac{1}{t}+\delta]dt}=[e^{\int(\frac{1}{t}-\delta)dt},e^{\int(\frac{1}{t}+\delta)dt}]=[te^{-\delta t},te^{\delta t}]=[\mathcal{I}_1(t),\mathcal{I}_2(t)],$$

at $t = t_0 = 1$, $\mathcal{I}(t_0) = [\mathcal{I}_1(t_0), \mathcal{I}_2(t_0)] = [e^{-\delta}, e^{\delta}]$ and hence its solution from (22) is

$$\begin{aligned}
\mathcal{U}(t) &= ([\mathcal{I}_1(t), \mathcal{I}_2(t)])^{-1}\left\{\int \mathcal{Q}(t)[\mathcal{I}_1(t), \mathcal{I}_2(t)]dt + \mathcal{U}(t_0)[\mathcal{I}_1(t_0), \mathcal{I}_2(t_0)]\right.\\
&\qquad\qquad\left. - \left(\int \mathcal{Q}(t)[\mathcal{I}_1(t), \mathcal{I}_2(t)]dt\right)_{t=t_0}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\int 2[te^{-\delta t}, te^{\delta t}]dt + [0,1,2,3][e^{-\delta}, e^{\delta}] - \left(\int 2[te^{-\delta t}, te^{\delta t}]dt\right)_{t=1}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\left[2\int te^{-\delta t}dt, 2\int te^{\delta t}dt\right] + [0,1,2,3][e^{-\delta}, e^{\delta}]\right.\\
&\qquad\qquad\left. - \left(\left[2\int te^{-\delta t}dt, 2\int te^{\delta t}dt\right]\right)_{t=1}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\left[2\left(t\frac{e^{-\delta t}}{-\delta} - \frac{e^{-\delta t}}{\delta^2}\right), 2\left(t\frac{e^{\delta t}}{\delta} - \frac{e^{\delta t}}{\delta^2}\right)\right] + [0,1,2,3][e^{-\delta}, e^{\delta}]\right.\\
&\qquad\qquad\left. - \left(\left[2\left(t\frac{e^{-\delta t}}{-\delta} - \frac{e^{-\delta t}}{\delta^2}\right), 2\left(t\frac{e^{\delta t}}{\delta} - \frac{e^{\delta t}}{\delta^2}\right)\right]\right)_{t=1}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\left[2\left(t\frac{e^{-\delta t}}{-\delta} - \frac{e^{-\delta t}}{\delta^2}\right), 2\left(t\frac{e^{\delta t}}{\delta} - \frac{e^{\delta t}}{\delta^2}\right)\right] + [0,1,2,3][e^{-\delta}, e^{\delta}]\right.\\
&\qquad\qquad\left. - \left(\left[2\left(\frac{e^{-\delta}}{-\delta} - \frac{e^{-\delta}}{\delta^2}\right), 2\left(\frac{e^{\delta}}{\delta} - \frac{e^{\delta}}{\delta^2}\right)\right]\right)\right\}
\end{aligned}$$

is the required solution.

Example 4.4. Consider non-homogeneous differential equation $\frac{d\mathcal{U}(t)}{dt} + \left[\frac{1}{t} - \delta, \frac{1}{t} + \delta\right]\mathcal{U}(t) = [t-\epsilon, t+\epsilon]$ with starting condition $\mathcal{U}_0 = \mathcal{U}(t_0) = \mathcal{U}(1) = [0,1,2,3]$ is trapezoidal fuzzy number. Here $\mathcal{P}(t) = [\mathcal{P}_1(t,\propto), \mathcal{P}_2(t,\propto)] = \left[\frac{1}{t} - \delta, \frac{1}{t} + \delta\right]$ and $\mathcal{Q}(t) = [t-\epsilon, t+\epsilon]$ are real valued fuzzy functions then, an integrating factor is

$$\mathcal{I}(t) = e^{\int [\frac{1}{t}-\delta, \frac{1}{t}+\delta]dt} = [e^{\int(\frac{1}{t}-\delta)dt}, e^{\int(\frac{1}{t}+\delta)dt}] = [te^{-\delta t}, te^{\delta t}] = [\mathcal{I}_1(t), \mathcal{I}_2(t)],$$

