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Research Article

Polynomial Extensions of Fuzzy Baer and Fuzzy Quasi-Baer Subrings

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Abstract. This paper introduces and investigates the notion of fuzzy quasi-Baer subrings, establishing fundamental properties relative to classical fuzzy algebraic structures. We demonstrate a hierarchical relationship between fuzzy Baer and fuzzy quasi-Baer subrings, proving that while every fuzzy Baer subring is quasi-Baer, the converse fails in general. The study focuses on polynomial extensions of these structures, demonstrating that the quasi-Baer property is preserved under polynomial extensions, unlike the Baer property. A key result establishes that when a polynomial extension of a fuzzy subring is quasi-Baer, the original fuzzy subring must necessarily be Baer. These findings reveal important structural constraints in fuzzy ring theory and provide new insights into the behavior of annihilator conditions in fuzzy algebraic systems. The work extends classical Baer and quasi-Baer ring theory to the fuzzy setting, while highlighting crucial distinctions between these concepts in polynomial extensions. Our results contribute to the growing body of knowledge in fuzzy algebra and open new directions for research in fuzzy homological algebra and categorical generalizations of noncommutative ring-theoretic properties.

Keywords. Fuzzy subring, Fuzzy Baer subring, Fuzzy quasi-Baer subring

Mathematics Subject Classification (2020). Primary 03E72, 08A72; Secondary 16W10

1. Introduction

The origin of fuzzy mathematics can be traced back to the pioneering work of L. A. Zadeh [20], who introduced the concept of a fuzzy set in his foundational paper on fuzzy set theory. In this framework, a fuzzy subset is characterized by a membership function $\mu : E \to [0,1]$ that assigns to each element $x \in E$; a degree of membership. This generalization of classical set theory, which accommodates partial membership, has since become a cornerstone in the study of fuzzy algebra, fuzzy topology, and related disciplines.

The algebraic study of fuzzy structures began with Rosenfeld's seminal work [17] on fuzzy subgroups, later extended to *fuzzy subrings* by Liu [12]. Subsequent research diversified this framework through generalizations such as Banerjee's *intuitionistic fuzzy subrings* [3], which introduced dual membership and non-membership degrees, and Melliani *et al*. [15] approach using fuzzy points. Further advancements included Maheswari's *bipolar fuzzy subrings* [13], Rasuli's *Q-fuzzy* and *anti-Q-fuzzy subrings* [16] based on *t-norms* and *t-conorms*, and Dogra and Pal [6] studied *picture fuzzy subrings* which incorporated a neutral membership component. In ordered semirings, Mikhled [1] developed *complex fuzzy soft rings* and ideals, analyzing their properties under homomorphisms, while Ameer [8] advanced the field with *complex intuitionistic fuzzy Lie superalgebras*, enabling structural analysis with applications in physics and mathematics. Building on these developments, Satyanarayana *et al*. [18] introduced *fuzzy neutrosophic prime ideals* (FNPIs) in commutative BCK-algebras, establishing that not every fuzzy neutrosophic ideal is an FNPI and deriving level-cut-based criteria for their identification. Kute *et al*. [11] introduced *fuzzy polynomial and fuzzy matrix subrings*, investigated their properties using fuzzy zero divisors, established a McCoy theorem for fuzzy polynonial subrings, and explored the notions of *fuzzy Armendariz subrings and fuzzy reduced subrings*. Collectively, these contributions have significantly expanded the toolbox of *fuzzy algebraic structures*, enhancing the modeling of uncertainty in algebraic systems.

Parallel to these developments, the study of Baer and quasi-Baer rings has played a crucial role in ring theory. Kaplansky [9] originally introduced Baer rings in the context of functional analysis, defining them as rings in which the left (resp. right) annihilator of any subset is generated by an idempotent. Clark [5] later generalized this notion to quasi-Baer rings, where the annihilator condition is imposed on left (resp. right) ideals. These concepts have since been extensively studied by Armendariz [2], Banerjee [4], and Waphare and Khairnar[19] with applications ranging from operator algebras to noncommutative ring theory.

