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Research Article

Independent Perfect Secure Domination in Certain Cartesian Product Graphs

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Abstract. In a graph $G = (V(G), E(G))$, a set $S \subseteq V(G)$ is a dominating set of G when every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S. A dominating set S is called an independent perfect secure dominating set of G if S is an independent set and if for each vertex $w \in V(G) \setminus S$, there exists a unique vertex $v \in S$ such that v is adjacent to w and $(S \setminus \{v\}) \cup \{w\}$ is a dominating set of G. The minimum cardinality of an independent perfect secure dominating set of G is called the independent perfect secure domination number of G. In this paper, we determine the exact value of the independent perfect secure domination number of $G \square H$, where G is a complete graph and H belongs to specific graph classes such as complete graphs, complete bipartite graphs, paths, and cycles. Upper bounds for the independent perfect secure domination number of certain other Cartesian product graphs are also dealt with in this paper.

Keywords. Independent perfect secure dominating set, Cartesian product

Mathematics Subject Classification (2020). 05C69

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1. Introduction

Relationships between entities in a system can be modeled using graphs. In this regard, it is possible to identify a set of representative vertices that facilitate functions such as communication, monitoring, and service delivery. Often, it is desirable for every non-representative vertex to be adjacent to at least one of the selected representative vertices. This requirement leads to the formulation of a dominating set in the graph. There are many variants of domination, and each is tailored to specific problems and their applications.

The graphs considered in this paper are finite, simple, and undirected. A vertex w is said to be a *neighbor* of a vertex v in G if w is adjacent to v. The set $N_G(v)$, comprising the neighbors of v, is called the *open neighborhood* of v in G. The *closed neighborhood* of v in G is the set $N_G[v] = \{v\} \cup N_G(v)$. For $X \subseteq V(G)$, $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $N_G[X] = N_G(X) \cup X$. We use the expressions $N(v)$, $N[v]$, $N(X)$, and $N[X]$ for $N_G(v)$, $N_G[v]$, $N_G(X)$, and $N_G[X]$, respectively, when the graph G is known from the context. Consider a set $U \subseteq V(G)$ and a vertex $u \in U$. The vertices in $\mathrm{pn}(u,U) = N[u] \setminus N[U \setminus \{u\}]$ are called the *private neighbors of u with respect to U*. The set $\mathrm{epn}(u,U) = \mathrm{pn}(u,U) \setminus \{u\}$ consists of the vertices that are the *external private neighbors of u with respect to U*. A subset $H \subseteq V(G)$ is called a *clique* of G if $G[H]$ is a complete graph.

A set $S \subseteq V(G)$ is a dominating set when $N[S] = V(G)$, and S is an efficient dominating set if $|N[v] \cap S| = 1$ for every $v \in V(G)$. The study on secure domination in graphs was initiated by Cockayne *et al.* [7]. A set $S \subseteq V(G)$ is a secure dominating set if for each vertex $u \in V(G) \setminus S$, there exists a vertex $v \in S \cap N(u)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. Then, u is said to be *S-defended* by v. When the set S is clear from the context, it can be said that u is *defended* by v or that v *defends* u. This variant of domination has been explored by Burger *et al.* [2,3], Chen *et al.* [5], Cockayne [6], Grobler and Mynhardt [9], Li and Xu [16], Merouane and Chellali [17], Wang *et al.* [21]. Rashmi *et al.* [18] introduced the concept of perfect secure domination. Results related to the computational complexity of the perfect secure domination problem can be found in [4]. Later, the concept of independent perfect secure domination was introduced by Thomas and Cynthia [19].

A dominating set S of a graph G is an independent perfect secure dominating set of G if S is an independent set and if for each vertex $u \in V(G) \setminus S$, there exists a unique vertex $v \in S$ such that $v \in N(u)$ and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. Some graphs may not have an independent perfect secure dominating set. When a graph has an independent perfect secure dominating set, the minimum cardinality of an independent perfect secure dominating set gives the independent perfect secure domination number of G, which is denoted by $\gamma_{ips}(G)$. An independent perfect secure dominating set with cardinality $\gamma_{ips}(G)$ is called a γ_{ips}-set of G.

For graphs A and B, the Cartesian product of the graphs $A \,\square\, B$ (Hammack *et al.* [10]) is a graph with vertex set $V(A) \times V(B)$ and two vertices (a,b) and (a',b') are adjacent if $a = a'$ and $bb' \in E(B)$, or $aa' \in E(A)$ and $b = b'$. The graphs A and B are called *factors* of the product $A \,\square\, B$. The number of vertices in $A \,\square\, B$ is $|V(A)||V(B)|$. For simple graphs A and B, $A \,\square\, B \cong B \,\square\, A$ and $K_1 \,\square\, A \cong A$. We observe that the Cartesian product method is one of the important methods for designing large-scale interconnection networks (see, Xu [22]). The graph obtained by this method is found to preserve many desirable properties of its factors.

Proposition 1.1 ([11]). *Let $S \subseteq V(G)$. The vertex $v \in S$ defends $u \in V(G) \setminus S$ if and only if $\mathrm{epn}(v,S) \cup \{v\} \subseteq N[u]$, and S is a secure dominating set of G if and only if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $G[\{u,v\} \cup \mathrm{epn}(v,S)]$ is a complete graph.*

Proposition 1.2 ([8]). *For any two graphs G and H, $\gamma(G \,\square\, H) \geq \min\{|V(G)|, |V(H)|\}$.*

Theorem 1.3 ([20]). $\min\{|V(G)|, |V(H)|\} \leq \gamma_s(G \,\square\, H) \leq \min\{|V(G)|\gamma_s(H), |V(H)|\gamma_s(G)\}$.

Lemma 1.4 ([20]). *Let n be an integer. If $2 \leq |V(H)| \leq n$, then $\gamma_s(K_n \,\square\, H) = |V(H)|$.*

Proposition 1.5 ([20]). *For any integers $n \geq 3$ and $n' \geq 3$, $\gamma_s(K_n \,\square\, P_{n'}) = \gamma_s(K_n \,\square\, C_{n'}) = n'$.*

Conjecture 1.6 ([20]). *For any integer $n \geq 2$, $\gamma_s(P_n \,\square\, K_2) = \lceil \frac{3n+1}{4} \rceil$.*

Furthermore, for $n \geq 3$, $\gamma_s(C_n \,\square\, K_2) = \begin{cases} \lceil \frac{3n}{4} \rceil + 1, & \text{if } n \equiv 4 \pmod 8 \\ \lceil \frac{3n}{4} \rceil, & \text{otherwise.} \end{cases}$

2. Main Results

2.1 Pairs of Graphs Whose Cartesian Product Graph Admits an Independent Perfect Secure Dominating Set

In this section, we exhibit a family of graphs $\mathcal{C}_H$ for an arbitrary graph H such that the product graph $G\square H$ has an independent perfect secure dominating set for every $G \in \mathcal{C}_H$. If H is a graph that has an efficient dominating set, we exhibit another graph F such that the product graph $H\square F$ has an independent perfect secure dominating set.

