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Research Article

Research on Geometry of Gravitation by Distinguishing Between Spatial and Temporal Part of Spherically Symmetric Metric

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Abstract. We prove a theorem that in case of a spherically symmetric metric the curvature tensor of the spatial metric (by ignoring the temporal component) must be 0, because otherwise it leads to a paradox. We consider gravitation, assuming that a point-mass body with mass M radially translates each point by a 3-vector with magnitude GM/c^2. The temporal metric coefficient g_{44} is uniquely determined by $\boldsymbol{g}$, so the Riemann curvature tensor is non-zero. The experimental tests apply and the basic results remain the same as in GR. The obtained equations for n-body problem differ from the Einstein-Infeld-Hoffmann equations by Lorentz invariant addends. While in GR the curvature scalar $\mathcal{R}$ is proportional to the density of matter, here $\mathcal{R}$ is proportional to the density of energy and also to the density of matter.

Keywords. 4D metric, Field equations, Experimental tests of General Relativity

Mathematics Subject Classification (2020). 53B50, 83D05, 83F05

1. Introduction

In [4], Einstein contemplates the concepts of foundations of physics that lend themselves to a mathematical formulation. With the reflection on how relativity freed the gravitational physics of the problems with coordinate systems, he is aware that “it cannot be claimed that those parts of the general relativity theory which can to-day be regarded as final have furnished physics with a complete and satisfactory foundation. In the first place, the total field appears in it to be composed of two logically unconnected parts, the gravitational and the electromagnetic", and furthermore, discusses the independence of relativity and quantum theories. However, just as the physics-freeing concepts were algebraic (group theory) in the core of the differential geometry of relativity, it should be expected that developments and improvements of the relativity can be achieved by evolving in that mathematical branch. Such framework and the global scheme in a system of axioms are given by Trenčevski [8]. From physical viewpoint, it is important to describe the particles as transmitters of the interactions, but from mathematical viewpoint, that is a secondary problem. The time dilation, the anomalous galactic rotation on its periphery etc., in the considerations of gravitation naturally emerge in that framework, since it is based on conversion of constraints in the space (S), space rotations (SR) and the time (T) (Trenčevski and Celakoska [9]), where S, T and SR are homeomorphic to S^3 and can be considered as the group of quaternions with module 1 (locally isomorphic to $SO(3,\mathbb{R})$). The space T is necessary for motion in arbitrary direction and magnitude, while the space SR is necessary for the angular velocity with arbitrary direction and magnitude.

If SR is not admitted, we would not be able to turn and if T is not admitted, everything would be static. The group of Lorentz transformations $O_+^\uparrow(1,3)$, isomorphic to the complex Lie group $SO(3,\mathbb{C})$ is the Lie group which connects the spaces S and T, considered in 6×6 matrix form. The previous comments about space "S" and time "T" refer to a chosen value of the 1-parametric time.

This concept of multidimensional representation of the spacetime together with construction of a non-linear connection was applied in [9] considering gravitation, however, it can be adapted in the standard General Relativity which describes the spacetime as a 4D pseudo-Riemannian manifold. The obvious choice of the metric

$$(g_{ij}) = \operatorname{diag}\left(1+\frac{2GM}{rc^2}, 1+\frac{2GM}{rc^2}, 1+\frac{2GM}{rc^2}, g_{44}\right)$$

in presence of a central source of gravitation gives that the space lengths increase in all directions by the same factor $1+\frac{2GM}{rc^2}$. However, if we analyse a series of small adjacent spheres placed on the equator around a planet (Figure 1), we easily come to contradiction.

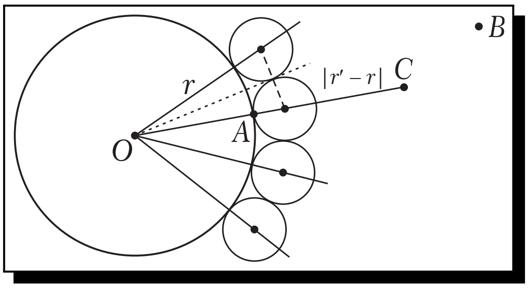

Figure 1. The radial distance between A and C is the same for the observers A, B, and C, however, it is not compatible with the known space metric

Namely, the condition that there is exactly one point of contact between any two neighbouring spheres is satisfied for both a local observer A on the planet and an observer B far from the planet. So, if r is the distance between the planet's center (origin) and the center of any of the spheres according to the observer A, then from the metric (g_{ij}) the radial distance according to the observer B should be $r\left(1+\frac{GM}{rc^2}\right) = r + \frac{GM}{c^2}$, otherwise the spheres would not touch each other. If there is an observer C collinear with A and the coordinate origin O, on radial distance $r' \approx r$ then both A and C observe the distance $|r'-r|$ between them. However, B

sees $\left|\left(r'+\frac{GM}{c^2}\right)-\left(r+\frac{GM}{c^2}\right)\right|=|r'-r|$, but not $|r'-r|\left(1+\frac{GM}{rc^2}\right)$, which is a contradiction.