Building on these foundations, this paper bridges the gap between classical ring-theoretic properties and fuzzy algebra by introducing fuzzy quasi-Baer subrings and investigating their polynomial extensions. Our results establish a hierarchical relationship between fuzzy Baer and fuzzy quasi-Baer subrings, demonstrating that while every fuzzy Baer subring is quasi-Baer, the converse does not hold in general. Furthermore, we prove that the quasi-Baer property is preserved under polynomial extensions for fuzzy subrings, whereas the Baer property is not — a distinction that highlights fundamental structural differences between these two classes. Additionally, we show that if the polynomial extension of a fuzzy subring is quasi-Baer, then the original fuzzy subring must necessarily be Baer, revealing a key constraint imposed by polynomial constructions. These findings not only deepen the theoretical understanding of fuzzy algebraic systems but also open new avenues for research into their homological and categorical aspects within noncommutative ring-theoretic frameworks.

Notation. Let R be an associative ring with identity, and let $S \subseteq R$. The *left annihilator* of S in R is denoted by

$$\ell_R(S) = \{r \in R \mid rS = 0\},$$

while the *right annihilator* of S is given by

$$r_R(S) = \{r \in R \mid Sr = 0\}.$$

2. Preliminaries

We recall fundamental definitions and concepts related to fuzzy algebraic structures, particularly fuzzy subrings, ideals, annihilators, and their extensions to polynomial and power series rings.

Definition 2.1 ([12])**.** A mapping $\mathsf{F}: R \to [0,1]$ is called a *fuzzy subring* of a ring R if for all $v, w \in R$:

(i) $\mathsf{F}(v-w) \geq \min\{\mathsf{F}(v), \mathsf{F}(w)\}$,

(ii) $\mathsf{F}(vw) \geq \min\{\mathsf{F}(v), \mathsf{F}(w)\}$.

Definition 2.2 ([12])**.** A mapping $\mathsf{F}: R \to [0,1]$ is called a *left* (*or right*) *fuzzy ideal* of a ring R if for all $v, w \in R$:

(i) $\mathsf{F}(v-w) \geq \min\{\mathsf{F}(v), \mathsf{F}(w)\}$,

(ii) $\mathsf{F}(vw) \geq \mathsf{F}(w)$ (or $\mathsf{F}(vw) \geq \mathsf{F}(v)$),

for all $v, w \in R$. If F is both left and right fuzzy ideal, it is called a *fuzzy ideal*. Equivalently, F satisfies:

(i) $\mathsf{F}(v-w) \geq \min\{\mathsf{F}(v), \mathsf{F}(w)\}$,

(ii) $\mathsf{F}(vw) \geq \max\{\mathsf{F}(v), \mathsf{F}(w)\}$.

Definition 2.3 ([14])**.** Let $\mathcal{L}: \mathsf{F} \to [0,1]$ be a fuzzy subring. A *fuzzy subset* $\phi: \mathsf{F}' \to [0,1]$, where $\mathsf{F}' \subseteq \mathsf{F}$, satisfies:

$$\phi(x) \leq \mathcal{L}(x), \quad \text{for all } x \in \mathsf{F}'.$$

We denote this by $\phi|_{\mathsf{F}'}$, emphasizing its restriction to F'.

Definition 2.4 ([20])**.** For a fuzzy subset F of R, the *α-cut* ($\alpha \in [0,1]$) is the crisp set:

$$\mathsf{F}_\alpha = \{x \in R \mid \mathsf{F}(x) \geq \alpha\}.$$

This provides a bridge between fuzzy and classical set theory.

Definition 2.5 ([14])**.** Let F be a fuzzy subring of R.

(i) The *left fuzzy annihilator* $\mathcal{L}(\mathsf{F})$ is defined as:

$$\mathcal{L}(\mathsf{F})(x) = \begin{cases} \sup\{t\}, & \text{if } x \in l(\mathsf{F}_t), \\ 0, & \text{otherwise,} \end{cases}$$

where $l(\mathsf{F}_t)$ is the left annihilator of the crisp set F_t.

(ii) The *right fuzzy annihilator* $\mathcal{R}(\mathsf{F})$ is defined analogously, replacing $l(\mathsf{F}_t)$ with $r(\mathsf{F}_t)$.

