Communications in Mathematics and Applications
Vol. 16, No. 3, pp. 703–716, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
http://www.rgnpublications.com
DOI: 10.26713/cma.v16i3.3371

Research Article

Pythagorean Fuzzy (l,u)-Level Cuts and Their Applications to Fuzzy Normed Subrings

Amal Kumar Adak*[1] and Gauri Kant Kumar[2]

[1] *Department of Mathematics, Ganesh Dutt College (affiliated to Lalit Narayan Mithila University), Begusarai 851101, Bihar, India*
[2] *Department of Mathematics, Lalit Narayan Mithila University, Darbhanga 846008, Bihar, India*
***Corresponding author:** amaladak17@gmail.com

Received: September 10, 2025 **Revised:** October 2, 2025 **Accepted:** October 11, 2025

Abstract. When attempting to quantify an ill-quantity, the *Pythagorean Fuzzy Set* (PFS) notion is essential. This study provides a (l,u)-level cut of Pythagorean Fuzzy Sets (PFSs). We define (l,u)-level cuts of a PFS (P) as crisp sets consisting of elements (x) for which the membership value is greater than (l) and non-membership value is less than (u). The concept of Pythagorean fuzzy normed subring is introduced and its properties are investigated. Additionally, the relationship between PFNSRs and Pythagorean fuzzy (l,u)-level sets is analyzed.

Keywords. Intuitionistic fuzzy set, Pythagorean fuzzy set, t-norm, s-norm, (l,u)-level sets, Pythagorean fuzzy normed subring

Mathematics Subject Classification (2020). 16Y30, 03E72, 16Y99

1. Introduction

Zadeh [15] introduced *Fuzzy Sets* (FSs) in 1965. Fuzzy sets are extensively used in many different industries. They offer a useful approach to communicating imprecise notions in a common language and provide a boarder means to quantify uncertainties more effectively. It is well known that most systems built on 'two-valued logic' or 'crisp set theory' have difficulty processing ambiguous and imprecise data. Fuzzy sets can therefore be utilised to deliver more practical answers to a wider range of issues.

In real-world scenarios, it is not always valid to assume that degree of non-membership is simply complement of degree of membership, as classical fuzzy sets are restricted to membership values between 0 and 1. To address this limitation, Atanassov [5] extended the concept of fuzzy set theory and introduced *Intuitionistic Fuzzy Sets* (IFSs). An IFS incorporates not only a degree of membership (μ) but also a degree of non-membership (ν), along with a degree of hesitation or uncertainty, defined as uncertainty $= 1-(\mu+\nu)$. This extension enhances the reliability and applicability of fuzzy modeling. In IFSs, each element is characterized by a pair of values (μ,ν), where μ and ν denote membership and non-membership degrees, respectively, subject to condition $\mu+\nu \le 1$.

Yager [12], and Yager and Abbasov [13] expanded the condition $\mu+\nu<1$ to $\mu^2+\nu^2 \le 1$. They also presented a class of PFSs with membership values represented by ordered pairs (μ,ν) that satisfy necessary condition $\mu^2+\nu^2 \le 1$. These PFSs can be applied in multicriterion decision-making and have different aggregation procedures. *Pythagorean Membership Values* (PMVs) are also part of space of all *Intuitionistic Membership Values* (IMVs), although PMVs are not required to be IMVs. For example, PFSs can be used when $\mu = 0.7$ and $\nu = 0.4$, however, IFSs cannot be used in this case since $\mu+\nu>1$, but $\mu^2+\nu^2 \le 1$. Because PFSs are broader than IFSs, they may address a boarder range of real world issues involving uncertainty and imprecision. IFSs are an extension of PFSs with the added benefit of not having underlying constraints. Researchers have recently interest in PFSs (Adak and Kumar [2]), as they provide an effective tool for modeling uncertain data.

Since natural language claims are modeled by FSs in many applications, ambiguity is frequently unavoidable. For instance, a patient's blood pressure or temperature may be measured very precisely, but symptoms like sleeplessness or headaches can only be expressed in natural language. The fuzzy logic of level sets studies operations with these kinds of arguments. This provides academics with sufficient motivation to re-examine several ideas and findings on level sets within a boarder fuzzy network. 'Different kinds of cut sets of interval-valued IFSs' were shown by Adak *et al*. [4], who also introduce the ideas of decomposition theorems on these findings. '(α,β)-Cut of Intuitionistic Fuzzy Ideals' was presented by Basnet [7]. Some results on 'level sets and minimal volume sets of probability density functions' are presented by Garcia *et al*. [9]. The *Lebesgue measure of α-cuts technique for determining the height of the membership function* was presented by Pap and Surla [11]. 'Three novel cut sets of FSs and new theories of FSs' were introduced by Yuan *et al*. [14].

Rest of the paper is organized as: Section 2 contains introductions and definitions of terms such as FSs, IFSs, PFSs, t-norm, and s-norm. Section 3 presents the idea of the (l,u)-level set of PFSs and highlights some of the key ideas of the PFNSR. There is a conclusion in Section 4.

2. Preliminaries and Definitions

We review relevant concepts of FSs, IFSs, and PFSs in this section.

Definition 2.1 ([15])**.** A fuzzy set F in X is defined as

$$F = \{\langle v, \widehat{\rho}_F(v)\rangle : v \in X\},$$

where $\widehat{\rho}_F : X \to [0,1]$ is a membership function.

The complement of ρ is $\bar{\rho}(v) = 1-\rho(v)$, for all $v \in X$ and denoted by $\bar{\rho}$.

Definition 2.2 ([5])**.** An *Intuitionistic Fuzzy Set* (IFS) A in X is

$$A = \{\langle v, \widehat{\rho}_A(v), \widehat{\sigma}_A(v)\rangle : v \in X\},$$

where $\widehat{\rho}_A(x)$ is MG and $\widehat{\sigma}_A(v)$ is the NMG of $v \in X$, respectively.
Also, $\widehat{\rho}_A : X \to [0,1]$, $\widehat{\sigma}_A : X \to [0,1]$ and satisfies

$$0 \le \widehat{\rho}_A(v) + \widehat{\sigma}_A(v) \le 1, \quad \text{for all } v \in X.$$

The indeterminacy $h_A(v) = 1 - \widehat{\rho}_A(v) - \widehat{\sigma}_A(v)$.