Taking this paradox into consideration, the gravitation in [8] is axiomatically defined by: *A point-mass body with mass M radially translates each point by* 3*-vector with magnitude GM/c^2*. Let's place the point-mass body M in the coordinate origin. Then, a point $A(x,y,z)$ translates to $A'(x,y,z)(1+GM/(rc^2))$, where $r=\sqrt{x^2+y^2+z^2}$. Clearly, the radial distance between any two points remains unchanged, but a vector $\boldsymbol{l}$ of a very small magnitude, orthogonal to radial direction would transform into $\boldsymbol{l}(1+GM/(rc^2))$. Let P and Q be two very close points of a test body and let $\overrightarrow{PQ}=(a,b,c)$. The non-commutativity of the two translations, where one of them is due to the gravitation and the other one is the translation for vector (a,b,c), actually gives the Newtonian law of gravitation. Namely, if the gravitational translation is applied firstly on a test body placed in the point A and then follows an infinitesimal translation by a vector (a,b,c), we arrive at the neighbouring point B':

$$A(x,y,z)\to(x,y,z)\left(1+\frac{GM}{rc^2}\right)\to B'\left((x,y,z)\left(1+\frac{GM}{rc^2}\right)+(a,b,c)\right).$$

Conversely, if the translation by a vector (a,b,c) is applied first and the gravitational translation follows after, then we arrive at the point B:

$$A(x,y,z)\to(x+a,y+b,z+c)\to B\left((x+a,y+b,z+c)\left(1+\frac{GM}{r'c^2}\right)\right),$$

where $r'=\sqrt{(x+a)^2+(y+b)^2+(z+c)^2}\approx r$. The non-commutativity yields the directed angle

$$\angle\overrightarrow{BOB'}=\frac{\overrightarrow{OB}\times\overrightarrow{OB'}}{|\overrightarrow{OB}|\cdot|\overrightarrow{OB'}|},$$

thus, analogously to the other three natural interactions [8], the obtained acceleration is

$$\boldsymbol{g}=\frac{c^2}{2}\operatorname{rot}\angle\overrightarrow{BOB'}=-c^2\frac{GM}{rc^2}\frac{(x,y,z)}{r\left(r+\frac{GM}{c^2}\right)}\approx-\frac{GM}{r^3}\boldsymbol{r}, \tag{1.1}$$

where the operator rotor is applied with respect to the coordinates a,b,c. In this paper, we continue to study the gravitation according to the mentioned definition.

2. Induced Space Metric

The previous considerations mean that the obvious choice of the metric requires a modification. Assume that the gravitational body is at the center of a coordinate system. Let x,y,z be the coordinates in Euclidean coordinate system assuming that the gravitational body is absent, and X,Y,Z are coordinates after the appearance of the gravitational body. Starting from the equalities

$$X=x\left(1+\frac{GM}{rc^2}\right),\quad Y=y\left(1+\frac{GM}{rc^2}\right),\quad Z=z\left(1+\frac{GM}{rc^2}\right), \tag{2.1}$$

where $r=\sqrt{x^2+y^2+z^2}$, it is easy to obtain that $r=R-GM/c^2$, where $R=\sqrt{X^2+Y^2+Z^2}$ and its inverse transformation is

$$x=X\left(1+\frac{GM}{Rc^2}\right),\quad y=Y\left(1+\frac{GM}{Rc^2}\right),\quad z=Z\left(1+\frac{GM}{Rc^2}\right). \tag{2.2}$$

Now, the metric with respect to the coordinates (X,Y,Z) can be found by differentiating (2.2) and replacing in $(ds)^2=(dx)^2+(dy)^2+(dz)^2$. Namely,

$$(ds)^2=(dx)^2+(dy)^2+(dz)^2$$

$$= g_{11}(dX)^2 + g_{22}(dY)^2 + g_{33}(dZ)^2 + 2g_{12}dXdY + 2g_{13}dXdZ + 2g_{23}dYdZ.$$

The expressions are large, however, by dismissing the components of order c^{-4} we obtain

$$g_{ij} \approx \left(1 - \frac{GM}{Rc^2}\right)^2 \delta_{ij} + 2X^i X^j \frac{GM}{R^3c^2}, \tag{2.3}$$

while in sphere coordinates the metric is

$$(ds)^2 = (dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2],$$

$$(ds)^2 = (dR)^2 + \left(1 - \frac{GM}{Rc^2}\right)^2 R^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2],$$

and also in the form

$$(ds)^2 = \left(1 - \frac{GM}{Rc^2}\right)^2 \left[\frac{1}{(1 - GM/(Rc^2))^2}(dR)^2 + R^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2]\right].$$

The following theorem gives the conditions for a space metric in gravitation free from the contradiction mentioned in the Introduction.