Example 2.6. Let $R = M_2(\mathbb{Z}_2)$ denote the ring of all 2×2 matrices over the finite field $\mathbb{Z}_2$. Consider the subset

$$U = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \subseteq R.$$

Define a fuzzy subring $\mathsf{F} : U \to [0,1]$ as follows:

$$\mathsf{F}(A) = \begin{cases} 0.85, & \text{if } A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ 0.21, & \text{if } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Let $\mathcal{L}(\mathsf{F}) : R \to [0,1]$ denote the left fuzzy annihilator of F, defined by

$$\mathcal{L}_R(\mathsf{F})(A) = \begin{cases} \sup\{t\}, & \text{if } A \in l_R(U), \\ 0, & \text{if } A \notin l_R(U), \end{cases}$$

$$\mathcal{L}_R(\mathsf{F})(A) = \begin{cases} 0.85, & \text{if } A \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}, \\ 0, & \text{if } A \notin l_R(U). \end{cases}$$

Similarly, define the right fuzzy annihilator $\mathcal{R}_R(\mathsf{F}) : R \to [0,1]$ by

$$\mathcal{R}_R(\mathsf{F})(A) = \begin{cases} \sup\{t\}, & \text{if } A \in r_R(U), \\ 0, & \text{if } A \notin r_R(U), \end{cases}$$

$$\mathcal{R}_R(\mathsf{F})(A) = \begin{cases} 0.85, & \text{if } A \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \\ 0, & \text{if } A \notin r_R(U). \end{cases}$$

Hence, it is observed that $\mathcal{L}_R(\mathsf{F}) \neq \mathcal{R}_R(\mathsf{F})$, demonstrating that the left and right fuzzy annihilator of a fuzzy subring may differ in general.

Definition 2.7 ([7])**.** Let F be a fuzzy subring of R. For $p(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, the *fuzzy polynomial subring* $\phi : R[x] \to [0,1]$ is defined by:

$$\phi(p(x)) = \min\{\mathsf{F}(a_i) \mid 0 \le i \le n\}.$$

This extends fuzzy subring F to the polynomial ring $R[x]$.

Definition 2.8 ([7])**.** For $p(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$, the *fuzzy power series subring* $\omega : R[[x]] \to [0,1]$ is given by:

$$\omega(p(x)) = \inf\{\mathsf{F}(a_i) \mid i \ge 0\}.$$

This generalizes the polynomial case to formal power series.

Definition 2.9 ([10])**.** A fuzzy subring F of a ring R is said to be generated by an element $a \in R$ if and only if

$$\{x \in R \mid \mathsf{F}(x) = \mathsf{F}(0)\} = aR.$$

Definition 2.10 ([10])**.** A fuzzy subring F of a ring R is said to be a fuzzy Baer subring of R if, for every fuzzy subset $\mu \subseteq \mathsf{F}$, the left (or right) fuzzy annihilator $\mathcal{L}_R(\mu)$ is generated by an idempotent of R.

3. Fuzzy Quasi-Baer Subring

This section introduces the notion of a fuzzy quasi-Baer subring, extending the framework established by Kute *et al.* [10], where the concept of a fuzzy Baer subring was developed using the notions of the fuzzy left (or right) annihilator (Medhi and Saikia [14]) and fuzzy subrings generated by elements. The primary objective of this section is to establish and analyze fundamental properties of fuzzy quasi-Baer subrings. To support and contextualize these theoretical developments, illustrative examples are presented, which also serve to highlight the scope and limitations of the proposed results.

Definition 3.1. A fuzzy subring F of R is termed a *left fuzzy quasi-Baer subring* if for every left fuzzy ideal subset η of F (defined over $S \subseteq R$), the left fuzzy annihilator $\mathcal{L}_R(\eta)$ is generated by an idempotent in R.

Definition 3.2. A fuzzy subring F of R is termed a *right fuzzy quasi-Baer subring* if for every right fuzzy ideal subset η of F (defined over $S \subseteq R$), the right fuzzy annihilator $\mathcal{R}_R(\eta)$ is generated by an idempotent in R.

A subring F is said to be a *fuzzy quasi-Baer subring* if it is both left and right symmetric.

Example 3.3. Let $\mathsf{F} : \mathbb{R} \to [0,1]$ be a fuzzy subring defined by the membership function:

$$\mathsf{F}(x) = \begin{cases} 0.91, & \text{if } x \in \mathbb{Z}, \\ 0.35, & \text{otherwise.} \end{cases}$$

For any fuzzy ideal subset $\eta \subseteq \mathsf{F}$ over $S \subseteq \mathbb{R}$, the left fuzzy annihilator is given by:

$$\mathcal{L}_{\mathbb{R}}(\eta)(x) = \begin{cases} 0.91, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{x \mid \mathcal{L}_{\mathbb{R}}(\eta)(x) = \mathcal{L}_{\mathbb{R}}(\eta)(0)\} = \{0\}$ is generated by the idempotent $0 \in \mathbb{R}$. Hence, F is left fuzzy quasi-Baer subring of $\mathbb{R}$.