In the following theorem, we consider H to be a graph whose vertex set is partitioned into the minimum number of non-empty sets $I_1, I_2, \dots, I_k$, such that I_i is an independent set for each $i \in \{1,2,\dots,k\}$. Then, $V(H) = \bigcup_{j=1}^{k} I_j$ and consider $I_j = \{v_a^j \mid a \in \{1,2,\dots,|I_j|\}\}$, for each $j \in \{1,2,\dots,k\}$. Let $\mathcal{C}_H$ denote a family of graphs with the following properties: For any graph $G \in \mathcal{C}_H$, let $V(G) = \{u_1^1, u_2^1, \dots, u_{n_1}^1\} \cup \{u_1^2, u_2^2, \dots, u_{n_2}^2\} \cup \dots \cup \{u_1^p, u_2^p, \dots, u_{n_p}^p\}$, where $p, n_1, n_2, \dots, n_p$ are integers satisfying $p \geq 1$ and $k < n_1 \leq n_2 \leq \dots \leq n_p$. For each $t \in \{1,2,\dots,p\}$, $N_G[u_q^t] = \{u_1^t, u_2^t, \dots, u_{n_t}^t\}$, $\{u_1^t, u_2^t, \dots, u_{n_t}^t\} \subseteq N_G[u_r^t]$, and $N_G[u_r^t] \subseteq \{u_1^t, u_2^t, \dots, u_{n_t}^t\} \cup \left(\bigcup_{z=1}^{p} \{u_{k+1}^z, u_{k+2}^z, \dots, u_{n_z}^z\}\right)$, where $1 \leq q \leq k$ and $k+1 \leq r \leq n_t$.

Theorem 2.1. *For $G \in \mathcal{C}_H$, $G\square H$ has an independent perfect secure dominating set and $\gamma_{ips}(G\square H) \leq p|V(H)|$.*

Proof. The set S is defined as,

$$S = \{(u_1^b, v_c^1) \mid b \in \{1,2,\dots,p\}, c \in \{1,2,\dots,|I_1|\}\} \cup \{(u_2^b, v_c^2) \mid b \in \{1,2,\dots,p\}, c \in \{1,2,\dots,|I_2|\}\} \\ \cup \dots \cup \{(u_k^b, v_c^k) \mid b \in \{1,2,\dots,p\}, c \in \{1,2,\dots,|I_k|\}\}.$$

We claim that S is an independent perfect secure dominating set of $G\square H$. It is easy to verify that S is an independent dominating set of $G\square H$. For any $(u_x^d, v_y^x) \in S$, we have $\{(u_{k+1}^d, v_y^x), (u_{k+2}^d, v_y^x), \dots, (u_{n_d}^d, v_y^x)\} \subseteq \mathrm{epn}((u_x^d, v_y^x), S)$ and $\mathrm{epn}((u_x^d, v_y^x), S) \subseteq A_{xy}^d$, where $A_{xy}^d = \{(u_s^d, v_y^x) \mid 1 \leq s \leq n_d, s \neq x\}$. Thus, each vertex in A_{xy}^d is S-defended by $(u_x^d, v_y^x) \in S$. Also, the neighbors in $N_{G\square H}((u_x^d, v_y^x)) \setminus A_{xy}^d$ cannot be S-defended by (u_x^d, v_y^x). Then, S is an independent perfect secure dominating set of $G\square H$. $\square$

In the following illustration, graph H is a cycle C_4 with $I_1 = \{v_1^1, v_2^1\}$ and $I_2 = \{v_1^2, v_2^2\}$. We exhibit a graph $G \in \mathcal{C}_H$ in Figure 2. Figure 3 shows an independent perfect secure dominating set of $G\square H$.

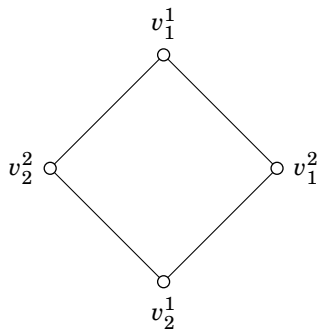

Figure 1. Graph H

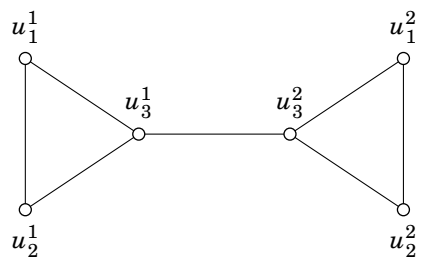

Figure 2. Graph G

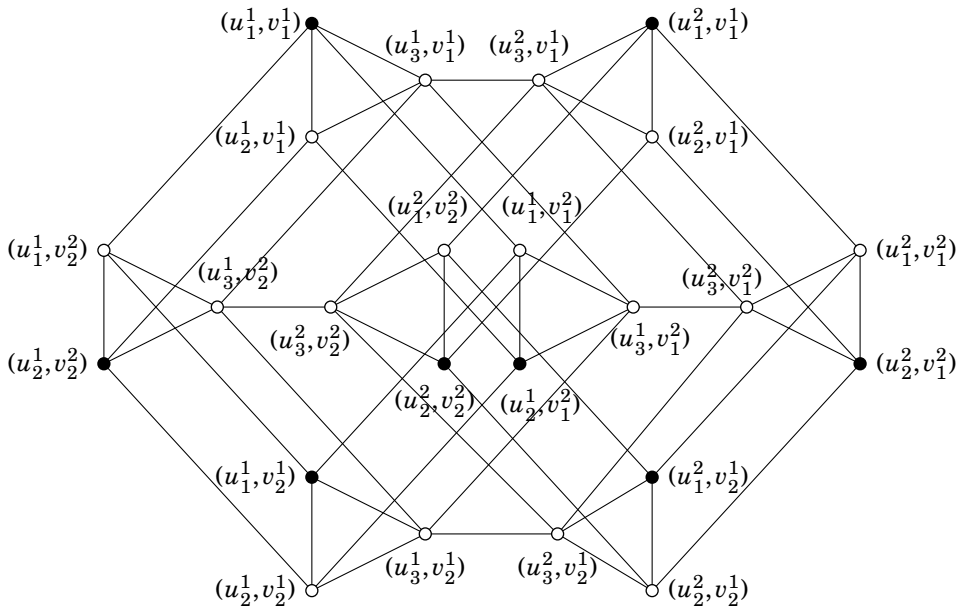

Figure 3. The shaded vertices represent the vertices in an independent perfect secure dominating set of $G \square H$

Corollary 2.2. *Let H be a graph whose vertex set $V(H)$ is partitioned into the minimum number of non-empty sets $I_1, I_2, \ldots, I_k$, such that I_i is an independent set for each $i \in \{1,2,\ldots,k\}$. $K_f \square H$ has an independent perfect secure dominating set when $f > k$ and $\gamma_{ips}(K_f \square H) \le |V(H)|$.*

Proof. By substituting $p = 1$ in Theorem 2.1, we obtain that $G \in \mathcal{C}_H$ is a complete graph. Hence, the result follows from Theorem 2.1. □

Let P be a non-trivial connected graph that has an efficient dominating set. In the following theorem, we exhibit another graph F, distinct from $\mathcal{C}_P$, such that the product graph $P \square F$ has an independent perfect secure dominating set.

