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Research Article

Some New Results on the Generalized Double Laplace Transform and Its Properties

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Abstract. Integral transformations play a fundamental role in various scientific and engineering domains, providing powerful tools for solving differential, integral, and functional equations. Among these, the Laplace transform is one of the most widely used, and many other emerging integral transforms are direct generalizations or modifications of it. This paper focuses on an extension of the classical Laplace transform to a new generalized double Laplace transform framework. The proposed transform is defined in terms of arbitrary, strictly increasing kernel functions, enabling greater flexibility in handling problems with non-uniform scaling in multiple variables. Fundamental properties of the transform, including linearity, shifting, change of scale, and convolution theorems, are established. Moreover, the study establishes that the classical double Laplace transform arises as a particular case within the framework of the generalized version. The derived framework broadens applicability to a wider range of *partial differential equations* (PDEs) in mathematics and physics.

Keywords. Two-dimensional Laplace transform, Generalized Laplace transform, Convolution theorem, Integral transforms, Partial differential equations

Mathematics Subject Classification (2020). 44A10, 26A33, 35A22, 35C05, 44A35

1. Introduction

The Laplace transform, developed by Pierre-Simon Laplace (1749–1827), became a key tool for efficiently tackling mathematical problems. As a result, many other integral transforms have been derived from the Laplace transform, including the Fourier transform (Bracewell [3]), the Aboodh transform (Aboodh [1]), the Kamal transform (Hassan [9]), the Sumudu transform (Watugala [20]), the Elzaki transform (Elzaki [7]), the ARA transform (Saadeh *et al*. [17]), and the Modified Laplace transform (Saif *et al*. [18]), among others. In addition, extensions of these transforms to double-variable cases have been studied. The notable examples include the double Laplace transform (Debnath [4]), the double Sumudu transform (Watugala [21]), the double Elzaki transform (Idrees *et al*. [11]), and the modified double Laplace transform (Borawake and Hiwarekar [2]), each contributing to the solution of more complex two variable problems. Hiwarekar [10] investigated the triple Laplace transform, highlighting its properties and potential applications in solving complex differential equations.

Classical double Laplace transforms have been extensively studied and applied to solve a wide range of partial differential equations, including the wave, heat, and telegraph equations (Debnath [4], Debnath and Bhatta [5], and Dhunde and Waghmare [6]). It plays a key role in addressing problems across various disciplines such as oscillation theory (Pradhan *et al*. [16]), heat conduction in solids (Kazim *et al*. [13]), analysis of stationary and real-time signals (Sharma and Meena [19]). These studies focus on transforms with the standard exponential kernel $e^{-(px+qy)}$, which, while powerful, are limited in adaptability to problems involving non-uniform or nonlinear scaling of variables.

In recent years, fractional and conformable variants of the double Laplace transform have emerged, such as those proposed by Ozkan and Kurt [15], where conformable fractional derivatives are incorporated to handle fractional-order PDEs. In addition, the one-dimensional generalized Laplace transform with respect to another function, investigated by Fahad *et al*. [8], introduced greater flexibility in modeling processes with variable time scales. More recently, Jarad and Abdeljawad [12] developed generalized fractional derivatives and Laplace transforms in a related setting, contributing to the extension of transform techniques to fractional calculus. Furthermore, Lemnaouar and El Hakki [14] formulated a generalized double Laplace transform with respect to another function and demonstrated its applicability to fractional partial differential equations.

Although these contributions have significantly advanced the theory and applications of Laplace-type transforms, they remain restricted in important ways. Classical double Laplace transforms are confined to linear scaling, while conformable and fractional variants rely on fixed power-law kernels. The generalized one-dimensional versions allow arbitrary scaling functions but are limited to single-variable problems, and the existing two-variable generalizations by Lemnaouar and El Hakki [14] are situated within fractional calculus, focusing on specific kernel choices — for instance, they computed the double generalized Laplace transform of bivariate Mittag-Leffler functions, while applying the theory to fractional partial differential equations. Hence, there is a need for a two-dimensional framework that incorporates arbitrary smooth and strictly monotonic kernel functions, enabling both variables to be scaled independently.