Similarly, an analogous argument shows that F is also a right fuzzy quasi-Baer subring. Hence, F is a fuzzy quasi-Baer subring of $\mathbb{R}$.

Example 3.4. Let $\eta : \mathbb{Z}_4 \to [0,1]$ be a fuzzy subring of $\mathbb{Z}_4$ defined by the membership function:

$$\eta(x) = \begin{cases} 0.95, & \text{if } x \in \{0,2\}, \\ 0.23, & \text{otherwise.} \end{cases}$$

Consider a fuzzy ideal $\mu \subseteq \eta$ with the left fuzzy annihilator $\mathcal{L}_{\mathbb{Z}_4}(\mu)$ given by:

$$\mathcal{L}_{\mathbb{Z}_4}(\mu)(x) = \begin{cases} 0.75, & \text{if } x \in \{0,2\}, \\ 0.15, & \text{otherwise.} \end{cases}$$

Observe that the set

$$\{x \in \mathbb{Z}_4 \mid \mathcal{L}_{\mathbb{Z}_4}(\mu)(x) = \mathcal{L}_{\mathbb{Z}_4}(\mu)(0)\} = \{0,2\}$$

is not generated by any idempotent element in $\mathbb{Z}_4$, since $\mathbb{Z}_4$ has no non-trivial idempotents. Therefore, η is not a fuzzy quasi-Baer subring of $\mathbb{Z}_4$.

Theorem 3.1. *Every fuzzy Baer subring of a ring R is a fuzzy quasi-Baer subring.*

Proof. Let F be a fuzzy Baer subring of R. Consider any left (or right) fuzzy ideal subset $\eta \subseteq \mathsf{F}$ of a subset $S \subseteq R$. From Definition 2.10, it follows directly that $\mathcal{L}_R(\eta)$ is generated by an idempotent of R. Hence, F is a fuzzy quasi-Baer subring of R. □

The converse, however, does not hold in general, as demonstrated by the following example.

Example 3.5. Consider the ring $R = U(\mathbb{Z}_2)$ of all upper triangular 2×2 matrices over $\mathbb{Z}_2$:

$$U(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{Z}_2 \right\}.$$

Define a fuzzy subring $\mathsf{F} : R \to [0,1]$ by the membership function:

$$\mathsf{F}(A) = \begin{cases} 0.8, & \text{if } A = \mathbf{0}, \\ 0.5, & \text{if } A = I, \\ 0.2, & \text{otherwise}, \end{cases}$$

where $\mathbf{0}$ and I denote the zero and identity matrices, respectively.

Claim 1: F is a fuzzy quasi-Baer subring of R.
Let $\eta \subseteq \mathsf{F}$ be a fuzzy ideal defined on the subring:

$$V(\mathbb{Z}_2) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \,\middle|\, a,b \in \mathbb{Z}_2 \right\},$$

with membership function:

$$\eta(A) = \begin{cases} p, & \text{if } A = \mathbf{0}, \text{ where } 0 \le p \le 0.8, \\ q, & \text{otherwise, with } 0 \le q \le 0.2. \end{cases}$$

The left fuzzy annihilator $\mathcal{L}_R(\eta)$ is given by:

$$\mathcal{L}_R(\eta)(A) = \begin{cases} p, & \text{if } A \in \{\mathbf{0}, E\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The set

$$\{A \in R \mid \mathcal{L}_R(\eta)(A) = \mathcal{L}_R(\eta)(\mathbf{0})\} = \{\mathbf{0}, E\}$$

is generated by the idempotent E, proving that F is a left fuzzy quasi-Baer subring. Additionally, it can be verified that $\mathcal{R}_R(\phi)$ is generated by the idempotent $\mathbf{0} \in R$. Hence, F is a fuzzy quasi-Baer subring of R.

Claim 2: F is not a fuzzy Baer subring of R.
Consider the fuzzy subset $\phi \subseteq \mathsf{F}$ defined on the subring:

$$W(\mathbb{Z}_2) = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \,\middle|\, b \in \mathbb{Z}_2 \right\},$$

with membership function:

$$\phi(A) = \begin{cases} r, & \text{if } A = \mathbf{0}, \\ s, & \text{otherwise}, \end{cases}$$

where $0 \le r \le 0.8$ and $0 \le s \le 0.2$. The left fuzzy annihilator satisfies:

$$\mathcal{L}_R(\phi)(A) = \begin{cases} p, & \text{if } A \in \{\mathbf{0}, E, N, N+E\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Since N cannot be generated by any idempotent in R, it follows that F is not a left fuzzy Baer subring. Hence, F is not a fuzzy Baer subring of R.