Theorem 2.1. *In an arbitrary metric theory of gravitation only the space metric in spherically symmetric form of the following type:*

$$(ds)^2 = (d\rho)^2 + \rho^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2],$$

where ρ is a function of R, is free from the contradiction. Hence, the curvature tensor must be 0.

This theorem holds also in case for several spherical coordinates, for example if we have 3 spherical coordinates the space metric should be

$$(ds)^2 = (d\rho)^2 + \rho^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2 + \sin^2(\theta)\sin^2(\phi)(d\psi)^2].$$

Proof. It is sufficient to prove the theorem in case of two spherical coordinates, because the proof in the general case follows the same reasoning. Let us start from the general metric

$$(ds)^2 = \frac{(dR)^2}{u^2} + \frac{R^2}{v^2}[(d\theta)^2 + \sin^2(\theta)(d\phi)^2] = \left(\frac{R}{rv}\right)^2 \left[\left(\frac{r\frac{dR}{dr}}{R}\right)^2 \frac{v^2}{u^2}(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2]\right],$$

where u and v as functions of R are of type $a_0 + \frac{a_1}{R} + \frac{a_2}{R^2} + \cdots$ and let f be a solution of the following equation

$$\frac{rf'(r)}{f(r)} = \frac{u(f(r))}{v(f(r))}.$$

Since u/v is also of the same type $a_0 + \frac{a_1}{R} + \frac{a_2}{R^2} + \cdots$, we can find a solution $f(r)$ in the following form $f(r) = r\left(b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots\right)$, by the method of unknown coefficients $b_0, b_1, b_2, \cdots$ where $b_0 \neq 0$. Then, using the change $R = f(r)$, the metric becomes isotropic

$$(ds)^2 = \left(\frac{f(r)}{rv}\right)^2 [(dr)^2 + r^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2]].$$

Moreover, since $v = v(f(r))$ it is easy to see that v is of the form $c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \cdots$, where $c_0 \neq 0$, and finally, also $f(r)/(rv)$ is of the form $d_0 + \frac{d_1}{r} + \frac{d_2}{r^2} + \cdots$, where $d_0 \neq 0$. Let's consider a series of small spheres placed on the equator as in Figure 1. It is obvious that if we change the coordinate system, the touching points will remain the same. According to the arguments in the Introduction, it must be $d_1 = 0$ in order to avoid the contradiction in the first order of approximation. Similarly, in order to avoid the contradiction in the higher order of the approximation it must also be that $d_2 = 0$, $d_3 = 0, \cdots$. So, $f(r)/(rv) = C$ is a non-zero constant.

If we place $\rho = Cr$, we obtain the required metric. Conversely, the above metric leads to flat space (in curvilinear coordinates) and so, it is not contradictory. This completes the proof. □

A special type of non-contradictory metric is $\rho = R - GM/c^2$, which we used previously. This 3D metric is flat – the Riemann-Christoffel tensor is 0 and the torsion tensor is 0, however supplementing it with one time dimension, the Ricci tensor becomes non-zero.

3. The Metric in 4D

The fourth component of the metric is $g_{44} = -(1+U/c^2)^2$, where U is the potential energy and it is approximately $-GM/R$. In the case of gravitation, from the eq. (1.1) the acceleration is given by $\frac{GM}{Rr} = \frac{GM}{R^2(1-\frac{GM}{Rc^2})}$ and by integrating,

$$U = c^2 \ln\left(1 - \frac{GM}{Rc^2}\right). \tag{3.1}$$

So,

$$g_{44} = -\left(1 + \ln\left(1 - \frac{GM}{Rc^2}\right)\right)^2. \tag{3.2}$$

Assuming that the mass rests in our coordinate system, $g_{14} = g_{24} = g_{34} = g_{41} = g_{42} = g_{43} = 0$ holds, so the metric is now completely determined. In matrix form it can be obtained from

$$\begin{aligned}(ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2\left(1 + \frac{U}{c^2}\right)^2 \\ &= \left(d\left(X\left(1 - \frac{GM}{Rc^2}\right)\right)\right)^2 + \left(d\left(Y\left(1 - \frac{GM}{Rc^2}\right)\right)\right)^2 + \left(d\left(Z\left(1 - \frac{GM}{Rc^2}\right)\right)\right)^2 - \left(1 + \ln\left(1 - \frac{GM}{Rc^2}\right)\right)^2 c^2(dt)^2.\end{aligned}$$

In general, if the mass is not in rest with respect to our coordinate system, then X, Y, Z will depend on the time a bit, so $g_{14}, g_{24}, g_{34}, g_{41}, g_{42}, g_{43}$ may not be 0. Also, the coefficient $1 + \ln\left(1 - \frac{GM}{Rc^2}\right)$ gives only a relative speed of the time running, not absolute, with the points far from gravitational sources.