The present study addresses this gap by introducing a two-dimensional generalized Laplace transform with respect to arbitrary, strictly increasing functions Ψ and ξ, defined for $\Psi,\xi \in C^1([0,\infty))$ with $\Psi(0)=\xi(0)=0$ having positive derivatives. We developed the fundamental properties, including linearity, shifting, change of scale, convolution theorems, facilitating direct application to both classical and fractional partial differential equations. These results can be used as a tool for applied mathematics, physics, and engineering problems. While Jarad and Abdeljawad [12] formulated a one-dimensional generalized fractional-derivative Laplace transform and Lemnaouar and El Hakki [14] considered a fractional double transform with fixed kernel functions, the present study extends these approaches by introducing a generalized double Laplace transform with independently chosen smooth and strictly monotonic kernels for each variable.

2. Notations, Definitions and Basic Results

Our discussion starts with the classical Laplace transform and then proceeds to the definition of its generalized form.

Definition 2.1. (Laplace Transform; Debnath and Bhatta [5]) The classical Laplace transform of a function $f:[0,\infty)\to\mathbb{R}$ is defined by

$$\mathcal{L}\{f(x)\}(s)=F(s)=\int_0^\infty e^{-sx}f(x)\,dx, \tag{2.1}$$

whenever the integral converges for the complex parameter s.

The Laplace transform with respect to another function was introduced by Jarad and Abdeljawad [12] and is defined as follows:

Definition 2.2. Suppose $f,g:[a,\infty)\to\mathbb{R}$ are real-valued functions, where g is continuously differentiable on $[a,\infty)$ and satisfying $g'(x)>0$. Then, the Laplace transform of f with respect to g is given by

$$\mathcal{L}_g\{f(x)\}(s)=\int_a^\infty e^{-s(g(x)-g(a))}f(x)g'(x)\,dx, \tag{2.2}$$

provided that the integral converges for the corresponding values of s.

Definition 2.3. (Generalized Laplace Transform; Fahad and Rehman [8]) for a function f, the generalized Laplace transform is defined by

$$\mathcal{L}_\Psi\{f(x)\}=\bar{F}(s)=\int_0^\infty e^{-s\Psi(x)}\Psi'(x)f(x)dx, \tag{2.3}$$

where this integral converges for all $s\in\mathbb{C}$ with $\Psi(0)=0$, Ψ is a non-negative, increasing function. The generalized Laplace transform is denoted by the operator $\mathcal{L}_\Psi$, which is called as Laplace transform with respect to Ψ.

Remark 2.4. The formulation in Definition 2.3 provides a unifying perspective on earlier transforms. Choosing $\Psi(x)=x$ yields the standard Laplace transform (Definition 2.1), while taking $\Psi(x)=g(x)-g(a)$ reproduces the transform in Definition 2.2, where the change of variable is governed by a strictly increasing differentiable function g. Hence, the generalized Laplace transform with respect to Ψ covers earlier constructions, recovering the classical transform as well as the function-based variant.

The generalized double Laplace transform defined by Lemnaouar and El Hakki [14] is given as follows:

Definition 2.5. Let $u:[a,\infty)\times[b,\infty)\to\mathbb{C}$ be piecewise continuous and of exponential order on each finite rectangle. Let $(g_1,g_2)\in C^n([a,\infty))\times C^n([b,\infty)$ be strictly increasing functions with $g_1'(x)>0$, $g_2'(y)>0$, for every $(x,y)\in[a,\infty)\times[b,\infty)$. The two-dimensional generalized Laplace transform of u with respect to g_1 and g_2 is defined by

$$\mathcal{L}_{g_1,g_2}\{u(x,y)\}(p,q) = \int_a^\infty \int_b^\infty e^{-[p(g_1(x)-g_1(a))+q(g_2(y)-g_2(b))]} u(x,y) g_1'(x) g_2'(y)\, dx\, dy, \qquad (2.4)$$

where $p,q \in \mathbb{C}$ are such that the double integral converges.