Theorem 3.2. *A fuzzy subring* F *of a ring* R *is a fuzzy quasi-Baer subring if and only if for every fuzzy ideal* η *of* F*, there exists an idempotent* $e \in R$ *such that the set of fuzzy ideal generated by* η *is contained in* eR*.*

Proof. Assume that F is a fuzzy quasi-Baer subring of R, and let η be a fuzzy ideal of F defined over a subset $S \subseteq R$. Since F is fuzzy quasi-Baer, the left fuzzy annihilator $\mathcal{L}_R(\eta)$ is generated by an idempotent $f \in R$, i.e.,

$$\{x \in R \mid \mathcal{L}_R(\eta)(x) = \mathcal{L}_R(\eta)(0)\} = fR.$$

Define $e = 1 - f$. For any $x \in S$ such that $\eta(x) = \eta(0)$, it follows that $x \in \eta_t$ for some $t = \sup_{z \in S} \eta(z)$. Since $fRx = 0$, we have $fx = 0$, which implies

$$x = (1 - f)x = ex \in eR.$$

Hence, the set of fuzzy ideal generated by η is contained in eR.

Conversely, assume that for every fuzzy ideal η of F, there exists an idempotent $e \in R$ such that the set of fuzzy ideal generated by η is contained in eR. To prove that F is a fuzzy quasi-Baer subring, it is sufficient to show that the left and right fuzzy annihilators of η are generated by idempotents.

Let $x \in R$ satisfy $\mathcal{L}_R(\eta)(x) = \mathcal{L}_R(\eta)(0)$. Then $x \in l(\eta_t)$ for some $t = \sup_{z \in S} \eta(z)$. Since $\eta_t \subseteq eR$, we have $xeR = 0$, implying $ex = 0$. Therefore,

$$x = (1 - e)x \in (1 - e)R.$$

Since $1 - e$ is an idempotent, it follows that $\mathcal{L}_R(\eta)$ is generated by an idempotent. Similarly, the right fuzzy annihilator $\mathcal{R}_R(\eta)$ is also generated by an idempotent in R. Thus, F is indeed a fuzzy quasi-Baer subring of R. □

4. Polynomial Extension of Fuzzy Baer and Fuzzy Quasi-Baer Subring

This section examines the behavior of fuzzy Baer and fuzzy quasi-Baer subrings under polynomial extensions. We demonstrate that while the polynomial extension of a fuzzy Baer subring may not retain the Baer property, the quasi-Baer property is preserved. The results are supported by an explicit example and two key theorems.

Theorem 4.1. *Let* F *be a fuzzy subring of a ring* R*, and let* ϕ *be a fuzzy polynomial subring of* $R[x]$ *induced by* η*. If* ϕ *is a fuzzy quasi-Baer subring of* $R[x]$*, then* F *is a fuzzy Baer subring of* R*.*

Proof. Let μ be an arbitrary fuzzy subset of F defined over a subset $S \subseteq R$. We can naturally extend μ to a fuzzy subset of ϕ. Since ϕ is a fuzzy quasi-Baer subring of $R[x]$, the left annihilator $\mathcal{L}_{R[x]}(\mu)$ is generated by an idempotent polynomial:

$$e(x) = \sum_{i=0}^{n} e_i x^i \in R[x], \quad \text{where } e(x)^2 = e(x).$$

The idempotence of $e(x)$ implies that its constant term e_0 satisfies $e_0^2 = e_0$, making e_0 an idempotent in R. Now, consider any $a \in R$. Since $ae(x)R[x] = 0$, evaluating at $x = 0$ yields $ae_0 = 0$. Thus, e_0 annihilates μ_t for some $t = \sup_{z \in R} \mu(z)$, meaning

$$e_0 \in l_R(\mu_t) \quad \text{and thus} \quad e_0 R \subseteq l_R(\mu_t).$$

For the reverse inclusion, take $b \in l_R(\mu_t)$. Since $l_R(\mu_t) = l_{R[x]}(\mu_t) \cap R$ and $l_{R[x]}(\mu_t) = e(x)R[x]$, there exists $h(x) \in R[x]$ such that:

$$b = e(x)h(x).$$

Evaluating at $x = 0$ gives $b = e_0 h(0) \in e_0 R$. Therefore,

$$l_R(\mu_t) \subseteq e_0 R.$$

Combining both inclusions, $l_R(\mu_t) = e_0 R$ is obtained. This shows that the left annihilator $\mathcal{L}_R(\mu)$ is generated by the idempotent $e_0 \in R$. By a similar argument, it can be shown that the right annihilator $\mathcal{R}_R(\mu)$ is generated by an idempotent in R, proving that F is a fuzzy Baer subring of R. □