Theorem 2.3. *Let P be a non-trivial connected graph with T as an efficient dominating set. Let $V(P) \setminus T$ be partitioned into non-empty sets $I_1, I_2, \ldots, I_k$, for a fixed integer $k \in \{1,2,\ldots,|V(P) \setminus T|\}$ such that I_i is an independent set for each $i \in \{1,2,\ldots,k\}$. Then, there exists a graph $F \notin \mathcal{C}_P$ with $3(k+2)$ vertices such that $P \square F$ has an independent perfect secure dominating set and $\gamma_{ips}(P \square F) \le 3|V(P)|$.*

Proof. Let the vertices in T be $\{s_1, s_2, \ldots, s_{|T|}\}$. For each $j \in \{1,2,\ldots,k\}$, consider $I_j = \{u_z^j \mid z \in \{1,2,\ldots,|I_j|\}\}$. Then, $V(P) = T \cup \left(\bigcup_{j=1}^{k} I_j\right)$. Let a graph F with $V(F) = \{v_1, v_2, \ldots, v_6\} \cup \{w_1, w_2, \ldots, w_{3k}\}$ be obtained as follows: Assume that the subgraph induced by each of the sets $\{w_1, w_2, \ldots, w_k\}$, $\{w_{k+1}, w_{k+2}, \ldots, w_{2k}\}$, and $\{w_{2k+1}, w_{2k+2}, \ldots, w_{3k}\}$ is a complete graph on k vertices.
Let

- $N_F(v_1) = \{w_1, w_2, \ldots, w_k, w_{k+1}, \ldots, w_{2k}\}$
- $N_F(v_2) = \{w_1, w_2, \ldots, w_k, v_3, v_4\}$
- $N_F(v_3) = \{w_1, w_2, \ldots, w_k, v_2, v_4\}$

- $N_F(v_4) = \{w_{k+1}, w_{k+2}, \ldots, w_{2k}, v_2, v_3\}$
- $N_F(v_5) = \{w_{k+1}, w_{k+2}, \ldots, w_{2k}, w_{2k+1}, \ldots, w_{3k}, v_6\}$
- $N_F(v_6) = \{w_{2k+1}, w_{2k+2}, \ldots, w_{3k}, v_5\}$

For the graph $P \square F$, consider the set

$$\begin{aligned} S = &\{(s_c, v_d) \mid c \in \{1,2,\ldots,|T|\}, d \in \{1,2,5\}\} \cup \{(u_x^q, w_q) \mid q \in \{1,2,\ldots,k\}, x \in \{1,2,\ldots,|I_q|\}\} \\ &\cup \{(u_x^q, w_{k+q}) \mid q \in \{1,2,\ldots,k\}, x \in \{1,2,\ldots,|I_q|\}\} \\ &\cup \{(u_x^q, w_{2k+q}) \mid q \in \{1,2,\ldots,k\}, x \in \{1,2,\ldots,|I_q|\}\}. \end{aligned}$$

We claim that S is an independent perfect secure dominating set of $P \square F$. It is easy to verify that S is an independent dominating set of $P \square F$. Also,

- For $a \in \{1,2,\ldots,|T|\}$, $\operatorname{epn}((s_a, v_1), S) = \emptyset$
- For $a \in \{1,2,\ldots,|T|\}$, $\operatorname{epn}((s_a, v_2), S) = \{(s_a, v_3), (s_a, v_4)\}$
- For $a \in \{1,2,\ldots,|T|\}$, $\{(s_a, v_6)\} \subseteq \operatorname{epn}((s_a, v_5), S)$ and $\operatorname{epn}((s_a, v_5), S) \subseteq \{(s_a, v_6), (s_a, w_{2k+1}), (s_a, w_{2k+2}), \ldots, (s_a, w_{3k})\}$
- For $b \in \{1,2,\ldots,k\}$ and $p \in \{1,2,\ldots,|I_b|\}$, $\{(u_p^b, v_3)\} \subseteq \operatorname{epn}((u_p^b, w_b), S)$ and $\operatorname{epn}((u_p^b, w_b), S) \subseteq \{(u_p^b, v_3), (u_p^b, w_1), (u_p^b, w_2), \ldots, (u_p^b, w_k)\} \setminus S$
- For $b \in \{k+1, k+2, \ldots, 2k\}$ and $p \in \{1,2,\ldots,|I_{b-k}|\}$, $\{(u_p^{b-k}, v_4)\} \subseteq \operatorname{epn}((u_p^{b-k}, w_b), S)$ and $\operatorname{epn}((u_p^{b-k}, w_b), S) \subseteq \{(u_p^{b-k}, v_4), (u_p^{b-k}, w_{k+1}), (u_p^{b-k}, w_{k+2}), \ldots, (u_p^{b-k}, w_{2k})\} \setminus S$
- For $b \in \{2k+1, 2k+2, \ldots, 3k\}$ and $p \in \{1,2,\ldots,|I_{b-2k}|\}$, $\{(u_p^{b-2k}, v_6)\} \subseteq \operatorname{epn}((u_p^{b-2k}, w_b), S)$ and $\operatorname{epn}((u_p^{b-2k}, w_b), S) \subseteq \{(u_p^{b-2k}, v_6), (u_p^{b-2k}, w_{2k+1}), (u_p^{b-2k}, w_{2k+2}), \ldots, (u_p^{b-2k}, w_{3k})\} \setminus S$

Table 1. Vertices in $V(P \square F) \setminus S$ that are S-defended by vertices in S

Vertices in S	Vertices that are S-defended
(s_a, v_1), $1 \le a \le \lvert T\rvert$	$N_{P \square F}((s_a, v_1))$
(s_a, v_2), $1 \le a \le \lvert T\rvert$	$\{(s_a, v_3), (s_a, v_4)\}$
(s_a, v_5), $1 \le a \le \lvert T\rvert$	$\{(s_a, v_6), (s_a, w_{2k+1}), (s_a, w_{2k+2}), \ldots, (s_a, w_{3k})\}$
(u_p^b, w_b), $1 \le b \le k$, $1 \le p \le \lvert I_b\rvert$	$\{(u_p^b, v_2), (u_p^b, v_3), (u_p^b, w_1), (u_p^b, w_2), \ldots, (u_p^b, w_k)\} \setminus S$
(u_p^{b-k}, w_b), $k+1 \le b \le 2k$, $1 \le p \le \lvert I_{b-k}\rvert$	$\{(u_p^{b-k}, v_4), (u_p^{b-k}, w_{k+1}), (u_p^{b-k}, w_{k+2}), \ldots, (u_p^{b-k}, w_{2k})\} \setminus S$
(u_p^{b-2k}, w_b), $2k+1 \le b \le 3k$, $1 \le p \le \lvert I_{b-2k}\rvert$	$\{(u_p^{b-2k}, v_5), (u_p^{b-2k}, v_6), (u_p^{b-2k}, w_{2k+1}), (u_p^{b-2k}, w_{2k+2}), \ldots, (u_p^{b-2k}, w_{3k})\} \setminus S$