4. Analogy With the Einstein Field Equations

In order to find the Ricci tensor and the curvature scalar with respect to the coordinate system (X, Y, Z) it is enough to begin with (x, y, z) coordinates where the metric is in the form $(1, 1, 1, -u^2)$ and then apply a coordinate transformation on the space coordinates. Since the goal is calculation of the curvature scalar $\mathcal{R}$, which does not change between coordinate systems, we can directly check with respect to the metric $(1, 1, 1, -u^2)$ that

$$\mathcal{R}_{ij} = \frac{1}{u}\frac{\partial^2 u}{\partial x^i \partial x^j}, \quad i, j \in \{1, 2, 3\},$$

$$\mathcal{R}_{14} = \mathcal{R}_{24} = \mathcal{R}_{34} = \mathcal{R}_{41} = \mathcal{R}_{42} = \mathcal{R}_{43} = 0, \ \ \mathcal{R}_{44} = -u\nabla^2 u,$$

which gives

$$\mathcal{R} = \frac{2}{u}\nabla^2 u, \text{ i.e., } \nabla^2 u = \frac{1}{2}\mathcal{R}\sqrt{|g|}. \tag{4.1}$$

The existence of a coordinate system with metric of form $(1, 1, 1, -u^2)$ in many-body system is a separate discussion in a section below. Now, using $\nabla^2 u = 4\pi G\rho_{\mathrm{m}}/c^2$ where ρ_{m} is the matter density,

$$\mathcal{R} = 8\pi G\rho_{\rm m}/c^2, \tag{4.2}$$

by ignoring the components of smaller order regarding c. This scalar refers to mass, so it is more appropriate to mark it as $\mathcal{R}_{\rm m}$. Now, let us upgrade eq. (4.2). The potential $u = 1+\ln\left(1-\frac{GM}{Rc^2}\right)$ in (x,y,z) coordinates is given by

$$u = 1 - \frac{GM}{rc^2} + \frac{G^2M^2}{2r^2c^4} + \cdots. \tag{4.3}$$

We consider the decomposition $u = 1+u_{\rm m}+u_{\rm e}$, where $u_{\rm m}$ refers to mass, $u_{\rm e}$ refers to energy, and where $u_{\rm e} \ll u_{\rm m}$ on large distance from the body. Note that $u_{\rm m} - u_{\rm e} = -\frac{GM}{rc^2}$ is the known potential for $r \neq 0$. Hence, applying Laplace operator we obtain $\Delta u_{\rm m} - \Delta u_{\rm e} = \Delta\left(-\frac{GM}{rc^2}\right) = 0$ and thus $\Delta u_{\rm m} = \Delta u_{\rm e}$. Having this in mind, from (4.3), we obtain

$$\Delta u_{\rm m} + \Delta u_{\rm e} = \Delta u_{\rm m} - \Delta u_{\rm e} + \Delta\left(\frac{G^2M^2}{2r^2c^4}\right) = \Delta u_{\rm m} - \Delta u_{\rm e} + \frac{G^2M^2}{r^4c^4}, \;\; 2\Delta u_{\rm e} = \frac{G^2M^2}{r^4c^4},$$

and the curvature scalar for the energy according to (4.1) is $\mathcal{R}_{\rm e} = \frac{G^2M^2}{r^4c^4} = \frac{a^2}{c^4}$, where a is the Newtonian acceleration, ignoring the components of smaller order regarding c. Using that $a^2 = 8\pi G\rho_{\rm e}$ where $\rho_{\rm e}$ is the density of energy (see the Remark 4.2), as a generalization of (4.2) we obtain the total curvature scalar

$$\mathcal{R} = \frac{2}{1+u_{\rm m}+u_{\rm e}}\Delta(u_{\rm m}+u_{\rm e}) \approx \frac{2\Delta u_{\rm m}}{1+u_{\rm m}} + 2\Delta u_{\rm e} = \mathcal{R}_{\rm m} + \mathcal{R}_{\rm e} = \frac{8\pi G\rho_{\rm m}}{c^2} + \frac{8\pi G\rho_{\rm e}}{c^4}, \tag{4.4}$$

where $\mathcal{R}_{\rm m} = \mathcal{R}|_{\rm e=0}$. The right side of this equation is closely related to the equality $\rho_{\rm e} = \rho_{\rm m}c^2$, which is a consequence from the relation $E = mc^2$.