Sometimes, the condition $0 \le \rho(v) + \sigma(v) \le 1$ may not be true. For example, $0.5 + 0.7 = 1.2 > 1$, but $0.5^2 + 0.7^2 < 1$, or $0.6 + 0.6 = 1.2 > 1$, but $0.6^2 + 0.6^2 < 1$. To demonstrate this issue, Yager [12], and Yager and Abbasov [13] proposed Pythagorean fuzzy sets in 2013.

Definition 2.3 ([12])**.** A Pythagorean fuzzy set $\widehat{P}$ in X is

$$\widehat{P} = \{\langle v, \widehat{\rho}_{\widehat{P}}(v), \widehat{\sigma}_{\widehat{P}}(v)\rangle \mid v \in X\},$$

where $\widehat{\rho}_{\widehat{P}}(v) : X \to [0,1]$ indicates MG and $\widehat{\sigma}_{\widehat{P}}(v) : X \to [0,1]$ represents NMG of $v \in X$ with

$$0 \le (\widehat{\rho}_{\widehat{P}}(v))^2 + (\widehat{\sigma}_{\widehat{P}}(v))^2 \le 1,$$

for all $v \in X$.
Indeterminacy $h_{\widehat{P}}(v) = \sqrt{1 - (\widehat{\rho}_{\widehat{P}}(v))^2 - (\widehat{\sigma}_{\widehat{P}}(v))^2}$.

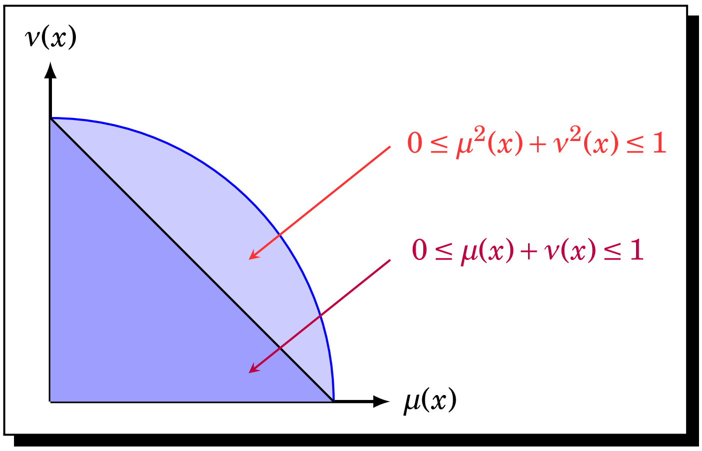

Figure 1. Comparison spaces for intuitionistic fuzzy and pythagorean membership grades

2.1 Some Operation of Pythagorean Fuzzy sets

Two PFNSRs, $\widehat{P}_1$ and $\widehat{P}_2$, are considered, then the following operations and relations can be defined as

$$\widehat{P}_1 \subseteq \widehat{P}_2 \text{ iff } \widehat{\rho}_{\widehat{P}_1}(v) \le \widehat{\rho}_{\widehat{P}_2}(v) \text{ and } \widehat{\sigma}_{\widehat{P}_1}(v) \ge \widehat{\sigma}_{\widehat{P}_2}(v), \text{ for all } v \in E,$$

$$\widehat{P}_1 = \widehat{P}_2 \text{ iff } \widehat{\rho}_{\widehat{P}_1}(v) = \widehat{\rho}_{\widehat{P}_2}(v) \text{ and } \widehat{\sigma}_{\widehat{P}_1}(v) = \widehat{\sigma}_{\widehat{P}_2}(v), \text{ for all } v \in E,$$

$$\widehat{P}_1 \cap \widehat{P}_2 = \{\langle v \cdot \min(\widehat{\rho}_{\widehat{P}_1}(v), \widehat{\rho}_{\widehat{P}_2}(v)), \max(\widehat{\sigma}_{\widehat{P}_1}(v), \widehat{\sigma}_{\widehat{P}\widehat{P}_2}(v))\rangle : v \in E\},$$

$$\widehat{P}_1 \cup \widehat{P}_2 = \{\langle v \cdot \max(\widehat{\rho}_{\widehat{P}_1}(v), \widehat{\rho}_{\widehat{P}_2}(v)), \min(\widehat{\sigma}_{\widehat{P}_1}(v), \widehat{\sigma}_{\widehat{P}_2}(v))\rangle : v \in E\}.$$

Definition 2.4. Two PFSs, $\widehat{P}$ and $\widehat{Q}$, are considered in R.
Then $\widehat{P}+\widehat{Q}$ is

$$\widehat{P}+\widehat{Q}=\{(v,\widehat{\rho}_{\widehat{P}+\widehat{Q}}(v),\widehat{\sigma}_{\widehat{P}+\widehat{Q}}(v)):v\in X\},$$

where

$$\widehat{\rho}_{\widehat{P}+\widehat{Q}}(v)=\begin{cases}\bigvee_{v=y+z}(\widehat{\rho}_{\widehat{P}}(y)\wedge\widehat{\rho}_{\widehat{Q}}(z)), & \text{if } v=y+z,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\widehat{\sigma}_{\widehat{P}+\widehat{Q}}(v)=\begin{cases}\bigwedge_{v=y+z}(\widehat{\rho}_{\widehat{P}}(y)\vee\widehat{\rho}_{\widehat{Q}}(z)), & \text{if } v=y+z,\\ 1, & \text{otherwise.}\end{cases}$$

Definition 2.5. Let $\widehat{P}$ and $\widehat{Q}$ be two PFSs in R.
Then $\widehat{P}\circ\widehat{Q}$ is

$$\widehat{P}\circ\widehat{Q}=\{(v,\widehat{\rho}_{\widehat{P}\oplus\widehat{Q}}(v),\widehat{\sigma}_{\widehat{P}\otimes\widehat{Q}}(v)):v\in X\},$$

where

$$\widehat{\rho}_{\widehat{P}\oplus\widehat{Q}}(v)=\begin{cases}\bigvee_{v=yz}(\widehat{\rho}_{\widehat{P}}(y)\wedge\widehat{\rho}_{\widehat{Q}}(z)), & \text{if } v=yz,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\widehat{\sigma}_{\widehat{P}\otimes\widehat{Q}}(v)=\begin{cases}\bigwedge_{v=yz}(\widehat{\rho}_{\widehat{P}}(y)\vee\widehat{\rho}_{\widehat{Q}}(z)), & \text{if } v=yz,\\ 1, & \text{otherwise.}\end{cases}$$