From the above table, it is evident that all the vertices in $V(P \square F) \setminus S$ are S-defended by their neighbors in S and that each vertex in $V(P \square F) \setminus S$ is S-defended by a unique vertex in S. Hence S is an independent perfect secure dominating set of $P \square F$ and $\gamma_{ips}(P \square F) \le 3|V(P)|$. $\square$

In the following illustration, we consider the graph P to be a path P_3 with $T = \{s_1\}$ as an efficient dominating set. We observe that $V(P) \setminus T$ is an independent set. $V(P) \setminus T$ can also be partitioned into two independent sets. In the illustration, $V(P) \setminus T$ has been partitioned into two independent sets I_1 and I_2 where $I_1 = \{u_1^1\}$ and $I_2 = \{u_1^2\}$. The graph F with 12 vertices is shown in Figure 5. Figure 6 shows an independent perfect secure dominating set of $P \square F$.

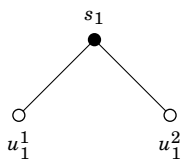

Figure 4. Graph P with the shaded vertex as an efficient dominating set

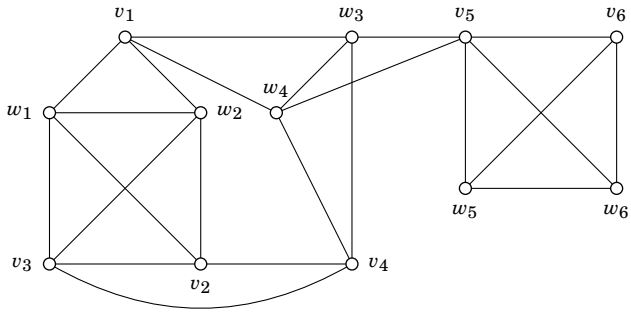

Figure 5. Graph F

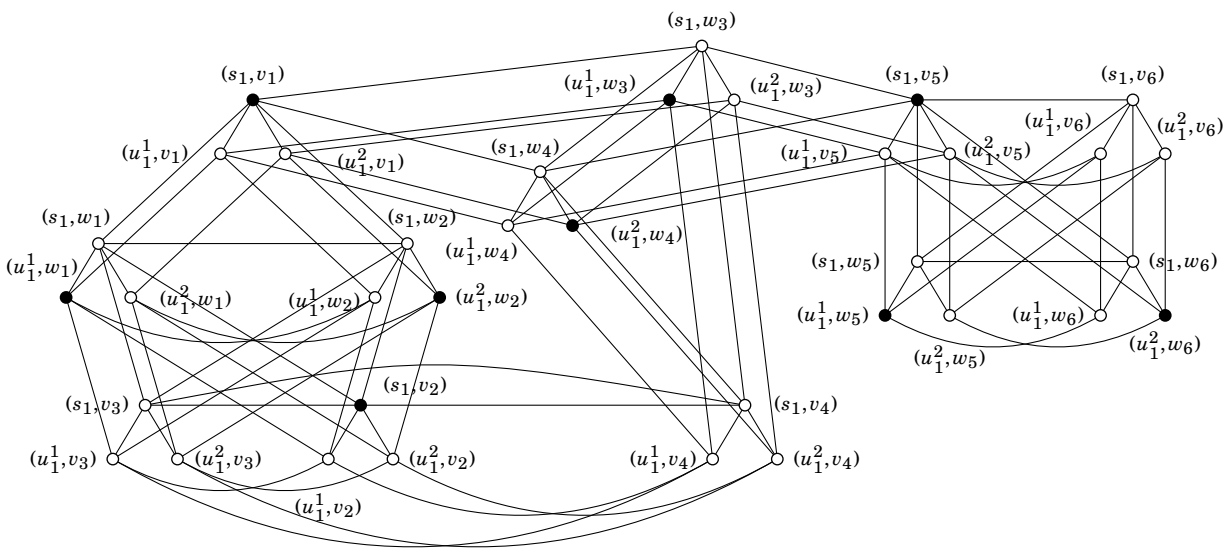

Figure 6. The shaded vertices represent the vertices in S for $P \square F$ when $k = 2$

2.2 Independent Perfect Secure Domination in Certain Cartesian Product Graphs With Complete Graphs as One Factor

In this section, we determine the exact value of the independent perfect secure domination number for the graphs $K_z \square K_{z'}$, $K_z \square P_l$, $K_z \square C_t$, and $K_z \square K_{g,h}$ under the conditions that the integers $z \geq 2$, $z' \geq 2$, $l \geq 3$, $t \geq 4$, $g \geq 1$, and $h \geq 1$, provided that an independent perfect secure dominating set exists for the respective graphs. The graphs K_z, $K_{g,h}$, P_l, and C_t are complete graphs, complete bipartite graphs, paths, and cycles, respectively.

Theorem 2.4. *The independent perfect secure domination number of $K_p \square K_q$ is $\min\{p,q\}$ when $p \geq 2$, $q \geq 2$, and $p \neq q$. The graph $K_p \square K_p$ does not have an independent perfect secure dominating set when $p \geq 2$.*

Proof. For integers $p \geq 2$ and $q \geq 2$, assume $r = \min\{p,q\}$ and $s = \max\{p,q\}$. Let the vertices of K_r be $\{v_1, v_2, \ldots, v_r\}$ and those of K_s be $\{u_1, u_2, \ldots, u_s\}$. Consider $p \neq q$. From Theorem 1.3, we have $\gamma_{ips}(K_p \square K_q) \geq r$. Also, $\gamma_{ips}(K_p \square K_q) \leq r$ from Corollary 2.2. For $K_s \square K_r$, let $S = \{(u_i, v_i) \mid 1 \leq i \leq r\}$. It is observed that $\mathrm{epn}((u_i, v_i), S) = \{(u_j, v_i) \mid r+1 \leq j \leq s\}$, for every $i \in \{1, 2, \ldots, r\}$. Then, the vertices in $\{(u_a, v_i) \mid 1 \leq a \leq s\} \setminus S$ are S-defended by (u_i, v_i)

for every $i \in \{1,2,\dots,r\}$, and the vertices in $N_{K_p \square K_q}((u_i,v_i)) \setminus \{(u_a,v_i) \mid 1 \le a \le s\}$ are not adjacent to the external private neighbors of (u_i,v_i). Thus, each vertex in $V(K_p \square K_q) \setminus S$ is S-defended by a unique neighbor in S. Therefore, $\gamma_{ips}(K_p \square K_q) = \min\{p,q\}$.