The conclusion is that this curvature scalar $\mathcal{R}$ includes also the energy density $a^2/(8\pi G) = \rho_{\rm e}$, while $\mathcal{R}$ in the GR does not, it holds only $\nabla^2 U = 4\pi G\rho_{\rm m}$, [6]. So, the scalar eq. (4.4) is a replacement of the Einstein field equations and $\rho_{\rm m}$ is interchangeable with $\rho_{\rm e}/c^2$.

Remark 4.1. The total energy of a particle can be calculated as follows. Outside of the Schwarzschild radius $r = GM/c^2$ the energy is $E = \int \rho_{\rm e} dV = Mc^2/2$. On the other hand, we previously obtained $\Delta u_{\rm m} = \Delta u_{\rm e}$. So, the same energy appears also from the mass of the body, inside the Schwarzschild sphere, and thus, the total energy is $E = mc^2$. Since mass and energy are equivalent, we can conclude that *half of the energy $Mc^2/2$ is on the Schwarzschild sphere and the outside, while the same energy (in form of mass energy) $Mc^2/2$ is inside the Schwarzschild sphere and refers to the mass of the body*.

Remark 4.2. The energy density $\rho_{\rm e} = -a^2/(8\pi G)$ ([6], Sec. 106, using the specific signature) is negative. Indeed, the mass m_i of arbitrary body in presence of many other bodies becomes

$$\frac{m_i}{\prod\limits_{j\neq i}\left(1+\frac{Gm_i}{r_{ij}c^2}\right)} \tag{4.5}$$

which is smaller, where r_{ij} is the distance between the j-th body and the i-th body and this should be used to obtain $a^2/(8\pi G) = \rho_{\rm e}$ as in [11]. Multiplying (4.5) with $1/\sqrt{1-v^2/c^2}$ a mass M_i which is constant in a many-body system is obtained. The argument that a mass in a gravitational field decreases together with its loss of energy, is analogous to the argument in the case of moving body with velocity v, while its mass increases since $m/\sqrt{1-v^2/c^2} > m$ together with the increment of the kinetic energy. Indeed the quotient $m : (E/c^2)$ must remain unchanged, because otherwise the body will become a "time traveler".

5. Experimental Confirmation of the Proposed Model

There are two possibilities: Use the geodesic lines, or use separately (I) the influence from temporal non-homogeneity in X,Y,Z coordinates, i.e., $g_{44}=-u^2$ and (II) the influence from the coordinates X,Y,Z to the spatial metric (2.3) and then, sum the influences (I) and (II). We will use the second possibility.

(a) First let us consider the *perihelion shift*. In case of (I) we should find the PPN parameter β, while the PPN parameter $\gamma=0$. Following the standard PPN formalism for evaluating metric theories of gravity [11], from (3.2),

$$g_{44}=-\left(1+\ln\left(1-\frac{GM}{Rc^2}\right)\right)^2=-\left(1-\frac{2GM}{Rc^2}+0\cdot\frac{G^2M^2}{R^3c^4}+\cdots\right),$$

so, $\beta=0$. Thus, the general formula for the perihelion shift in PPN coordinates [7], yields

$$\Delta=\frac{2-\beta+2\gamma}{3}\frac{6\pi MG}{a(1-\epsilon^2)c^2}=\frac{4\pi MG}{a(1-\epsilon^2)c^2},$$

where a is the major semi-axis.

The other perturbation (II) is the impact of the lateral metric, which corresponds to the gravitational translation indicated by the angle $\angle\overrightarrow{BOB'}$ (see Section 1). If the total perihelion angle is given by $\oint\angle\overrightarrow{AOB}$ and $\oint\angle\overrightarrow{AOB'}=2\pi$, then

$$\begin{aligned}\oint\angle\overrightarrow{B'OB}&=\oint\frac{GM\rho}{c^2}\frac{(x,y,z)\times(da,db,dc)}{r^2}\\&=\oint\frac{GM\rho}{c^2}\frac{(x,y,z)}{r}\times\frac{(x+da,y+db,z+dc)}{r}\\&=\oint\frac{GM\rho}{c^2}d\vec{\varphi}\\&=\boldsymbol{n}_0\int_0^{2\pi}\frac{GM}{c^2}\frac{1+\epsilon\cos\varphi}{a(1-\epsilon^2)}d\varphi\\&=\frac{2\pi GM}{a(1-\epsilon^2)c^2}\boldsymbol{n}_0,\end{aligned}$$

where $\boldsymbol{n}_0$ is the unit vector orthogonal to the plane of motion. So, the total effect is $\Delta=\frac{6\pi MG}{a(1-\epsilon^2)c^2}$. In fact, the second effect is a consequence of the gravitational translation which extends a line segment orthogonal to the radial direction by the factor $1+\frac{GM}{Rc^2}$, i.e., the orbit is extended to $(2\pi+\Delta)R$.