Example 2.1. Let $Z_5=\{0,1,2,3,4\}$ and consider the ring $(Z_5,\times_5,+_5)$. $\widehat{P}$ and $\widehat{Q}$ be two PFSs, where

$$\widehat{P}=\{(0,\langle 0.5,0.3\rangle),(1,\langle 0.4,0.4\rangle),(2,\langle 0.7,0.2\rangle),(3,\langle 0.8,0.1\rangle),(4,\langle 0.6,0.6\rangle)\}$$

and

$$\widehat{Q}=\{(0,\langle 0.4,0.8\rangle),(1,\langle 0.5,0.7\rangle),(2,\langle 0.4,0.5\rangle),(3,\langle 0.5,0.6\rangle),(4,\langle 0.9,0.1\rangle)\}.$$

Then,

$$\widehat{P}+\widehat{Q}=\{(0,\langle 0.5,0.4\rangle),(1,\langle 0.7,0.2\rangle),(2,\langle 0.8,0.1\rangle),(3,\langle 0.6,0.5\rangle),(4,\langle 0.5,0.3\rangle)\}$$

and

$$\widehat{P}\circ\widehat{Q}=\{(0,\langle 0.5,0.6\rangle),(1,\langle 0.5,0.4\rangle),(2,\langle 0.4,0.6\rangle),(3,\langle 0.5,0.5\rangle),(4,\langle 0.6,0.6\rangle)\}.$$

Definition 2.6. Triangular norm (t-norm) is a mapping $T:[0,1]^2\to[0,1]$, satisfies:

(i) $T(x,T(y,z))=T(T(x,y),z)$, for all $x,y,z\in X$.

(ii) $T(x,y)=T(y,x)$, for all $x,y\in X$.

(iii) If $x_1,x_2,x_3,x_4\in[0,1]$ and $x_1\le x_2$, $x_3\le x_4$, then $T(x_1,x_3)\le T(x_2,x_4)$.

(iv) $T(x,1)=x$, for all $x\in X$.

The most commonly used t-norms are

(i) $T_M(x,y) = \min\{x,y\}$, for all $x,y \in [0,1]$.

(ii) $T_P(x,y) = \max\{x,y\}$, for all $x,y \in [0,1]$.

(iii) $T_L(x,y) = \max\{x+y-1,0\}$, for all $x,y \in [0,1]$.

Definition 2.7. Triangular conorm (t-conorm) is a mapping $S:[0,1]^2 \to [0,1]$, satisfies:

(i) $S(x,S(y,z)) = S(T(x,y),z)$, for all $x,y,z \in X$.

(ii) $S(x,y) = S(y,x)$, for all $x,y \in X$.

(iii) If $x_1,x_2,x_3,x_4 \in [0,1]$ and $x_1 \le x_2$, $x_3 \le x_4$, then $S(x_1,x_3) \le S(x_2,x_4)$.

(iv) $T(0,x) = x$, for all $x \in X$.

The most commonly used t-conorms are

(i) $S_M(x,y) = \max\{x,y\}$, for all $x,y \in [0,1]$.

(ii) $S(x,y) = x+y-xy$, for all $x,y \in [0,1]$.

Definition 2.8. A *Normed Ring* (NR) is a ring possess a norm $\| \ \|$, that is a non-negative real valued function $\| \ \| : NR \to \mathbb{R}$, satisfies

(i) $\|x\| = 0 \Longleftrightarrow x = 0$.

(ii) $\|x+y\| \le \|x\| + \|y\|$.

(iii) $\|xy\| \le \|x\|\|y\|$.

(iv) $\|x\| = \|-x\|$, for all $x,y \in NR$.

Definition 2.9. A PFNSR $\widehat{P} = \{(v, \widehat{\rho}_{\widehat{P}}(v), \widehat{\sigma}_{\widehat{P}}(v)) : v \in NR\}$ is called *Pythagorean Fuzzy Normed Subring* (PFNSR) of the normed ring $(NR,+,\cdot)$ if it satisfies

(i) $\rho_{\widehat{P}}(v-r) \ge \min\{\rho_{\widehat{P}}(v), \rho_{\widehat{P}}(r)\}$,

(ii) $\rho_{\widehat{P}}(vr) \ge \min\{\rho_{\widehat{P}}(v), \rho_{\widehat{P}}(r)\}$,

(iii) $\sigma_{\widehat{P}}(v-r) \le \max\{\sigma_{\widehat{P}}(v), \sigma_{\widehat{P}}(r)\}$,

(iv) $\sigma_{\widehat{P}}(vr) \le \max\{\sigma_{\widehat{P}}(v), \sigma_{\widehat{P}}(r)\}$, for all $v,r \in NR$.

Definition 2.10. Let NR be a normed ring. A Pythagorean fuzzy subring $\widehat{P}$ of NR is Pythagorean fuzzy normed normal subring of NR if

(i) $\rho_{\widehat{P}}(vr) = \rho_{\widehat{P}}(rv)$,

(ii) $\sigma_{\widehat{P}}(vr) = \sigma_{\widehat{P}}(rv)$, for all $v,r \in NR$.

3. Main Results

In many practical applications, it is often necessary for a given PFNSR to consider the subsets of elements whose assigned grades meet or exceed a prescribed threshold. In this context, level sets of PFSs play a crucial role. In this section, we introduce (l,u)-level sets of PFSs, establish several important results concerning these levels, and validate the findings with appropriate examples.

Definition 3.1. Let $\widehat{P} = \{(v, \widehat{\rho}_{\widehat{P}}(v)), \widehat{\sigma}_{\widehat{P}}(v) : v \in NR\}$ be a PFNSR of a normed ring NR. The (l,u)-level set of $\widehat{P}$ is

$$\mathcal{L}_{l,u}(\widehat{P}) = \{v \in NR : \widehat{\rho}_{\widehat{P}}(v) \geq l \text{ and } \widehat{\sigma}_{\widehat{P}}(v) \leq u\},$$

where $l+u \leq 1$ and $l,u \in [0,1]$.