For $K_p \square K_p$ where $p \ge 2$, it is observed that the vertices in an independent dominating set will not have external private neighbors. Then, each vertex that is not an element of the independent dominating set will not have a unique vertex to defend it. Therefore, $K_p \square K_p$ does not have an independent perfect secure dominating set for $p \ge 2$. □

Theorem 2.5. *For integers $p \ge 1$, $q \ge 1$, and $a \ge 3$, $\gamma_{ips}(K_a \square K_{p,q}) = p+q$. The graph $K_2 \square K_{p,q}$ does not have an independent perfect secure dominating set.*

Proof. Let the vertices of K_a be denoted by $\{u_1,u_2,\dots,u_a\}$. Let the vertices of $K_{p,q}$ be denoted by $\{v_1,v_2,\dots,v_p\} \cup \{w_1,w_2,\dots,w_q\}$, where the vertices $\{v_1,v_2,\dots,v_p\}$ belong to a particular partite set while the vertices $\{w_1,w_2,\dots,w_q\}$ belong to the other partite set. Assume S to be an independent perfect secure dominating set of $K_a \square K_{p,q}$.

Case 1. $K_a \square K_{p,q}$ when $a \ge 3$
Let $(u_{b'},v_b) \in S$ for some $b \in \{1,2,\dots,p\}$ and some $b' \in \{1,2,\dots,a\}$ when S exists. Then $(u_e,v_b) \notin S$ for every $e \in \{1,2,\dots,a\} \setminus \{b'\}$. The following arguments also hold true when considering a vertex $(u_{b'},w_{c'}) \in S$ for some $c' \in \{1,2,\dots,q\}$ and some $b' \in \{1,2,\dots,a\}$.

Claim 1. $\mathrm{epn}((u_{b'},v_b),S) \cap \{(u_{b'},w_g) \mid 1 \le g \le q\} = \emptyset$ when S exists.

Proof of the Claim. If $(u_{b'},w_y) \in \mathrm{epn}((u_{b'},v_b),S)$ for some $y \in \{1,2,\dots,q\}$, then each (u_j,v_b) should be S-defended by a vertex (u_j,w_{k_j}) such that $k_j \in \{1,2,\dots,q\} \setminus \{y\}$, and each k_j has distinct values from each other for every $j \in \{1,2,\dots,a\} \setminus \{b'\}$. Thus, $\{(u_j,w_{k_j}) \mid 1 \le j \le a, j \ne b'\} \subset S$. This is possible only when $q \ge a$. When $(u_{b'},w_y) \in \mathrm{epn}((u_{b'},v_b),S)$, $\{(u_i,w_y) \mid 1 \le i \le a\} \cap S = \emptyset$. Then, each (u_s,w_y) should be S-defended by a vertex (u_s,v_{t_s}) where $t_s \in \{1,2,\dots,p\} \setminus \{b\}$ for every $s \in \{1,2,\dots,a\} \setminus \{b'\}$. Also, each t_s should be distinct from each other for different values of s. This is possible only when $p \ge a$. Even when considering $p \ge a$ and $q \ge a$, it violates the fact that S must be an independent set as $(u_\alpha,w_{k_\alpha}) \in S$ and $(u_\alpha,v_{t_\alpha}) \in S$ would be adjacent for $\alpha \in \{1,2,\dots,a\} \setminus \{b'\}$. Thus, $\mathrm{epn}((u_{b'},v_b),S) \cap \{(u_{b'},w_g) \mid 1 \le g \le q\} = \emptyset$ when S exists. The claim holds for every vertex in S. □

Claim 2. $\mathrm{epn}((u_{b'},v_b),S) \ne \emptyset$ when S exists.

Proof of the Claim. Suppose $\mathrm{epn}((u_{b'},v_b),S) = \emptyset$, then $(u_{b'},v_b)$ S-defends all its neighbors. Consider a neighbor (u_r,v_b) where $r \in \{1,2,\dots,a\} \setminus \{b'\}$. Since $\mathrm{epn}((u_{b'},v_b),S) = \emptyset$, there exists a vertex $(u_r,w_h) \in S$ for some $h \in \{1,2,\dots,q\}$. Then, $\mathrm{epn}((u_r,w_h),S) \cap \{(u_r,v_{g'}) \mid 1 \le g' \le p\} = \emptyset$ from *Claim* 1. If $\mathrm{epn}((u_r,w_h),S) = \emptyset$ or if $\mathrm{epn}((u_r,w_h),S) \subset \{(u_i,w_h) \mid 1 \le i \le a, i \ne r\}$, we have $(u_{b'},w_h)$ to be S-defended by (u_r,w_h) and $(u_{b'},v_b)$, a contradiction. Thus, $\mathrm{epn}((u_{b'},v_b),S) \ne \emptyset$ when S exists. The claim holds for every vertex in S. □

From *Claim* 1 and *Claim* 2, $\mathrm{epn}((u_{b'},v_b),S) \subseteq \{(u_i,v_b) \mid 1 \le i \le a, i \ne b'\}$ and $\mathrm{epn}((u_{b'},v_b),S) \ne \emptyset$ for any vertex $(u_{b'},v_b) \in S$. Hence, $(u_{b'},v_b)$ defends the vertices in $\{(u_i,v_b) \mid 1 \le i \le a, i \ne b'\}$. Thus, $\gamma_{ips}(K_a \square K_{p,q}) \ge p+q$ when S exists. Then, $\{(u_1,v_1),\dots,(u_1,v_p)\} \cup \{(u_2,w_1),\dots,(u_2,w_q)\}$ is an independent perfect secure dominating set of $K_a \square K_{p,q}$ and $\gamma_{ips}(K_a \square K_{p,q}) = p+q$ when $a \ge 3$.

Case 2. $K_2 \square K_{p,q}$
Assume $K_2 \square K_{p,q}$ to have S as an independent perfect secure dominating set. Without loss of generality, let $(u_1, v_z) \in S$ for some $z \in \{1,2,\dots,p\}$. The following arguments are similar in the case when a vertex (u_1, w_l), (u_2, v_z), or (u_2, w_l) is assumed to be a vertex in S for some $l \in \{1,2,\dots,q\}$. If $\mathrm{epn}((u_1,v_z),S) = (u_2,v_z)$, then $S \cap \{(u_1,w_g),(u_2,w_g) \mid 1 \le g \le q\} = \emptyset$. Thus, for any $(u_1,v_{j'}) \in S$, $(u_2,v_{j'}) \in \mathrm{epn}((u_1,v_{j'}),S)$ and $(u_1,v_{k'}) \in \mathrm{epn}((u_2,v_{k'}),S)$ for any $(u_2,v_{k'}) \in S$. If a vertex $(u_1,w_{a'})$, with $1 \le a' \le q$, has to be S-defended, it must be adjacent to some vertex in $\{(u_2,v_{g'}) \mid 1 \le g' \le p\}$, a contradiction. Thus, $(u_2,v_z) \notin \mathrm{epn}((u_1,v_z),S)$.