(b) For the *light deflection*, let us set a mass in the point O, while A and C are two points on the light trajectory, which are very close to one another (Figure 2), $A(ct,r)$ and $C(ct+c\Delta t,r)$.

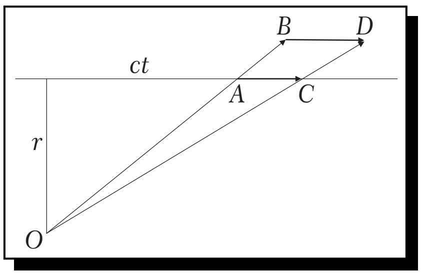

Figure 2. Light deflection: Gravitational translation of two neighbouring points on the light trajectory. The center of mass is at O

After the gravitational translation, these two points map to

$$B\left(ct+a_0\frac{ct}{\sqrt{r^2+c^2t^2}}, r+a_0\frac{r}{\sqrt{r^2+c^2t^2}}\right)$$

and

$$D\left(c(t+\Delta t)+a_0\frac{c(t+\Delta t)}{\sqrt{r^2+c^2(t+\Delta t)^2}}, r+a_0\frac{r}{\sqrt{r^2+c^2(t+\Delta t)^2}}\right),$$

where $a_0 = GM/c^2$. By simple calculation,

$$\alpha = \angle(\overrightarrow{BD},\overrightarrow{AC}) \approx -a_0\frac{rct}{(r^2+c^2t^2)^{3/2}}.$$

So, the complete angle obtained in this manner is defined with respect to the influence (II) of the X, Y and Z coordinates on the spatial metric,

$$\alpha = \int_{-\infty}^{\infty}\frac{GMrtdt}{c(r^2+c^2t^2)^{3/2}} = \frac{GM}{rc^2}.$$

The influence (I) is the classical Newtonian effect and gives the same angle. Thus the total angle of the light deflection is $2GM/(rc^2)$.

(c) The *geodetic precession* is also a consequence of (I) $g_{44} \approx -1+GM/Rc^2$ and in the same way as in the GR it leads to $(\boldsymbol{v}\times\boldsymbol{a})/(2c^2)$, and (II) the influence from the coordinates X,Y,Z to the spatial metric (2.3). The second influence is a direct consequence of the gravitational translation indicated by the angle

$$\angle\overrightarrow{B'OB} = \frac{GM}{r^3c^2}(x,y,z)\times(a,b,c).$$

Differentiating by t,

$$\begin{aligned}\frac{d}{dt}\angle\overrightarrow{B'OB} &= \frac{GM}{r^3c^2}(x,y,z)\times\left(\frac{da}{dt},\frac{db}{dt},\frac{dc}{dt}\right)\\ &= \frac{GM}{r^3c^2}(\boldsymbol{r}\times\boldsymbol{v})\\ &= -\frac{GM}{r^3c^2}(\boldsymbol{v}\times\boldsymbol{r})\\ &= \frac{\boldsymbol{v}\times\boldsymbol{a}}{c^2},\end{aligned}$$

thus, the total geodetic precession is $\frac{3}{2}\frac{\boldsymbol{v}\times\boldsymbol{a}}{c^2}$.

(d) The *time-delay* of signals in GR is a consequence of the PPN parameter $\gamma = 1$. This experiment shows that an observer far from the massive gravitational body observes that the light velocity c is slower for a coefficient $(1-GM/(Rc^2))^2$. It is slower for a coefficient $k_t = 1-GM/(Rc^2)$ due to the slower time, and the same coefficient $k_s = 1-GM/(Rc^2)$ due to the space, because it observes enlarged distances for coefficient $1+GM/(Rc^2)$. The terms of order c^{-4} are neglected. So, it is sufficient to prove that the observer sees the lengths larger for coefficient $1+GM/(Rc^2)$. It just follows from the equation (2.1) as a homothety for coefficient $1+GM/(Rc^2)$. Indeed, all lengths on distance close to R are enlarged for the same coefficient $1+GM/(Rc^2)$. Note that we do not need to differentiate the equation (2.1) in this case, while we need to differentiate it if we want to find the curvature tensors and curvature scalar.