Theorem 3.1. *Let $\widehat{P}$ be a PFNSR. Then $\mathcal{L}_{l,u}$ is a PFNSR of NR if $\widehat{\rho}_{\widehat{P}}(0) \geq l$ and $\widehat{\sigma}_{\widehat{P}}(0) \leq u$ with $l+u \leq 1$.*

Proof. Clearly, $\mathcal{L}_{l,u}(\widehat{P}) \neq \emptyset$, since $0 \in \mathcal{L}_{l,u}(\widehat{P})$. Let $v_1, v_2 \in \mathcal{L}_{l,u}(\widehat{P})$.
Then,

$$\widehat{\rho}_{\widehat{P}}(v_1) \geq l, \quad \widehat{\rho}_{\widehat{P}}(v_2) \geq l$$

and

$$\widehat{\sigma}_{\widehat{P}}(v_1) \leq u, \quad \widehat{\sigma}_{\widehat{P}}(v_2) \leq u.$$

Then

$$\widehat{\rho}_{\widehat{P}}(v_1 - v_2) \geq \min\{\widehat{\rho}_{\widehat{P}}(v_1), \widehat{\rho}_{\widehat{P}}(v_2)\} \geq l$$

and

$$\widehat{\sigma}_{\widehat{P}}(v_1 - v_2) \leq \max\{\widehat{\sigma}_{\widehat{P}}(v_1), \widehat{\sigma}_{\widehat{P}}(v_2)\} \leq u.$$

Thus,

$$v_1 - v_2 \in \mathcal{L}_{l,u}(\widehat{P}).$$

Therefore, $\mathcal{L}_{l,u}$ is a subring.
Now, let $r \in NR$,

$$\widehat{\rho}_{\widehat{P}}(v_1 r) \geq \min\{\widehat{\rho}_{\widehat{P}}(v_1), \widehat{\rho}_{\widehat{P}}(r)\} \geq \widehat{\rho}_{\widehat{P}}(v_1) \geq l$$

and

$$\widehat{\rho}_{\widehat{P}}(r v_1) \geq \min\{\widehat{\rho}_{\widehat{P}}(r), \widehat{\rho}_{\widehat{P}}(v_1)\} \geq \widehat{\rho}_{\widehat{P}}(v_1) \geq l.$$

Also,

$$\widehat{\sigma}_{\widehat{P}}(v_1 r) \leq \max\{\widehat{\sigma}_{\widehat{P}}(v_1), \widehat{\sigma}_{\widehat{P}}(r)\} \leq \widehat{\sigma}_{\widehat{P}}(v_1) \leq u$$

and

$$\widehat{\sigma}_{\widehat{P}}(r v_1) \leq \max\{\widehat{\sigma}_{\widehat{P}}(r), \widehat{\sigma}_{\widehat{P}}(v_1)\} \leq \widehat{\sigma}_{\widehat{P}}(v_1) \leq u.$$

Thus, $v_1 r$ and $r v_1 \in \mathcal{L}_{l,u}(\widehat{P})$.
Therefore, $\mathcal{L}_{l,u}(\widehat{P})$ is a PFNSR of NR. □

Theorem 3.2. *If $\widehat{P}$ is a PFNSR in NR, then $\mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{\theta,\gamma}(\widehat{P})$ if $l \geq \theta$ and $u \leq \gamma$.*

Proof. Let $v \in \mathcal{L}_{l,u}(\widehat{P})$. Then $\widehat{\rho}_{\widehat{P}}(v) \geq l$ and $\widehat{\sigma}_{\widehat{P}}(v) \leq u$.
As $l \geq \theta$ and $u \leq \gamma$, then

$$\widehat{\rho}_{\widehat{P}}(v) \geq l \geq \theta \quad \text{and} \quad \widehat{\sigma}_{\widehat{P}}(v) \leq u \leq \gamma.$$

Therefore,

$$v \in \mathcal{L}_{\theta,\gamma}(\widehat{P}).$$

Hence,

$$\mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{\theta,\sigma}(\widehat{P}).$$ □

Example 3.1. Let $\widehat{P}$ be a PFSs and

$$\widehat{P} = \{(0,\langle 0.7,0.2\rangle),(1,\langle 0.6,0.4\rangle),(2,\langle 0.8,0.4\rangle),(3,\langle 0.5,0.6\rangle),(4,\langle 0.8,0.3\rangle)\},$$

for $l = 0.5$, $u = 0.2$; and $\theta = 0.4$, $\gamma = 0.7$, so that $l \geq \theta$ and $u \leq v$.
We have $L_{l,u}(\widehat{P}) = \{0\}$ and $L_{\theta,\gamma}(\widehat{P}) = \{0,1,2,3,4\}$. Clearly,

$$L_{l,u}(\widehat{P}) \subseteq L_{\theta,\gamma}(\widehat{P}).$$

Theorem 3.3. *If $l + u \leq 1$, then $\mathcal{L}_{1-u,u}(\widehat{P}) \subseteq \mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{l,1-l}(\widehat{P})$.*

Proof. As $l + u \leq 1$, then $1 - u \geq l$. Then,

$$\mathcal{L}_{1-u,u}(\widehat{P}) \subseteq \mathcal{L}_{l,u}(\widehat{P}).$$

Also, $u \leq 1 - l$, then

$$\mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{l,1-l}(\widehat{P}).$$

Therefore,

$$\mathcal{L}_{1-u,u}(\widehat{P}) \subseteq \mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{l,1-l}(\widehat{P}).$$ □

Theorem 3.4. *Let $\mathcal{L}_{l,u}(\widehat{P})$ be a Pythagorean fuzzy normed normal subring with $l + u \leq 1$ and $l,u \in [0,1]$, then $\widehat{P}$ is a PFNSR in NR.*

Proof. Let $v_1, v_2 \in NR$ and $l = \widehat{\rho}_{\widehat{P}}(v_1) \wedge \widehat{\rho}_{\widehat{P}}(v_2)$ and $u = \widehat{\sigma}_{\widehat{P}}(v_1) \vee \widehat{\sigma}_{\widehat{P}}(v_2)$.
$\widehat{\rho}_{\widehat{P}}(v_1) \geq l$, $\widehat{\sigma}_{\widehat{P}}(v_1) \leq u$ and $\widehat{\rho}_{\widehat{P}}(v_2) \geq l$, $\widehat{\sigma}_{\widehat{P}}(v_2) \leq u$.
Then, $v_1, v_2 \in \mathcal{L}_{l,u}(\widehat{P})$. Since, $\mathcal{L}_{l,u}(\widehat{P})$ is an ideal of NR, so $v_1 - v_2 \in \mathcal{L}_{l,u}(\widehat{P})$.
Therefore,

$$\widehat{\rho}_{\widehat{P}}(v_1 - v_2) \geq l = \{\widehat{\rho}_{\widehat{P}}(v_1) \wedge \widehat{\rho}_{\widehat{P}}(v_2)\}$$

and

$$\widehat{\sigma}_{\widehat{P}}(v_1 - v_2) \leq u = \{\widehat{\sigma}_{\widehat{P}}(v_1) \vee \widehat{\sigma}_{\widehat{P}}(v_2)\}.$$

Assume, $\lambda = \widehat{\rho}_{\widehat{P}}(v_1) \vee \widehat{\rho}_{\widehat{P}}(v_2)$. Without loss of generality, let $\lambda = \widehat{\rho}_{\widehat{P}}(v_1)$.
Since,

$$\widehat{\rho}_{\widehat{P}}(v_1) + \widehat{\sigma}_{\widehat{P}}(v_1) \leq 1,$$

so

$$\widehat{\sigma}_{\widehat{P}}(v_1) \leq 1 - \widehat{\rho}_{\widehat{P}}(v_1) \leq 1 - \lambda,$$

then

$$v_1 \in \mathcal{L}_{\lambda,1-\lambda}(\widehat{P}).$$

Since, $\mathcal{L}_{\lambda,1-\lambda}(\widehat{P})$ is an ideal of NR, so $v_1 v_2 \in \mathcal{L}_{\lambda,1-\lambda}(\widehat{P})$.