The next example shows that the polynomial extension of a fuzzy Baer subring need not retain the Baer property.

Example 4.1. Let $R = M_2(\mathbb{Z})$ be the ring of 2×2 matrices over the integers $\mathbb{Z}$. Define a fuzzy subring F on R as follows:

$$\mathsf{F}(A) = \begin{cases} 0.8, & \text{if } A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ 0.5, & \text{if } A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \text{ where } a \neq 0, \\ 0.2, & \text{otherwise.} \end{cases}$$

Since for every $t \in \operatorname{Im}\mathsf{F}$, the t-level cut F_t is a subring of R, it follows that F is indeed a fuzzy subring of R.

Claim: F is a fuzzy Baer subring of R.
To prove that F is a fuzzy Baer subring of R, consider any fuzzy subset $\eta \subseteq \mathsf{F}$ over $M_2(S)$ for some subset $S \subseteq \mathbb{Z}$. For any $A \in M_2(S)$, the left fuzzy annihilator of η is defined as:

$$\mathcal{L}_R(\eta)(A) = \begin{cases} \sup\{p\}, & \text{if } A \in l(\eta_p), \\ 0, & \text{if } A \notin l(\eta_p), \end{cases}$$

where $p \in \operatorname{Im}\mathsf{F}$. Since R is a Baer ring, for any $t \in \operatorname{Im}\eta$, the left annihilator $l(\eta_t)$ is generated by an idempotent matrix $E \in R$, i.e., $l(\eta_p) = ER$. Consequently, the left fuzzy annihilator of η is generated by the idempotent E. An analogous result holds for the right fuzzy annihilator $\mathcal{R}_R(\eta)$. Consequently, both left and right fuzzy annihilators are generated by idempotents in R, establishing that F is a fuzzy Baer subring of R.

Claim: The polynomial extension of a fuzzy Baer subring is not necessarily a fuzzy Baer subring. Let ϕ be the polynomial extension of F over $R[x]$. Consider any fuzzy subset $\mu \subseteq \phi$ over $M_2(T)[x] \subseteq R[x]$, where $T = \{0, 1, 2\} \subseteq \mathbb{Z}$. Let

$$p(x) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x \in M_2(T)[x].$$

We have $\phi(p(x)) = 0.2$. The right fuzzy annihilator of $p(x) \in M_2(T)[x]$ over $R[x]$ is given by

$$\mathcal{R}_{R[x]}(\mu)(p(x)) = \begin{cases} \sup\{t\}, & \text{if } p(x) \in r(\mu_t), \\ 0, & \text{if } p(x) \notin r(\mu_t). \end{cases}$$

Since $\mu_t \subseteq M_2(T)[x]$, it can be easily observed that

$$r(\mu_t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the right fuzzy annihilator of μ is not generated by a nonzero idempotent in $R[x]$. Hence, ϕ is not a fuzzy Baer subring of $R[x]$.

Theorem 4.2. *Let* F *be fuzzy quasi-Baer subring of* R. *If* ϕ *is fuzzy polynomial subring of* $R[x]$ *over* F *then* ϕ *is fuzzy quasi-Baer subring of* $R[x]$.