6. The n-Body Problem

Let's consider a multibody system of gravitationally interacting bodies with masses concentrated in points. The motion of the j-th body, influenced by all other bodies in the system can be written in the form

$$\frac{d}{ds}\left(\frac{dx^i}{ds}\right)^{(j)} = -\sum_{l\neq j}(\Gamma^i_{jk})^{(l)}\left(\frac{dx^j}{ds}\right)^{(j)}\left(\frac{dx^k}{ds}\right)^{(j)},$$

where $(dx^i/ds)^{(j)}$ denote the corresponding 4-velocities and $(\Gamma^i_{jk})^{(l)}$ are the Christoffel symbols based on the l-th body. Here, $\sum_{l\neq j}(\Gamma^i_{jk})^{(l)}$ does not have to be induced by only one metric tensor, similarly as it is the case with the *Einstein-Infeld-Hoffmann Equations* (EIHE) (Eddington and Clark [3], and Einstein *et al*. [5]). In this way, the metric tensor has limited range. Let r_{ij} be the distance between the i-th and the j-th body and the masses of the corresponding bodies are M_i and M_j as in Remark 4.2. We want to determine the value ds for the j-th body.

In the first approximation, $\sum_{i\neq j} U_i = -\sum_{i\neq j}\frac{GM_i}{r_{ij}}$, so by writing

$$\sum_{i\neq j} U_i = -\sum_{i\neq j}\frac{GM_i}{r_{ij}} + \frac{\Phi_j}{c^2} \quad \text{and} \quad u = \sqrt{1-\frac{v_j^2}{c^2}} - \sum_{i\neq j}\frac{GM_i}{r_{ij}c^2} + \frac{\Phi_j}{c^4},$$

Φ_j is determined by

$$\frac{2}{u}\nabla^2 u = \mathcal{R} = \frac{8\pi G\rho_m}{c^2} = \frac{8\pi G\rho_e}{c^4} = \frac{\left(\text{grad}\left(-\sum_{i\neq j}\frac{GM_i}{r_{ij}}\right)\right)^2}{c^4},$$

$$\nabla^2\Phi_j = \frac{1}{2}\left(\text{grad}\sum_{i\neq j}\frac{-GM_i}{r_{ij}}\right)^2.$$

From here, Φ_j is well determined by the Green's function, so

$$ds = dt\left(\sqrt{1-\frac{v_j^2}{c^2}} - \sum_{i\neq j}\frac{GM_i}{(r_{ij}-GM_i/c^2)c^2} + \frac{\Phi_j}{c^4}\right).$$

Let us, now, determine the equations of motion starting from the metric $(1,1,1,-u^2)$. This problem was solved in [11], where a spin connection was used and an antisymmetric tensor field, analogous to an electromagnetic model in flat space. It is also solved in [10] for $n=1$, but close to the Schwarzschild radius. That model corresponds to this model, for example, the gravitational acceleration $\frac{GM(x,y,z)}{r^2(r+GM/c^2)}$ from (1.1) is the same without approximation. It uses only Euclidean coordinates x,y,z and the corresponding equations of motion, analogous to the EIHE are the following

$$\begin{aligned}\frac{d^2\boldsymbol{r}_j}{dt^2} = \sum_{i\neq j}\Bigg\{&-\frac{(\boldsymbol{r}_j-\boldsymbol{r}_i)Gm_i}{r_{ij}^3}\left[1-\frac{3}{2}\frac{[\boldsymbol{v}_i\cdot(\boldsymbol{r}_j-\boldsymbol{r}_i)]^2}{r_{ij}^2c^2} - \frac{G(m_i+2m_j)}{r_{ij}c^2} - \sum_{k\neq i,j}\left(\frac{Gm_k}{r_{ki}c^2}+\frac{Gm_k}{r_{kj}c^2}\right)\right.\\ &\left.+\frac{\boldsymbol{v}_i^2}{c^2} - 2\frac{\boldsymbol{v}_i\cdot\boldsymbol{v}_j}{c^2} + \frac{(\boldsymbol{v}_i-\boldsymbol{v}_j)^2}{c^2}\right] - \frac{Gm_i}{r_{ij}^3c^2}(\boldsymbol{r}_j-\boldsymbol{r}_i)\times((\boldsymbol{r}_j-\boldsymbol{r}_i)\times\dot{\boldsymbol{v}}_i)\\ &+\frac{3}{2}\frac{Gm_i}{r_{ij}^3c^2}(\boldsymbol{v}_j-\boldsymbol{v}_i)[(\boldsymbol{r}_j-\boldsymbol{r}_i)\cdot(\boldsymbol{v}_j-\boldsymbol{v}_i)] + \frac{Gm_i}{r_{ij}^3c^2}(\boldsymbol{v}_j-\boldsymbol{v}_i)[(\boldsymbol{r}_j-\boldsymbol{r}_i)\cdot\boldsymbol{v}_j]\Bigg\}.\end{aligned} \tag{6.1}$$

They are Lorentz-covariant equations, obtained without Lagrangians, and they differ from the EIHE only in Lorentz-invariant addends, offering an alternative to the standard high-order post-Newtonian Hamiltonian formulations [1].