Thus,

$$\widehat{\rho}_{\widehat{P}}(v_1v_2) \geq \lambda = \widehat{\rho}_{\widehat{P}}(v_1) \vee \widehat{\rho}_{\widehat{P}}(v_2).$$

Similarly, it can be proven that

$$\widehat{\sigma}_{\widehat{P}}(v_1v_2) \leq \widehat{\sigma}_{\widehat{P}}(v_1) \wedge \widehat{\sigma}_{\widehat{P}}(v_2).$$

Hence, $\widehat{P}$ is a PFNSR of NR. □

Theorem 3.5. *Let $\widehat{P}$ and $\widehat{Q}$ be two PFNSR of NR. If $\widehat{P} \subseteq \widehat{Q}$, then $\mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{l,u}(\widehat{Q})$.*

Proof. Let $v \in \mathcal{L}_{l,u}(\widehat{P})$, then $\widehat{\rho}_{\widehat{P}}(v) \geq l$ and $\widehat{\sigma}_{\widehat{P}}(v) \leq u$. As $\widehat{P} \subseteq \widehat{Q}$ implies $\widehat{\rho}_{\widehat{Q}}(v) \geq \widehat{\rho}_{\widehat{P}}(v) \geq l$ and $\widehat{\sigma}_{\widehat{Q}}(v) \leq \widehat{\sigma}_{\widehat{P}}(v) \leq u$.
Hence, $\widehat{\rho}_{\widehat{Q}}(v) \geq l$ and $\widehat{\sigma}_{\widehat{Q}}(v) \leq u$. So, $v \in \mathcal{L}_{l,u}(\widehat{Q})$.
Therefore, $\mathcal{L}_{l,u}(\widehat{P}) \subseteq \mathcal{L}_{l,u}(\widehat{Q})$. □

Example 3.2. Consider two PFSs $\widehat{P}$ and $\widehat{P}$ such that

$$\widehat{P} = \{(0,\langle 0.8,0.2\rangle),(1,\langle 0.7,0.6\rangle),(2,\langle 0.9,0.2\rangle),(3,\langle 0.6,0.2\rangle),(4,\langle 0.8,0.5\rangle)\},$$

$$\widehat{Q} = \{(0,\langle 0.9,0.1\rangle),(1,\langle 0.8,0.2\rangle),(2,\langle 0.9,0.12\rangle),(3,\langle 0.7,0.1\rangle),(4,\langle 0.8,0.2\rangle)\},$$

where $\widehat{P} \subseteq \widehat{Q}$; for $l = 0.4$, $u = 0.6$; $L_{l,u}(\widehat{P}) = \{0,1\}$ and $L_{l,u}(\widehat{Q}) = \{0,1,2,3,4\}$.
Therefore, $L_{l,u}(\widehat{P}) \subseteq L_{l,u}(\widehat{Q})$.

Theorem 3.6. *If $\widehat{P}_1$ and $\widehat{P}_2$ are two PFNSR of NR. Then,*

$$\mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1) \cap \mathcal{L}_{l,u}(\widehat{P}_2).$$

Proof. Since, $\widehat{P}_1 \cap \widehat{P}_2 \subseteq \widehat{P}_1$ and $\widehat{P}_1 \cap \widehat{P}_2 \subseteq \widehat{P}_2$.
Therefore,

$$\mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) \text{ and } \mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_2).$$

Hence

$$\mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) \cap \mathcal{L}_{l,u}(\widehat{P}_2). \tag{3.1}$$

Also, let $v \in \mathcal{L}_{l,u}(\widehat{P}_1) \cap \mathcal{L}_{l,u}(\widehat{P}_2)$ implies $v \in \mathcal{L}_{l,u}(\widehat{P}_1)$ and $v \in \mathcal{L}_{l,u}(\widehat{P}_2)$.
Then,

$$\widehat{\rho}_{(}v) \geq l,\ \widehat{\rho}_{\widehat{P}_2}(v) \geq l \text{ and } \widehat{\sigma}_P(v) \leq u,\ \widehat{\sigma}_{\widehat{P}_2}(v) \leq u,$$

$$\min\{\widehat{\rho}_{\widehat{P}_1}(v),\widehat{\rho}_{\widehat{P}_2}(v)\} \geq l \text{ and } \max\{\widehat{\sigma}_{\widehat{P}_1}(v),\widehat{\sigma}_{\widehat{P}_2}(v)\} \leq u$$

$$\widehat{\rho}_{\widehat{P}_1 \cap \widehat{P}_2}(v) \geq l \text{ and } \widehat{\sigma}_{\widehat{P}_1 \cap \widehat{P}_2} \leq u.$$

Therefore,

$$v \in \mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2).$$

Thus,

$$\mathcal{L}_{l,u}(\widehat{P}_1) \cap \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2). \tag{3.2}$$

Hence, from equations (3.1) and (3.2), we have

$$\mathcal{L}_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1) \cap \mathcal{L}_{l,u}(\widehat{P}_2).$$ □

Example 3.3. Two PFSs $\widehat{P}_1$ and $\widehat{P}_2$, where

$$\widehat{P}_1 = \{(0,\langle 0.6,0.4\rangle),(1,\langle 0.8,0.2\rangle),(2,\langle 0.5,0.6\rangle),(3,\langle 0.9,0.1\rangle),(4,\langle 0.6,0.7\rangle)\},$$
$$\widehat{P}_2 = \{(0,\langle 0.5,0.4\rangle),(1,\langle 0.6,0.4\rangle),(2,\langle 0.9,0.2\rangle),(3,\langle 0.7,0.6\rangle),(4,\langle 0.8,0.4\rangle)\},$$

then

$$\widehat{P}_1 \cap \widehat{P}_2 = \{(0,\langle 0.5,0.4\rangle),(1,\langle 0.6,0.4\rangle),(2,\langle 0.5,0.6\rangle),(3,\langle 0.7,0.6\rangle),(4,\langle 0.6,0.7\rangle)\},$$

for $l = 0.4, u = 0.6$;

$$L_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) = \{0,1,2,3\},$$
$$L_{l,u}(\widehat{P}_1) = \{0,1,2,3\},\ L_{l,u}(\widehat{P}_2) = \{0,1,2,3,4\}$$
$$\Longrightarrow \quad L_{l,u}(\widehat{P}_1) \cap L_{l,u}(\widehat{P}_2) = \{0,1,2,3\}$$