Suppose that $(u_1,w_c) \in \mathrm{epn}((u_1,v_z),S)$ for some $c \in \{1,2,\dots,q\}$. Then, $(u_2,w_c) \notin S$ and in order to S-defend (u_2,w_c), we have $(u_2,v_d) \in S$ for some $d \in \{1,2,\dots,p\} \setminus \{z\}$. This is possible only if $p \ge 2$. In order to S-defend (u_2,v_z), we have $(u_2,w_{r'}) \in S$ for some $r' \in \{1,2,\dots,q\} \setminus \{c\}$. This is possible only if $q \ge 2$. The adjacency between (u_2,v_d) and $(u_2,w_{r'})$ leads to a contradiction. Thus, $\mathrm{epn}((u_1,v_z),S) \cap \{(u_1,w_x) \mid 1 \le x \le q\} = \emptyset$.

The only remaining possibility to be discussed is that $\mathrm{epn}((u_1,v_z),S) = \emptyset$. Then, $|N_{K_2 \square K_{p,q}}((u_2,v_z)) \cap S| \ge 2$. Hence a vertex (u_2,w_k) will be a vertex in S for some $k \in \{1,2,\dots,q\}$. The instance $\mathrm{epn}((u_2,w_k),S) = \emptyset$ can be eliminated as (u_2,v_z) must be S-defended by a unique vertex in the dominating set S. Thus, we consider $\mathrm{epn}((u_2,w_k),S) = (u_2,v_f)$ for some $f \in \{1,2,\dots,p\} \setminus \{z\}$. Then, $(u_1,v_f) \notin S$ and forces a vertex (u_1,w_t) to be an element in S, where $t \in \{1,2,\dots,q\} \setminus \{k\}$. The adjacency between (u_1,v_z) and (u_1,w_t) leads to a contradiction. Hence, $\mathrm{epn}((u_1,v_z),S) = \emptyset$ is also not feasible. Therefore, $K_2 \square K_{p,q}$ does not have an independent perfect secure dominating set. □

Theorem 2.6. *For integers $p \ge 3$ and $q \ge 3$, $\gamma_{ips}(K_p \square P_q) = q$.*

Proof. Let the vertices of K_p be $\{u_1,u_2,\dots,u_p\}$. Let the vertices of P_q be $\{v_1,v_2,\dots,v_q\}$ such that a vertex $v_l \in V(P_q)$ is adjacent to v_{l-1} for $2 \le l \le q$. Assume S to be an independent perfect secure dominating set of $K_p \square P_q$.

Claim. If S exists, $S \cap \{(u_i,v_f) \mid 1 \le i \le p\} \ne \emptyset$ for each $f \in \{1,2,\dots,q\}$.

Proof of the Claim. Suppose that S exists and $S \cap \{(u_i,v_j) \mid 1 \le i \le p\} = \emptyset$ for some $j \in \{1,2,\dots,q\}$. Then, at least p vertices in $\{(u_i,v_{j-1}) \mid 1 \le i \le p\} \cup \{(u_i,v_{j+1}) \mid 1 \le i \le p\}$ should be elements in S in order to S-defend the vertices in $\{(u_i,v_j) \mid 1 \le i \le p\}$. The set $\{(u_i,v_{j-1}) \mid 1 \le i \le p\}$ does not exist when $j = 1$ and the set $\{(u_i,v_{j+1}) \mid 1 \le i \le p\}$ does not exist when $j = q$. Since $p \ge 3$, $|S \cap \{(u_i,v_{j-1}) \mid 1 \le i \le p\}| \ge 2$ or $|S \cap \{(u_i,v_{j+1}) \mid 1 \le i \le p\}| \ge 2$. In both the cases, it contradicts the fact that S should be an independent set. □

From Proposition 1.5 and the above *Claim*, it is evident that $\gamma_{ips}(K_p \square P_q) \ge q$ when S exists. From Corollary 2.2, $\gamma_{ips}(K_p \square P_q) \le q$. Hence $\gamma_{ips}(K_p \square P_q) = q$. The set

$$S = \left\{(u_1, v_{2(r-1)+1}) \mid r \in \left\{1,2,\dots,\left\lceil \frac{q}{2} \right\rceil\right\}\right\} \cup \left\{(u_2, v_{2(k-1)+2}) \mid k \in \left\{1,2,\dots,\left\lfloor \frac{q}{2} \right\rfloor\right\}\right\}$$

is an independent perfect secure dominating set of $K_p \square P_q$. □

Theorem 2.7. *For integers $q \ge 3$ and $i \ge 1$, the graph $K_2 \square P_q$ has an independent perfect secure dominating set if and only if $q \equiv 1 \pmod 4$. Also, $\gamma_{ips}(K_2 \square P_{1+4i}) = 1 + 3i$.*