Remark 2.6. Definition 2.5 is a natural extension of Definition 2.2. Indeed, the generalized double Laplace transform can be obtained from the one-dimensional generalized Laplace transform (Definition 2.2).

3. Main Results

In the following section, we outline the fundamental definitions and theorems of the generalized double Laplace transform.

Definition 3.1. Consider $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$, and suppose Ψ and ξ are non-negative, strictly increasing functions satisfying $\Psi(0) = 0$ and $\xi(0) = 0$. The generalized double Laplace transform of f with respect to Ψ and ξ is denoted by $\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\} = \bar{\bar{F}}(p,q)$ and is

$$\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\} = \int_0^\infty \int_0^\infty e^{-p\Psi(x)-q\xi(y)} f(x,y)\Psi'(x)\xi'(y)\, dx\, dy, \qquad (3.1)$$

where this integral converges for all $p,\ q \in \mathbb{C}$.

Remark 3.2. Definition 3.1 can be viewed as a simplified form of the generalized double Laplace transform given in Definition 2.5. In particular, if we assume $a = b = 0$, with $g_1(0) = g_2(0) = 0$ and rename $g_1(x) = \Psi(x)$, $g_2(y) = \xi(y)$, then Definition 2.5 reduces to Definition 3.1. The imposed conditions $\Psi(0) = \xi(0) = 0$ are natural, since they ensure consistency with the one-dimensional generalized Laplace transform (Definition 2.3) and yield a cleaner, more tractable formulation. This setting not only harmonizes the theory but also makes it easier to establish properties and theorems, thereby enhancing both the theoretical development and practical applications of the transform.

Remark 3.3. The classical double Laplace transform [4] can be recovered as a special case of the generalized double Laplace transform by choosing $\Psi(x) = x$ and $\xi(y) = y$ in eq. (3.1). Thus, we have

$$\mathcal{L}_2\{f(x,y)\} = \bar{\bar{f}}(p,q) = \int_0^\infty \int_0^\infty f(x,y) e^{-(px+qy)}\, dx\, dy.$$

Definition 3.4. Let $f : [0,\infty) \times [0,\infty) \to \mathbb{R}$. We say that f is of exponential order $a > 0$ and $b > 0$, if there exist non-zero constants U, a, b with $x > X$, $y > Y$ such that,

$$|f(x,y)| \le U e^{a\Psi(x)+b\xi(y)}. \qquad (3.2)$$

In this case we may write, $f(x,y) = \mathcal{O}(e^{a\Psi(x)+b\xi(y)})$ as $x \to \infty$, $y \to \infty$.

A function $f(x,y)$ is therefore said to be of exponential order at infinity if its growth does not exceed $Ue^{a\Psi(x)+b\xi(y)}$, for some constants $U, a, b > 0$.

The next theorem provides a sufficient condition under which the generalized double Laplace transform of a function exists.

Theorem 3.1. *Let $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be a piecewise continuous and of exponential order for some constants $a>0$ and $b>0$. Suppose Ψ and ξ are non-negative, strictly increasing functions satisfying $\Psi(0)=0$ and $\xi(0)=0$. Hence, the generalized double Laplace transform of f exists whenever $p>a$ and $q>b$.*