Hence,

$$L_{l,u}(\widehat{P}_1 \cap \widehat{P}_2) = L_{l,u}(\widehat{P}_1) \cap L_{l,u}(\widehat{P}_2).$$

Theorem 3.7. $\mathcal{L}_{l,u}(\cap \widehat{P}_k | k \in K) = \cap\{\mathcal{L}_{l,u}(\widehat{P}_k) | k \in K\}$.

Proof. Let $v \in \mathcal{L}_{l,u}(\cap \widehat{P}_k) \Longleftrightarrow (\cap \widehat{\rho}_{\widehat{P}_k})(v) \geq l$ and

$$(\cap \widehat{\sigma}_{\widehat{P}_k})(v) \leq u$$
$$\Longleftrightarrow \quad \widehat{\rho}_{\widehat{P}_k}(v) \geq l \ \text{ and } \ \widehat{\sigma}_{\widehat{P}_k}(v) \leq u$$
$$\Longleftrightarrow \quad v \in \mathcal{L}_{l,u}(\widehat{P}_k), \quad \text{for all } k$$
$$\Longleftrightarrow \quad v \in \cap \mathcal{L}_{l,u}(\widehat{P}_k)$$

Therefore,

$$\mathcal{L}_{l,u}(\cap \widehat{P}_k | k \in K) = \cap\{\mathcal{L}_{l,u}(\widehat{P}_k) | k \in K\}.$$ □

Theorem 3.8. *If $\widehat{P}_1$ and $\widehat{P}_2$ are two PFNSR of NR. Then,*

$$\mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2),$$

equality holds if $l + u = 1$.

Proof. Since $\widehat{P}_1 \subseteq \widehat{P}_1 \cup \widehat{P}_2$ and $\widehat{P}_2 \subseteq \widehat{P}_1 \cup \widehat{P}_2$.
Therefore,

$$\mathcal{L}_{l,u}(\widehat{P}_1) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2) \text{ and } \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2).$$

Hence,

$$\mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2). \tag{3.3}$$

We now prove that $\mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2)$ if $l + u = 1$.
Let $v \in \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2)$, then $\widehat{\rho}_{(\widehat{P}_1 \cup \widehat{P}_2)}(v) \geq l$ and $\widehat{\rho}_{(\widehat{P}_1 \cup \widehat{P}_2)}(v) \leq u$.
So, $\max\{\widehat{\rho}_{\widehat{P}_1}(v), \widehat{\rho}_{\widehat{P}_2}(v)\} \geq l$ and $\min\{\widehat{\sigma}_{\widehat{P}_1}(v), \widehat{\sigma}_{\widehat{P}_2}(v)\} \leq u$.
If $\widehat{\rho}_{\widehat{P}_1}(v) \geq l$, then $\widehat{\sigma}_{\widehat{P}_1}(v) \leq 1 - \widehat{\rho}_{\widehat{P}_1}(v) \leq 1 - l = u$.
It follows that $v \in \mathcal{L}_{l,u}(\widehat{P}_1) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2)$.
Similarly, if $\widehat{\rho}_{\widehat{P}_2}(v) \geq l$, then $\widehat{\sigma}_{\widehat{P}_2}(v) \leq 1 - \widehat{\rho}_{\widehat{P}_2}(v) \leq 1 - l = u$.

It follows that $v \in \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2)$.
Therefore, $v \in \mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2)$.
Hence,

$$\mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2). \tag{3.4}$$

Hence by (3.3) and (3.4),

$$\mathcal{L}_{l,u}(\widehat{P}_1) \cup \mathcal{L}_{l,u}(\widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1 \cup \widehat{P}_2). \qquad \square$$

Example 3.4. Let $Z_5 = \{0,1,2,3,4\}$ and consider the ring $(Z_5, \times_5, +_5)$. $\widehat{P}_1$ and $\widehat{P}_2$ be two PFSs, where

$$\widehat{P}_1 = \{(0,\langle 0.6,0.2\rangle),(1,\langle 0.7,0.2\rangle),(2,\langle 0.5,0.5\rangle),(3,\langle 0.8,0.4\rangle),(4,\langle 0.6,0.4\rangle)\}$$
$$\widehat{P}_2 = \{(0,\langle 0.7,0.3\rangle),(1,\langle 0.6,0.3\rangle),(2,\langle 0.5,0.2\rangle),(3,\langle 0.8,0.2\rangle),(4,\langle 0.5,0.3\rangle)\}$$
$$\widehat{P}_1 \cup \widehat{P}_2 = \{(0,\langle 0.7,0.2\rangle),(1,\langle 0.7,0.2\rangle),(2,\langle 0.5,0.2\rangle),(3,\langle 0.8,0.2\rangle),(4,\langle 0.6,0.3\rangle)\}$$

for $l = 0.6$, $u = 0.3$;

$$L_{l,u}(\widehat{P}_1 \cup \widehat{P}_2) = \{0,1,3,4\},$$
$$L_{l,u}(\widehat{P}_1) = \{0,1\},\ L_{l,u}(\widehat{P}_2) = \{0,1,3\}$$
$$\Longrightarrow \quad L_{l,u}(\widehat{P}_1) \cup L_{l,u}(\widehat{P}_2) = \{0,1,3\}$$

Clearly,

$$L_{l,u}(\widehat{P}_1) \cup L_{l,u}(\widehat{P}_2) \subseteq L_{l,u}(\widehat{P}_1 \cup \widehat{P}_2).$$

Again, for $l = 0.6$, $u = 0.4$;

$$L_{l,u}(\widehat{P}_1 \cup \widehat{P}_2) = \{0,1,3,4\},$$
$$L_{l,u}(\widehat{P}_1) = \{0,1,3,4\},\ L_{l,u}(\widehat{P}_2) = \{0,1,3\}$$
$$\Longrightarrow \quad L_{l,u}(\widehat{P}_1) \cup L_{l,u}(\widehat{P}_2) = \{0,1,3,4\} = L_{l,u}(\widehat{P}_1 \cup \widehat{P}_2).$$