Proof. The vertices of the path are labeled as in Theorem 2.6. Let the vertices of K_2 be $\{u_1,u_2\}$. Let S be a minimum independent perfect secure dominating set of $K_2 \square P_q$, assuming such a set exists. If $S \cap \{(u_1,v_1),(u_2,v_1)\} = \emptyset$, then (u_1,v_2) and (u_2,v_2) would be forced to be elements in S. This, would then violate the fact that S should be an independent set. Hence, without loss of generality, let $(u_1,v_1) \in S$. Since the graph is triangle-free, $|\mathrm{epn}((u_k,v_j),S)| \le 1$ for any $(u_k,v_j) \in S$ where $k \in \{1,2\}$ and $j \in \{1,2,\dots,q\}$. If $\mathrm{epn}((u_1,v_1),S) = (u_1,v_2)$, then $(u_2,v_2) \notin S$ and it would force $(u_2,v_1) \in \mathrm{epn}((u_1,v_1),S)$. Then $|\mathrm{epn}((u_1,v_1),S)| > 1$, a contradiction. If $\mathrm{epn}((u_1,v_1),S) = (u_2,v_1)$, then $(u_1,v_2) \notin S$ and $(u_2,v_2) \notin S$. Thus, $(u_2,v_3) \in S$. Since $|\mathrm{epn}((u_1,v_1),S)| \le 1$, we have $(u_1,v_3) \in S$. This violates the fact that S must be an independent set. Thus, $\mathrm{epn}((u_1,v_1),S) = \emptyset$ and $(u_2,v_2) \in S$. The case where $\mathrm{epn}((u_2,v_2),S) = \emptyset$ can be eliminated as (u_2,v_1) and (u_1,v_2) must each be S-defended by a unique vertex in the dominating set S. Therefore, we have $\mathrm{epn}((u_2,v_2),S) = (u_2,v_3)$. Then, $(u_1,v_3) \notin S$. Hence, we consider the case when $q \ge 4$. $(u_1,v_4) \in S$ with $\mathrm{epn}((u_1,v_4),S) = (u_1,v_3)$. Since (u_2,v_4) cannot be an element of S, we consider the case of $K_2 \square P_q$ where $q \ge 5$. Then, $(u_2,v_5) \in S$ with $\mathrm{epn}((u_2,v_5),S) = \emptyset$. When $q = 5$, $S = \{(u_1,v_1),(u_1,v_4),(u_2,v_2),(u_2,v_5)\}$ is an independent perfect secure dominating set. Since S is independent, $(u_1,v_5) \notin S$ and $(u_2,v_6) \notin S$ when $q > 5$. Thus, $(u_1,v_6) \in S$ with $\mathrm{epn}((u_1,v_6),S) = (u_1,v_7)$ so that (u_1,v_5) and (u_2,v_6) are S-defended by (u_2,v_5) alone. Since (u_2,v_7) cannot be an element of S, we consider the case when $q \ge 8$. Then, $(u_2,v_8) \in S$ so that (u_2,v_7) is dominated. Since (u_1,v_8) cannot be an element of S, we consider $q \ge 9$. When $q = 9$, $S = \{(u_1,v_1),(u_1,v_4),(u_2,v_2),(u_2,v_5),(u_1,v_6),(u_2,v_8),(u_1,v_9)\}$ is an independent perfect secure dominating set. Further, $(u_2,v_{10}) \in S$ due to similar reasoning as $(u_1,v_6) \in S$ when $q \ge 10$. Repeating the procedure, we obtain

$$\begin{aligned} S = &\left\{(u_1,v_{8a+b}) \mid a \in \left\{0,1,\dots,\left\lceil \frac{q}{8} \right\rceil - 1\right\}, b \in \{1,4\}\right\} \\ &\cup \left\{(u_2,v_{8a+c}) \mid a \in \left\{0,1,\dots,\left\lceil \frac{q}{8} \right\rceil - 1\right\}, c \in \{2,5\}\right\} \\ &\cup \left\{(u_1,v_{8d+6}),(u_2,v_{8d+8}) \mid d \in \left\{0,1,\dots,\left\lfloor \frac{q}{8} \right\rfloor - 1\right\}\right\} \end{aligned}$$

when $q \equiv 5 \pmod 8$ and $q \ne 5$. When $q \equiv 1 \pmod 8$, we obtain

$$\begin{aligned} S = &\left\{(u_1,v_{8a+1}) \mid a \in \left\{0,1,\dots,\left\lceil \frac{q}{8} \right\rceil - 1\right\}\right\} \cup \left\{(u_1,v_{8d+f}) \mid d \in \left\{0,1,\dots,\left\lfloor \frac{q}{8} \right\rfloor - 1\right\}, f \in \{4,6\}\right\} \\ &\cup \left\{(u_2,v_{8d+h}) \mid d \in \left\{0,1,\dots,\left\lfloor \frac{q}{8} \right\rfloor - 1\right\}, h \in \{2,5,8\}\right\}. \end{aligned}$$

S forms an independent perfect secure dominating set for $K_2 \square P_{1+4i}$ and $\gamma_{ips}(K_2 \square P_{1+4i}) = 1+3i$ for $i = 1,2,\dots$. □

Theorem 2.8. *For integers $p \ge 4$ and $q \ge 4$, $\gamma_{ips}(K_p \square C_q) = q$.*

Proof. Let the vertices of K_p be $\{u_1,u_2,\dots,u_p\}$ and let the vertices of C_q be $\{v_1,v_2,\dots,v_q\}$ such that a vertex $v_j \in V(C_q)$ is adjacent to v_{j-1} for $2 \le j \le q$ and $v_1v_q \in E(C_q)$. For $a \in \{1,2,\dots,p\}$, $r \in \{1,2,\dots,q\}$, and $s \in \{1,2,\dots,q\}$, let us assume $(u_a,v_{r+s}) = (u_a,v_{r+s-q})$, if $r+s > q$ and $(u_a,v_{r-s}) = (u_a,v_{q+r-s})$ if $s \ge r$. Similar to Theorem 2.6, $\gamma_{ips}(K_p \square C_q) \ge q$ for $p \ge 4$ and $q \ge 4$ when an independent perfect secure dominating set exists. Let

$$S = \begin{cases} \left\{(u_1,v_{2(i-1)+1}),(u_2,v_{2(i-1)+2}) \mid i \in \left\{1,2,\dots,\left\lfloor \frac{q}{2} \right\rfloor\right\}\right\}, & \text{when } q \text{ is even,} \\ \left\{(u_1,v_{2(i-1)+1}),(u_2,v_{2(i-1)+2}) \mid i \in \left\{1,2,\dots,\left\lfloor \frac{q}{2} \right\rfloor\right\}\right\} \cup (u_3,v_q), & \text{when } q \text{ is odd.} \end{cases}$$

Since S is an independent perfect secure dominating set, $\gamma_{ips}(K_p \square C_q) = q$ when $p \geq 4$ and $q \geq 4$. □

Theorem 2.9. *For an even integer $q \geq 4$, the graph $K_3 \square C_q$ has an independent perfect secure dominating set and $\gamma_{ips}(K_3 \square C_q) = q$. The graph $K_3 \square C_q$ does not have an independent perfect secure dominating set when q is an odd integer and $q > 4$.*

Proof. The vertices of $K_3 \square C_q$, with $q \geq 4$ are labeled as in Theorem 2.8. Assume S to be an independent perfect secure dominating set of $K_3 \square C_q$, $q \geq 4$. Similar to Theorem 2.6, $\gamma_{ips}(K_3 \square C_q) \geq q$ when S exists. Also, $\{(u_1, v_{2(r-1)+1}), (u_2, v_{2(r-1)+2}) \mid 1 \leq r \leq \lfloor \frac{q}{2} \rfloor\}$ is an independent perfect secure dominating set when q is even and $q \geq 4$. Thus, $\gamma_{ips}(K_3 \square C_q) = q$ when q is even and $q \geq 4$.