Using Definitions 3.1 and 3.4, we have

$$\begin{aligned}
\left|\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\}\right| &= \left|\int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}f(x,y)\Psi'(x)\xi'(y)\,dx\,dy\right| \\
&\le \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}|f(x,y)|\Psi'(x)\xi'(y)\,dx\,dy \\
&\le U\int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}e^{a\Psi(x)+b\xi(y)}\Psi'(x)\xi'(y)\,dx\,dy \\
&= U\int_0^\infty e^{-(p-a)\Psi(x)}\Psi'(x)\,dx\int_0^\infty e^{-(q-b)\xi(y)}\xi'(y)\,dy \\
&= U\left[\frac{e^{-(p-a)\Psi(x)}}{-(p-a)}\right]_0^\infty\left[\frac{e^{-(q-b)\xi(y)}}{-(q-b)}\right]_0^\infty \\
&= \frac{U}{(p-a)(q-b)}\left[1-e^{-(p-a)\lim\limits_{x\to\infty}\Psi(x)}\right]\left[1-e^{-(q-b)\lim\limits_{y\to\infty}\xi(y)}\right] \\
&= \frac{U}{(p-a)(q-b)}.
\end{aligned} \tag{3.3}$$

The following result demonstrates how the generalized double Laplace transform relates to the classical double Laplace transform.

Theorem 3.2. *Let $f:[0,\infty)\times[0,\infty)\to\mathbb{R}$, and let Ψ and ξ be non-negative, strictly increasing functions satisfying $\Psi(0)=0$ and $\xi(0)=0$. For a function f whose generalized double Laplace transform exists, it can be expressed in a form that generalizes the classical double Laplace transform as follows:*

$$\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\} = \mathcal{L}_2\{f(\Psi^{-1}(x),\ \xi^{-1}(y))\}, \tag{3.4}$$

where $\mathcal{L}_2\{f\}$ is the classical double Laplace transform.

Remark 3.5. Theorem 3.2 shows that, under a suitable change of variables, the generalized double Laplace transform can be viewed as a reformulation of the classical Laplace transform. More precisely, by setting $u=\Psi(x)$ and $v=\xi(y)$ and applying their inverse functions, one obtains the expression in Theorem 3.2. This result highlights the consistency between the generalized and classical frameworks and ensures that many properties of the classical transform naturally extend to the generalized case.

Some properties of generalized double Laplace transform are as follows:

Theorem 3.3 (Linearity Property)**.** *Two functions $g(x,y)$ and $h(x,y)$ whose generalized double Laplace transform exists, then*

$$\mathcal{L}^2_{\Psi,\xi}\{\gamma g(x,y)+\delta h(x,y)\} = \gamma\mathcal{L}^2_{\Psi,\xi}\{g(x,y)\}+\delta\mathcal{L}^2_{\Psi,\xi}\{h(x,y)\}, \tag{3.5}$$

where γ and δ are constants.

Theorem 3.4 (Shifting Property). *If $f(x,y)$ satisfying the condition of piecewise continuous and exponential order, then*

$$\mathcal{L}^2_{\Psi,\xi}\{e^{-\gamma\Psi(x)-\delta\xi(y)}f(x,y)\} = \bar{\bar{F}}(p+\gamma,\ q+\delta). \tag{3.6}$$

Using definition 3.1, we have,

$$\begin{aligned}\mathcal{L}^2_{\Psi,\xi}\{e^{-\gamma\Psi(x)-\delta\xi(y)}f(x,y)\} &= \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}(e^{-\gamma\Psi(x)-\delta\xi(y)}f(x,y))\Psi'(x)\xi'(y)dxdy\\ &= \int_0^\infty\int_0^\infty e^{-(p+\gamma)\Psi(x)-(q+\delta)\xi(y)}f(x,y)\Psi'(x)\xi'(y)dxdy\\ &= \bar{\bar{F}}(p+\gamma,\ q+\delta).\end{aligned}$$

Theorem 3.5 (Change of Scale Property). *If $f(x,y)$ satisfying the condition of piecewise continuous and exponential order, then*

$$\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x),\delta\xi(y))\} = \frac{1}{\gamma\delta}\bar{\bar{F}}\left(\frac{p}{\gamma},\frac{q}{\delta}\right). \tag{3.7}$$