Therefore, if $l + u = 1$; then

$$L_{l,u}(\widehat{P}_1) \cup L_{l,u}(\widehat{P}_2) = L_{l,u}(\widehat{P}_1 \cup \widehat{P}_2).$$

Definition 3.2. Let $\widehat{P}_1$ and $\widehat{P}_2$ be two PFNSR in NR. Then,

$$\mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2) = \{(v, \widehat{\rho}_{\widehat{P}_1+\widehat{P}_2}(v) \geq l \text{ and } \widehat{\sigma}_{\widehat{P}_1+\widehat{P}_2}(v) \leq u) : v \in NR\}.$$

Theorem 3.9. *For any two Pythagorean fuzzy normed ideal $\widehat{P}_1$ and $\widehat{P}_2$ in NR, $\mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2)$ and equality holds if $l + u = 1$.*

Proof. Let $x = y + z \in \mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2)$ such that $y \in \mathcal{L}_{l,u}(\widehat{P}_1)$ and $z \in \mathcal{L}_{l,u}(\widehat{P}_2)$.
Then $\widehat{\rho}_{\widehat{P}_1}(y) \geq l$, $\widehat{\rho}_{\widehat{P}_2}(z) \geq l$ and $\widehat{\sigma}_{\widehat{P}_1}(y) \leq u$, $\widehat{\sigma}_{\widehat{P}_2}(z) \leq u$, $\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z) \geq l$ and $\widehat{\sigma}_{\widehat{P}_1}(y) \vee \widehat{\sigma}_{\widehat{P}_2}(z) \leq u$.
Therefore,

$$\bigvee_{x=y+z} (\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z)) \geq l \quad \text{and} \quad \bigwedge_{x=y+z} (\widehat{\sigma}_{\widehat{P}_1}(y) \vee \widehat{\sigma}_{\widehat{P}_2}(z)) \leq u.$$

So,

$$\widehat{\rho}_{\widehat{P}_1+\widehat{P}_2}(x) \geq l \quad \text{and} \quad \widehat{\sigma}_{\widehat{P}_1+\widehat{P}_2}(x) \leq u.$$

Therefore,

$$x \in \mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2).$$

Hence

$$\mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2). \tag{3.5}$$

For the second part, let $l + u = 1$ and let $x \in \mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2)$.

Then,

$$\widehat{\rho}_{\widehat{P}_1+\widehat{P}_2}(x) \geq l \text{ and } \widehat{\sigma}_{\widehat{P}_1+\widehat{P}_2}(x) \leq u.$$

Take,

$$\begin{aligned} & \widehat{\rho}_{\widehat{P}_1+\widehat{P}_2}(x) \geq l \\ \Longrightarrow \quad & \bigvee_{x=y+z} (\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z)) \geq l \\ \Longrightarrow \quad & \widehat{\rho}_{\widehat{P}_1}(w) \vee \widehat{\rho}_{\widehat{P}_2}(v) \geq l, \end{aligned}$$

for some $x = w + v$.

Therefore,

$$\widehat{\rho}_{\widehat{P}_1}(w) \geq l \text{ and } \widehat{\rho}_{\widehat{P}_2}(v) \geq l.$$

Also,

$$\widehat{\sigma}_{\widehat{P}_1}(w) \leq 1 - \widehat{\rho}_{\widehat{P}_1}(w) \leq 1 - l = u \text{ and } \widehat{\sigma}_{\widehat{P}_2}(v) \leq 1 - \widehat{\rho}_{\widehat{P}_2}(v) \leq 1 - l = u.$$

Thus,

$$\widehat{\sigma}_{\widehat{P}_1}(w) \leq u \text{ and } \widehat{\sigma}_{\widehat{P}_2}(v) \leq u.$$

Hence,

$$w \in \mathcal{L}_{l,u}(\widehat{P}_1) \text{ and } v \in \mathcal{L}_{l,u}(\widehat{P}_2).$$

Then

$$x = w + v \in \mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2).$$

Thus,

$$\mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2). \tag{3.6}$$

From equations (3.5) and 3.6, we get

$$\mathcal{L}_{l,u}(\widehat{P}_1 + \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1) + \mathcal{L}_{l,u}(\widehat{P}_2).$$ □

Definition 3.3. Let $\widehat{P}_1$ and $\widehat{P}_2$ be two PFNSRs in NR. Then,

$$\mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2) = \{(x, \widehat{\rho}_{\widehat{P}_1 \oplus \widehat{P}_2}(x) \geq l \text{ and } \widehat{\sigma}_{\widehat{P}_1 \otimes \widehat{P}_2}(x) \leq u) : x \in NR\}.$$

Theorem 3.10. *For any two PFNSRs $\widehat{P}_1$ and $\widehat{P}_2$ of a normed ring NR, $\mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2)$ and equality holds if $l + u = 1$.*

Proof. $\widehat{\rho}_{\widehat{P}_1 \oplus \widehat{P}_2}(x) = \bigvee_{x=yz} (\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z))$ and $\widehat{\sigma}_{\widehat{P}_1 \otimes \widehat{P}_2}(x) = \bigwedge_{x=yz} (\widehat{\sigma}_{\widehat{P}_1}(y) \vee \widehat{\sigma}_{\widehat{P}_2}(z))$.

Let $x = yz \in \mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2)$ such that

$$y \in \mathcal{L}_{l,u}(\widehat{P}_1) \text{ and } z \in \mathcal{L}_{l,u}(\widehat{P}_2).$$

So,

$$\widehat{\rho}_{\widehat{P}_1}(y) \geq l,\ \widehat{\sigma}_{\widehat{P}_1}(y) \leq u \ \text{ and } \ \widehat{\rho}_{\widehat{P}_2}(z) \geq l,\ \widehat{\sigma}_{\widehat{P}_2}(z) \leq u.$$

Hence,

$$\begin{aligned}\widehat{\rho}_{\widehat{P}_1 \oplus \widehat{P}_2}(x) &= \bigvee_{x=yz} (\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z)) \\ &\geq \widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z) \\ &\geq l\end{aligned}$$

and

$$\begin{aligned}\widehat{\sigma}_{\widehat{P}_1 \otimes \widehat{P}_2}(x) &= \bigwedge_{x=yz} (\widehat{\sigma}_{\widehat{P}_1}(y) \vee \widehat{\sigma}_{\widehat{P}_2}(z)) \\ &\leq \widehat{\sigma}_{\widehat{P}_1}(y) \vee \widehat{\sigma}_{\widehat{P}_2}(z) \\ &\leq u.\end{aligned}$$

Therefore,

$$x \in \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2).$$

Then,

$$\mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2). \tag{3.7}$$

To prove $\mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2)$, let $(l+u=1)$ and $(x \in \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2))$.