In order to obtain the equations with respect to X,Y,Z, everywhere in the right side of the equations of motion (6.1) the distances r_{pq} as well as the vectors $\boldsymbol{r}_p - \boldsymbol{r}_q$ should be multiplied by $1+G(m_p+m_q)/(r_{pq}c^2)$, because these distances are translated twice, for two masses m_p as well as of m_q. So,

$$r_{pq} = R_{pq}\left(1+\frac{G(m_p+m_q)}{R_{pq}c^2}\right)^{-1} \quad \text{and} \quad \boldsymbol{r}_p - \boldsymbol{r}_q = (\boldsymbol{R}_p - \boldsymbol{R}_q)\left(1+\frac{G(m_p+m_q)}{R_{pq}c^2}\right)^{-1}$$

and the addends of order c^{-4} are dismissed. In this case, also, the equations of motion differ from EIHE only in Lorentz-invariant addends, more precisely, there is only one new Lorentz-invariant addend

$$\sum_{i\neq j} 2\frac{(\boldsymbol{R}_j-\boldsymbol{R}_i)Gm_i}{R_{ij}^3}\cdot\frac{G(m_i+m_j)}{R_{ij}c^2} = \sum_{i\neq j} 2\frac{(\boldsymbol{R}_j-\boldsymbol{R}_i)G^2m_i(m_i+m_j)}{R_{ij}^4c^2}. \tag{6.2}$$

7. Conclusion

In this study, we point to the paradoxical situation appearing in presence of a central source of gravitation which, standardly, by the obvious choice of the metric $(g_{ij}) = \mathrm{diag}\big(1+\frac{2GM}{rc^2}, 1+\frac{2GM}{rc^2}, 1+\frac{2GM}{rc^2}, g_{44}\big)$, in presence of a central source of gravitation, gives that the space lengths increase in all directions by the same factor $1+\frac{2GM}{rc^2}$. So, we proved that in an arbitrary metric theory of gravitation only the space metric in symmetrical spherical form of the type $(ds)^2 = (d\rho)^2 + \rho^2[(d\theta)^2 + \sin^2(\theta)(d\phi)^2]$, is free from this paradox and so, the curvature tensor must be 0. This gave rise to new geometrical considerations on the field equations, which, on one hand proved different from the EIH case by Lorentz invariant addends, however, on the other hand, the standard tests of GR hold for this modeled 4D metric. This parallels other mathematical efforts to derive non-contradictory spherically symmetric solutions in modified gravitational frameworks [2]. We also discuss the n-body case using this geometric model of gravitation.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, *Living Reviews in Relativity* **17** (2014), article number 2, DOI: 10.12942/lrr-2014-2.

[2] S. Capozziello, A. Stabile and A. Troisi, Spherically symmetric solutions in $f(R)$ gravity via Noether symmetry approach, *Classical and Quantum Gravity* **24** (2007), 2153 – 2166, DOI: 10.1088/0264-9381/24/8/013.

[3] A. S. Eddington and G. L. Clark, The problem of n bodies in general relativity theory, *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **166**(927) (1938), 465 – 475, DOI: 10.1098/rspa.1938.0104.

[4] A. Einstein, Considerations concerning the fundaments of theoretical physics, *Nature* **145** (1940), 920 – 924, DOI: 10.1038/145920a0.

[5] A. Einstein, L. Infeld and B. Hoffmann, The gravitational equations and the problem of motion, *The Annals of Mathematics* **39**(1) (1938), 65 – 100, DOI: 10.34663/9783945561317-17.

[6] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th revised English edition, Butterworth Heinemann, Amsterdam, xiii + 183 pages (1994).

[7] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W.H. Freeman and Company, San Francisco, xxvii + 1110 pages (1973).

[8] K. Trenčevski, Global scheme of the basic interactions and their geometrical interpretations, *Filomat* **37**(25) (2023), 8521 – 8535, DOI: 10.2298/FIL2325521T.

[9] K. Trenčevski and E. Celakoska, Induced spin velocity of the Earth and its influence on the seasonal variation of the Earth's angular velocity, *European Physical Journal Plus* **135** (2020), article number 450, DOI: 10.1140/epjp/s13360-020-00455-z.

[10] K. Trenčevski and E. G. Celakoska, Representation of geodesics near the Schwarzschild radius, *Gravitation and Cosmology* **21**(1) (2015), 93 – 103, DOI: 10.1134/S0202289315010132.

[11] C. M. Will, The confrontation between general relativity and experiment, *Living Reviews in Relativity* **17** (2014), article number 4, DOI: 10.12942/lrr-2014-4.