at $t = t_0 = 1$, $\mathcal{I}(t_0) = [\mathcal{I}_1(t_0), \mathcal{I}_2(t_0)] = [e^{-\delta}, e^{\delta}]$ and hence its solution from (29) is

$$\begin{aligned}
\mathcal{U}(t) &= ([\mathcal{I}_1(t), \mathcal{I}_2(t)])^{-1}\left\{\int [\mathcal{Q}_1(t), \mathcal{Q}_2(t)][\mathcal{I}_1(t), \mathcal{I}_2(t)]dt + \mathcal{U}(t_0)[\mathcal{I}_1(t_0), \mathcal{I}_2(t_0)]\right.\\
&\qquad\qquad\left. - \left(\int [\mathcal{Q}_1(t), \mathcal{Q}_2(t)][\mathcal{I}_1(t), \mathcal{I}_2(t)]dt\right)_{t=t_0}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\int 2[[t-\epsilon, t+\epsilon]e^{-\delta t}, [t-\epsilon, t+\epsilon]e^{\delta t}]dt + [0,1,2,3][e^{-\delta}, e^{\delta}]\right.\\
&\qquad\qquad\left. - \left(\int 2[[t-\epsilon, t+\epsilon]e^{-\delta t}, [t-\epsilon, t+\epsilon]e^{\delta t}]dt\right)_{t=1}\right\}\\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\left\{\left[2\int [t-\epsilon, t+\epsilon]e^{-\delta t}dt, 2\int [t-\epsilon, t+\epsilon]e^{\delta t}dt\right] + [0,1,2,3][e^{-\delta}, e^{\delta}]\right.\\
&\qquad\qquad\left. - \left(\left[2\int [t-\epsilon, t+\epsilon]e^{-\delta t}dt, 2\int [t-\epsilon, t+\epsilon]e^{\delta t}dt\right]\right)_{t=1}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= [te^{-\delta t}, te^{\delta t}]^{-1}\Bigg\{\left[(t-\epsilon)\frac{2e^{-\delta t}}{-\delta} - \frac{2e^{-\delta t}}{\delta^2}, (t+\epsilon)\frac{2e^{\delta t}}{\delta} - \frac{2e^{\delta t}}{\delta^2}\right] + [0,1,2,3][e^{-\delta}, e^{\delta}] \\
&\qquad - \left(\left[(t-\epsilon)\frac{2e^{-\delta t}}{-\delta} - \frac{2e^{-\delta t}}{\delta^2}, (t+\epsilon)\frac{2e^{\delta t}}{\delta} - \frac{2e^{\delta t}}{\delta^2}\right]\right)_{t=1}\Bigg\} \\
&= [te^{-\delta t}, te^{\delta t}]^{-1}\Bigg\{\left[(t-\epsilon)\frac{2e^{-\delta t}}{-\delta} - \frac{2e^{-\delta t}}{\delta^2}, (t+\epsilon)\frac{2e^{\delta t}}{\delta} - \frac{2e^{\delta t}}{\delta^2}\right] + [0,1,2,3][e^{-\delta}, e^{\delta}] \\
&\qquad - \left(\left[(1-\epsilon)\frac{2e^{-\delta}}{-\delta} - \frac{2e^{-\delta}}{\delta^2}, (1+\epsilon)\frac{2e^{\delta}}{\delta} - \frac{2e^{\delta}}{\delta^2}\right]\right)\Bigg\}
\end{aligned}
$$

is the required solution.

5. Conclusion

In this article, we proposed a method for solving non-homogeneous first-order fuzzy differential equations with a preliminary condition given as a trapezoidal fuzzy number. We consider four possible cases for the functions involved in the differential equations. As an illustration, we solve a representative example using a trapezoidal fuzzy number as the starting condition and find a strong solution under all four cases and yield strong solutions. Furthermore, we suggest to extend this approach to first-order non-homogeneous fuzzy differential equations with various types of fuzzy numbers as initial conditions.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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