Using Definition 3.1, we have,

$$\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x),\ \delta\xi(y))\} = \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}\Psi'(x)\xi'(y)(f(\gamma\Psi(x),\ \delta\xi(y)))dxdy, \tag{3.8}$$

by substituting $u=\gamma\Psi(x)$ and $v=\delta\xi(y)$ in eq. (3.8), we get

$$\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x),\ \delta\xi(y))\} = \frac{1}{\gamma\delta}\int_0^\infty\int_0^\infty e^{-\frac{pu}{\gamma}-\frac{qv}{\delta}}f(u,v)dudv,$$

by using Definition 3.1 and using Theorem 3.2, we get

$$\begin{aligned}\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x),\ \delta\xi(y))\} &= \frac{1}{\gamma\delta}\mathcal{L}_2\{f(\Psi^{-1}(u),\ \xi^{-1}(v))\}\\ &= \frac{1}{\gamma\delta}\bar{\bar{F}}\left(\frac{p}{\gamma},\ \frac{q}{\delta}\right).\end{aligned}$$

Theorem 3.6. *If $f(x,y)=h(x)g(y)$, then*

$$\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\} = \mathcal{L}_\Psi\{h(x)\}\mathcal{L}_\xi\{g(y)\}. \tag{3.9}$$

Using Definition 3.1, we have

$$\begin{aligned}\mathcal{L}^2_{\Psi,\xi}\{f(x,y)\} &= \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}h(x)g(y)\Psi'(x)\xi'(y)dxdy\\ &= \int_0^\infty e^{-p\Psi(x)}h(x)\Psi'(x)dx\int_0^\infty e^{-q\xi(y)}g(y)\xi'(y)dy\\ &= \mathcal{L}_\Psi\{h(x)\}\mathcal{L}_\xi\{g(y)\}.\end{aligned}$$

Theorem 3.7.

$$\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x))\cdot g(\delta\xi(y))\} = \frac{1}{\gamma\delta}\bar{F}\left(\frac{p}{\gamma}\right)\bar{G}\left(\frac{q}{\delta}\right). \tag{3.10}$$

Using Definition 3.1, we have,

$$\mathcal{L}^2_{\Psi,\xi}\{f(\gamma\Psi(x))\cdot g(\delta\xi(y))\} = \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}f(\gamma\Psi(x))\cdot g(\delta\xi(y))\Psi'(x)\xi'(y)dxdy$$

$$= \int_0^\infty e^{-p\Psi(x)} f(\gamma\Psi(x))\Psi'(x)dx \int_0^\infty e^{-q\xi(y)} g(\delta\xi(y))\xi'(y)dy, \quad (3.11)$$

by substituting $u = \gamma\Psi(x)$ and $v = \delta\xi(y)$ in eq. (3.11), we get

$$= \frac{1}{\gamma}\int_0^\infty e^{-\frac{pu}{\gamma}} f(u)du \frac{1}{\delta}\int_0^\infty e^{-\frac{qv}{\delta}} g(v)dv$$

$$= \frac{1}{\gamma\delta}\mathcal{L}\{f(\Psi^{-1}(u))\}\mathcal{L}\{g(\xi^{-1}(v))\}.$$

By using Definition 3.1 and Theorem 3.2 this can also be written as

$$= \frac{1}{\gamma\delta}\bar{F}\left(\frac{p}{\gamma}\right)\bar{G}\left(\frac{q}{\delta}\right).$$

4. Convolution Theorem

Definition 4.1. Assume $g(x,y)$ and $h(x,y)$ are piecewise continuous functions and are of exponential order. Their (Ψ,ξ)-convolution is denoted by $(g**_{\Psi,\xi} h)(x,y)$, is given by:

$$(g**_{\Psi,\xi} h)(x,y) = \int_0^x \int_0^y g(\Psi^{-1}(\Psi(x)-\Psi(\tau)), \xi^{-1}(\xi(y)-\xi(\eta)))h(\tau,\eta)\Psi'(\tau)\xi'(\eta)d\tau d\eta. \quad (4.1)$$

Remark 4.2. In this definition, Ψ^{-1} and ξ^{-1} are the inverse functions of Ψ and ξ, respectively, and the terms $\Psi'(\tau)$ and $\xi'(\eta)$ account for the generalized measure in the integration. This form extends the classical convolution to the (Ψ,ξ)-Laplace transform setting, allowing the properties of convolution to hold in the generalized framework.