When q is odd and $q > 4$, $S \cap \{(u_i, v_j) \mid 1 \leq i \leq 3\} \neq \emptyset$ for every $j \in \{1, 2, \ldots, q\}$ as in the *Claim* of Theorem 2.6. Also, $|S \cap \{(u_i, v_j) \mid 1 \leq i \leq 3\}| = 1$ for every $j \in \{1, 2, \ldots, q\}$. The vertex $S \cap \{(u_i, v_j) \mid 1 \leq i \leq 3\}$ will S-defend the vertices in $\{(u_i, v_j) \mid 1 \leq i \leq 3\} \setminus S$ for every $j \in \{1, 2, \ldots, q\}$. We have $S \cap \{(u_1, v_k) \mid 1 \leq k \leq q\} \neq \emptyset$, $S \cap \{(u_2, v_k) \mid 1 \leq k \leq q\} \neq \emptyset$, and $S \cap \{(u_3, v_k) \mid 1 \leq k \leq q\} \neq \emptyset$ as the vertices in S should form an independent set. There will exist at least one vertex $(u_a, v_b) \in S$ for a fixed $a \in \{1, 2, 3\}$ and a fixed $b \in \{1, 2, \ldots, q\}$ that will not have any external private neighbor. Then, each vertex in $\{(u_a, v_{b-1}), (u_a, v_{b+1})\}$ will not be S-defended by a unique vertex in S. Thus, $K_3 \square C_q$ will not have an independent perfect secure dominating set when q is odd and $q > 4$. □

Theorem 2.10. *For integers $q \geq 4$ and $k \geq 1$, the graph $K_2 \square C_q$ has an independent perfect secure dominating set if and only if $q \equiv 0 \pmod 8$. Also, $\gamma_{ips}(K_2 \square C_{8k}) = 6k$.*

Proof. For $q \geq 4$, the vertices of $K_2 \square C_q$ are labeled as in Theorem 2.8. Let S be a minimum independent perfect secure dominating set of $K_2 \square C_q$, assuming such a set exists. Since the graph is triangle-free, $|\mathrm{epn}((u_l, v_i), S)| \leq 1$ for any vertex $(u_l, v_i) \in S$ where $l \in \{1, 2\}$ and $i \in \{1, 2, \ldots, q\}$. Without loss of generality, consider $(u_1, v_1) \in S$. Suppose that $\mathrm{epn}((u_1, v_1), S) = (u_2, v_1)$, then the vertices (u_1, v_3) and (u_2, v_3) would be forced to be elements in S, a contradiction. Hence, if $\mathrm{epn}((u_1, v_1), S) \neq \emptyset$, $\mathrm{epn}((u_1, v_1), S) \subseteq \{(u_1, v_2), (u_1, v_q)\}$. Without loss of generality, let $\mathrm{epn}((u_1, v_1), S) = (u_1, v_q)$. Then, $\{(u_2, v_2), (u_1, v_3)\} \subset S$ with $\mathrm{epn}((u_2, v_2), S) = \emptyset$ and $\mathrm{epn}((u_1, v_3), S) = (u_1, v_4)$. Since the set S is not an independent perfect secure dominating set of $K_2 \square C_4$, we consider the case when $q \geq 5$. Then, $(u_2, v_5) \in S$ with $\mathrm{epn}((u_2, v_5), S) = (u_2, v_4)$. This contradicts the fact that $\mathrm{epn}((u_1, v_1), S) = (u_1, v_5)$ for $q = 5$. Hence, we consider the case when $q \geq 6$. (u_1, v_5) can be S-defended only when $(u_1, v_6) \in S$. Thus, we consider the case when $q \geq 7$. If (u_1, v_5) has to be S-defended by (u_1, v_6), we must have $\mathrm{epn}((u_1, v_6), S) = \emptyset$. Hence, $(u_2, v_7) \in S$. If $q = 7$, each neighbor of (u_2, v_7) will not be S-defended by a unique vertex in the dominating set S. Therefore, we consider the case when $q \geq 8$. When $q = 8$, $S = \{(u_1, v_1), (u_2, v_2), (u_1, v_3), (u_2, v_5), (u_1, v_6), (u_2, v_7)\}$ is an independent perfect secure dominating set. By repeating the procedure, we conclude that $K_2 \square C_q$ has an independent perfect secure dominating set S when $q \equiv 0 \pmod 8$ and

$$S = \left\{(u_1, v_{8(j-1)+a}) \mid j \in \left\{1, 2, \ldots, \frac{q}{8}\right\}, a \in \{1, 3, 6\}\right\} \cup \left\{(u_2, v_{8(j-1)+b}) \mid j \in \left\{1, 2, \ldots, \frac{q}{8}\right\}, b \in \{2, 5, 7\}\right\}.$$

If $\mathrm{epn}((u_1,v_1),S) = \emptyset$, then $\{(u_2,v_q),(u_2,v_2)\} \subset S$ with $\mathrm{epn}((u_2,v_q),S) = (u_2,v_{q-1})$ and $\mathrm{epn}((u_2,v_2),S) = (u_2,v_3)$, so that the vertices in $N_{K_2 \square C_q}((u_1,v_1))$ are S-defended only by (u_1,v_1). Thus, we discuss the case where $q \geq 5$. We consider $q \geq 6$ as $(u_1,v_3) \notin S$ and $(u_1,v_4) \notin S$ for $q = 5$. When $q \geq 6$, $(u_1,v_4) \in S$ so that (u_1,v_3) is S-defended by (u_1,v_4). In order to S-defend (u_2,v_4) and (u_1,v_5), we require $(u_2,v_5) \in S$. Then, it violates the fact that $\mathrm{epn}((u_2,v_q),S) = (u_2,v_{q-1})$ when $q = 6,7$. Hence, we consider the case with $q \geq 8$. When $q = 8$, $S = \{(u_1,v_1),(u_2,v_2),(u_1,v_4),(u_2,v_5),(u_1,v_6),(u_2,v_8)\}$ is an independent perfect secure dominating set. By repeating the procedure, we conclude that $K_2 \square C_q$ has an independent perfect secure dominating set S when $q \equiv 0 \pmod 8$ and

$$S = \left\{(u_1, v_{8(j-1)+c}) \mid j \in \left\{1,2,\ldots,\frac{q}{8}\right\}, c \in \{1,4,6\}\right\}$$
$$\cup \left\{(u_2, v_{8(j-1)+d}) \mid j \in \left\{1,2,\ldots,\frac{q}{8}\right\}, d \in \{2,5,8\}\right\}.$$

In both the cases, S is a minimum independent perfect secure dominating set and $\gamma_{ips}(K_2 \square C_{8k}) = 6k$ for $k = 1,2,\ldots$. □

3. Conclusion

Many prominent graph structures can be obtained with the Cartesian product of some well-known graphs. Various domination parameters of Cartesian product graphs have been extensively studied. In this paper, we analyzed the existence of independent perfect secure dominating sets in some Cartesian product graphs that had complete graphs as a factor. The bound obtained for the independent perfect secure domination number was found to be sharp when the other factor was taken to be some cases of complete graphs, complete bipartite graphs, paths, or cycles. For a non-trivial connected graph that has an efficient dominating set, we have exhibited a graph of specific order such that their Cartesian product graph has an independent perfect secure dominating set. Further, an extensive study of independent perfect secure domination in the Cartesian product of other graphs can be conducted.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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