Proof. The fuzzy polynomial subring ϕ of $R[x]$ is defined by

$$\phi(f(x)) = \min_i \left\{ \mathsf{F}(a_i) \mid f(x) = \sum_i a_i x^i \right\},$$

where F is a fuzzy subring of R. Let $\psi \subseteq \phi$ be a fuzzy ideal of $S[x] \subseteq R[x]$. The left fuzzy annihilator of ψ for any polynomial $g(x) = \sum_i b_i x^i \in S[x]$ is given by

$$\mathcal{L}_{R[x]}(\psi)(g(x)) = \begin{cases} \sup\{t\}, & \text{if } g(x) \in l(\psi_t) \text{ for some } t \in \operatorname{Im}(\psi), \\ 0, & \text{otherwise.} \end{cases}$$

To establish the result, it is sufficient to show that $\mathcal{L}_{R[x]}(\psi)$ is generated by an idempotent in $R[x]$. Consider the fuzzy subset $\eta \subseteq \mathsf{F}$ over $S \subseteq R$ defined by

$$\eta(a) = \begin{cases} \psi(g(x)), & \text{if } a = b_0 \text{ (the lowest-degree coefficient of } g(x)), \\ 0, & \text{otherwise.} \end{cases}$$

We now prove that η is a fuzzy ideal of S. Let $r_0, s_0 \in S$ be nonzero elements. Then there exist polynomials $r(x) = r_0 + \sum_{i=1}^{k} r_i x^i$ and $s(x) = s_0 + \sum_{i=1}^{l} s_i x^i$ in $S[x]$ such that r_0 and s_0 are their respective lowest-degree coefficients. By the definition of η, we have

$$\eta(r_0 - s_0) = \psi(r(x) - s(x)) = \min\{\psi(r(x)), \psi(s(x))\} = \min\{\eta(r_0), \eta(s_0)\}$$

and

$$\eta(r_0 s_0) = \psi(r(x)s(x)) = \max\{\psi(r(x)), \psi(s(x))\} = \max\{\eta(r_0), \eta(s_0)\}.$$

Thus, η is a fuzzy ideal of S. Since η is a fuzzy ideal, $L_R(\eta)$ is generated by an idempotent $e \in R$, and we have

$$\{x \in S \mid \mathcal{L}_R(\eta)(x) = \mathcal{L}_R(\eta)(0)\} = Re.$$

Now, let $p(x) = \sum_{i=0}^{m} p_i x^i \in S[x]$ be such that

$$\mathcal{L}_{R[x]}(\psi)(p(x)) = \mathcal{L}_{R[x]}(\psi)(0) = u, \quad \text{where } u = \sup_{f(x) \in S[x]} \{\psi(f(x))\}.$$

This implies $p(x) \in l(\psi_u)$, and thus there exists a polynomial $q(x) = \sum_{i=0}^{n} q_i x^i \in R[x]$ with $\psi(q(x)) \geq u$ such that $p(x)q(x) = 0$. From this, we deduce $p_0 q_0 = 0$. Consequently, $q_0 \in \eta_u$

and $p_0 \in l(\eta_u)$. Therefore,

$$\mathcal{L}_R(\eta)(p_0) = \mathcal{L}_R(\eta)(0) = u.$$

Since η is a fuzzy ideal, it follows that $p_0 = ep_0$. Next, examining the coefficient of x in the product $p(x)q(x) = 0$, we obtain

$$p_0q_1 + p_1q_0 = 0.$$

Given that $p_0q_1 = 0$, it follows that $p_1q_0 = 0$, and thus $p_1 = ep_1$. Repeating this argument for each coefficient p_i, we conclude that $p_i = ep_i$ for all i. Hence,

$$p(x) = ep(x),$$

which shows that $p(x)$ is generated by the idempotent e. Therefore, $\mathcal{L}_{R[x]}(\psi)$ is generated by a nonzero idempotent e. A similar argument holds for the right fuzzy annihilator $\mathcal{R}_{R[x]}(\psi)$, and hence ϕ is a fuzzy quasi-Baer subring of $R[x]$. □

Theorem 4.3. *Let* F *be fuzzy quasi-Baer subring of* R. *If* ω *is fuzzy power series subring of* $R[[x]]$ *over* F *then* ω *is fuzzy quasi-Baer subring of* $R[[x]]$.

Proof. This follows directly from Theorem 4.2. □

5. Conclusion

This study establishes the fundamental properties of fuzzy quasi-Baer subrings, demonstrating that while every fuzzy Baer subring is quasi-Baer, the converse fails in general. The quasi-Baer property is shown to be preserved under polynomial extensions, unlike the Baer property, with the critical result that when such an extension is quasi-Baer, the base subring must be Baer. These findings extend classical Baer ring theory to fuzzy settings while revealing key distinctions in their behavior under algebraic constructions. The work provides new insights into annihilator conditions in fuzzy algebra and suggests potential applications in fuzzy module theory and homological algebra, offering a foundation for further research in noncommutative fuzzy algebraic structures.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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