The convolution satisfies the following set of properties. These are summarized in the theorem below.

Theorem 4.1. *Assume that g, h, and f are piecewise continuous functions and of exponential order over every finite interval, and let c and d be constants. Then,*

(i) $(g**_{\Psi,\xi} h)(x,y) = (h**_{\Psi,\xi} g)(x,y)$ (*Commutative*).

(ii) $[(g**_{\Psi,\xi} h)**_{\Psi,\xi} f](x,y) = [g**_{\Psi,\xi}(h**_{\Psi,\xi} f)](x,y)$ (*Associative*).

(iii) $[g**_{\Psi,\xi}(ch+df)](x,y) = c(g**_{\Psi,\xi} h)(x,y) + d(g**_{\Psi,\xi} f)(x,y)$ (*Distributive*).

We now present a result that proves a convolution theorem for the generalized double Laplace transform applied to two functions.

Theorem 4.2. *Assume that g and h are piecewise continuous functions and of exponential order over every finite interval. Then,*

$$\mathcal{L}^2_{\Psi,\xi}\{(g**_{\Psi,\xi} h)\}(x,y) = \mathcal{L}^2_{\Psi,\xi}\{g(x,y)\}\mathcal{L}^2_{\Psi,\xi}\{h(x,y)\}, \quad (4.2)$$

where $(g**_{\Psi,\xi} h)(x,y)$ is given by Definition 4.1.

We have, by Definition 3.1,

$$\mathcal{L}^2_{\Psi,\xi}\{(g**_{\Psi,\xi} h)\}(x,y) = \int_0^\infty\int_0^\infty e^{-p\Psi(x)-q\xi(y)}\Psi'(x)\xi'(y) \cdot\left[\int_0^x\int_0^y g(\Psi^{-1}(\Psi(x)-\Psi(\tau)), \xi^{-1}(\xi(y)-\xi(\eta)))h(\tau,\eta)\Psi'(\tau)\xi'(\eta)d\tau d\eta\right]dxdy,$$

by substituting $u=\Psi^{-1}(\Psi(x)-\Psi(\tau))$ and $v=\xi^{-1}(\xi(y)-\xi(\eta))$ in above integral, we get

$$\begin{aligned}\mathcal{L}^2_{\Psi,\xi}\{(g**_{\Psi,\xi}h)\}(x,y) &= \int_0^\infty\int_0^\infty h(\tau,\eta)\Psi'(\tau)\xi'(\eta)d\tau\,d\eta \\ &\quad\cdot\left[\int_0^\infty\int_0^\infty e^{-p(\Psi(u)+\Psi(\tau))-q(\xi(v)+\xi(\eta))}g(u,v)\Psi'(u)\xi(v)\right]du\,dv \\ &= \int_0^\infty\int_0^\infty e^{-p\Psi(u)-q\xi(v)}g(u,v)\Psi'(u)\xi(v)du\,dv \\ &\quad\cdot\int_0^\infty\int_0^\infty e^{-p\Psi(\tau)-q\xi(\eta)}h(\tau,\eta)\Psi'(\tau)\xi'(\eta)d\tau\,d\eta \\ &= \mathcal{L}^2_{\Psi,\xi}\{g(x,y)\}\mathcal{L}^2_{\Psi,\xi}\{h(x,y)\}.\end{aligned}$$

5. Generalized Double Laplace Transforms of Standard Functions

In this section, we present the generalized Laplace transforms of some standard functions. Here, Ψ and ξ are non-negative, strictly increasing functions with $\Psi(0)=0$ and $\xi(0)=0$, ensuring that the generalized double Laplace transform of f exists.