Then,

$$\widehat{\rho}_{\widehat{P}_1 \circ \widehat{P}_2}(x) \geq l \ \text{ and } \ \widehat{\sigma}_{\widehat{P}_1 \circ \widehat{P}_2}(x) \leq u.$$

Now,

$$\begin{aligned}&\widehat{\rho}_{\widehat{P}_1 \circ \widehat{P}_2}(x) \geq l \\ \Longrightarrow\ &\bigvee_{x=yz} (\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z)) \geq l \\ \Longrightarrow\ &\widehat{\rho}_{\widehat{P}_1}(y) \wedge \widehat{\rho}_{\widehat{P}_2}(z) \geq l.\end{aligned}$$

Then

$$\widehat{\rho}_{\widehat{P}_1}(y) \geq l \ \text{ and } \ \widehat{\rho}_{\widehat{P}_2}(z) \geq l.$$

Also,

$$\widehat{\sigma}_{\widehat{P}_1}(y) \leq 1 - \widehat{\rho}_{\widehat{P}_1}(y) \leq 1 - l = u \ \text{ and } \ \widehat{\sigma}_{\widehat{P}_2}(z) \leq 1 - \widehat{\rho}_{\widehat{P}_2}(z) \leq 1 - l = u.$$

Then,

$$\widehat{\sigma}_{\widehat{P}_1}(y) \leq u \ \text{ and } \ \widehat{\sigma}_{\widehat{P}_2}(z) \leq u.$$

Hence,

$$y \in \mathcal{L}_{l,u}(\widehat{P}_1) \ \text{ and } \ z \in \mathcal{L}_{l,u}(\widehat{P}_2).$$

So,

$$x \in \mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2).$$

Thus,

$$\mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2) \subseteq \mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2). \tag{3.8}$$

Therefore,

$$\mathcal{L}_{l,u}(\widehat{P}_1)\mathcal{L}_{l,u}(\widehat{P}_2) = \mathcal{L}_{l,u}(\widehat{P}_1 \circ \widehat{P}_2). \qquad \square$$

4. Conclusion

The concept of PFNSR is of importance for quantifying an ill-quantity. We introduced the concept of (l,u)-level sets of PFSs. In this paper, we build up a connection between PFNSR and (l,u)-level set of PFSs. Some important links between PFNSR with (l,u)-level sets of PFSs were established. Two new operations on PFSs are defined and applied these results on level sets. Future research may extend this work by introducing upper and lower level sets for various extensions of FSs-such as interval-valued fuzzy sets, type-2 FSs, fuzzy multi-sets, hesitant FSs, and Fermatean fuzzy sets-and applying them to semi-groups, ternary semigroups, and logical algebras to better handle imprecise and vague information.

Acknowledgments

The author would like to express their thanks to referees for their valuable suggestions and helpful comments in improving this paper.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] A. K. Adak and N. Kamal, Pythagorean fuzzy sets for credit risk assessment: A novel approach to predicting loan default, *Control and Optimization in Applied Mathematics* **10**(1) (2025), 175 – 191, DOI: 10.30473/coam.2025.73505.1287.

[2] A. K. Adak and D. Kumar, Some properties of Pythagorean fuzzy ideals of Γ-near-rings, *Palestine Journal of Mathematics* **11**(4) (2022), 336 – 346, URL: https://pjm.ppu.edu/sites/default/files/papers/PJM_December_2022_336_to_346..pdf.

[3] A. K. Adak and D. D. Salookolaei, Some properties of rough pythagorean fuzzy sets, *Fuzzy Information and Engineering* **13**(4) (2021), 420 – 435, DOI: 10.1080/16168658.2021.1971143.

[4] A. K. Adak, M. Bhowmik and M. Pal, Interval cut-set of generalized interval-valued intuitionistic fuzzy sets, *International Journal of Fuzzy System Applications* **2**(3) (2012), 35 – 50, DOI: 10.4018/ijfsa.2012070103.

[5] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20**(1) (1986), 87 – 96, DOI: 10.1016/S0165-0114(86)80034-3.

[6] B. De Baets, R. Mesiar, Triangular norms on product lattices, *Fuzzy Sets and Systems* **104**(1) (1999), 61 – 75, DOI: 10.1016/S0165-0114(98)00259-0.

[7] D. K. Basnet, (α,β)-Cut of intuitionistic fuzzy ideals, *International Journal of Algebra* **4**(27) (2010), 1329 – 1334.

[8] G. Deschrijver, C. Cornelis and E. E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, *IEEE Transactions on Fuzzy Systems* **12**(1) (2004), 45 – 61, DOI: 10.1109/TFUZZ.2003.822678.

[9] J. N. Garcia, Z. Kutalik, K. H. Cho and O. Wolkenhaue, Level sets and minimum volume sets of probability density functions, *International Journal of Approximate Reasoning* **34**(1) (2003), 25 – 47, DOI: 10.1016/S0888-613X(03)00052-5.

[10] E. F. Okumuş and S. Yamak, Relative TL-ideals in semigroups, *Punjab University Journal of Mathematics* **55**(9-10) (2023), 383 – 395, DOI: 10.52280/pujm.2023.55(9-10)04.

[11] E. Pap and D. Surla, Lebesgue measure of α-cuts approach for finding the height of the membership function, *Fuzzy Sets and Systems* **111**(3) (2000), 341 – 350, DOI: 10.1016/S0165-0114(98)00162-6.

[12] R. R. Yager, Pythagorean fuzzy subsets, in: *2013 Joint IFSA World Congress and NAFIPS Annual Meeting* (IFSA/NAFIPS) (Edmonton, AB, Canada, 2013), pp. 57 – 61, (2013), DOI: 10.1109/IFSA-NAFIPS.2013.6608375.

[13] R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers and decision making, *International Journal of Intelligent Systems* **28**(5) (2013), 436 – 452, DOI: 10.1002/int.21584.

[14] X.-H. Yuan, H. Li and E. S. Lee, Three new cut sets of fuzzy sets and new theories of fuzzy sets, *Computers & Mathematics with Applications* **57**(5) (2009), 691 – 701, DOI: 10.1016/j.camwa.2008.05.044.

[15] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.