(5.1) $\mathcal{L}^2_{\Psi,\xi}\{1\}=\dfrac{1}{pq}$.

(5.2) $\mathcal{L}^2_{\Psi,\xi}\{e^{a\Psi(x)+b\xi(y)}\}=\dfrac{1}{(p-a)(q-b)}$.

(5.3) $\mathcal{L}^2_{\Psi,\xi}\{e^{i(a\Psi(x)+b\xi(y))}\}=\dfrac{1}{(p-ia)(q-ib)}=\dfrac{(pq-ab)+i(aq+bp)}{(p^2+a^2)(q^2+b^2)}$.

(5.4) Consequently, $\mathcal{L}^2_{\Psi,\xi}\{\cos(a\Psi(x)+b\xi(y))\}=\dfrac{pq-ab}{(p^2+a^2)(q^2+b^2)}$.

(5.5) And, $\mathcal{L}^2_{\Psi,\xi}\{\sin(a\Psi(x)+b\xi(y))\}=\dfrac{aq+bp}{(p^2+a^2)(q^2+b^2)}$.

(5.6) $\mathcal{L}^2_{\Psi,\xi}\{\Psi(x)\xi(y)\}=\dfrac{1}{p^2q^2}$.

(5.7) For $n>0$, $\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^n(\xi(y))^n\}=\dfrac{(n!)^2}{(pq)^{n+1}}$.

In particular for $n=2$, we have $\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^2(\xi(y))^2\}=\dfrac{2!2!}{p^3q^3}=\dfrac{4}{(pq)^3}$.

(5.8) Consequently, for $m>0$, $n>0$, $\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^m(\xi(y))^n\}=\dfrac{m!n!}{p^{m+1}q^{n+1}}$.

(5.9) If $c>-1$, $d>-1$, we have $\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^c(\xi(y))^d\}=\dfrac{\Gamma(c+1)}{p^{c+1}}\cdot\dfrac{\Gamma(d+1)}{q^{d+1}}$, where the Euler gamma function $\Gamma(c)$ for $c>0$ is defined by the uniformly convergent integral $\Gamma(c)=\int_0^\infty s^{c-1}e^{-s}ds$.

(5.10) From (5.9) it follows that $\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^{c-1}(\xi(y))^{d-1}\}=\dfrac{\Gamma(c)}{p^c}\cdot\dfrac{\Gamma(d)}{q^d}$.

6. Illustrative Examples

Throughout the following illustrations, we assume that $\Psi,\xi:[0,\infty)\to\mathbb{R}$ are non-negative, increasing functions with $\Psi(0)=0$ and $\xi(0)=0$. According to Theorem 3.1, the generalized double Laplace transform exists whenever $f(x,y)$ is continuous on $[0,\infty)\times[0,\infty)$.

We now provide a few illustrative examples demonstrating the application of the main results.

Example 6.1. Using Theorem 3.1 and as an application of (5.1), we have

$$\mathcal{L}^2_{\Psi,\xi}\{1\}=\frac{1}{pq}.$$

Example 6.2. Using Theorem 3.3 and as an application of (5.4), we have

$$\mathcal{L}^2_{\Psi,\xi}\{2\cos(\ln(1+x))+3\cos(\sqrt{y})\}=\frac{2p}{p^2+1}+\frac{3q}{q^2+1}.$$

Example 6.3. Using Theorem 3.4 and as an application of (5.4), we have

$$\mathcal{L}^2_{\Psi,\xi}\{\cos(2\ln(1+x)+3y)\}=\frac{pq-6}{(p^2+4)(q^2+9)}.$$

Example 6.4. Using Theorem 3.4 and as an application of (5.2), we have

$$\mathcal{L}^2_{\Psi,\xi}\{e^{2\sin x+3y^2}\}=\mathcal{L}^2_{\Psi,\xi}\{e^{2\sqrt{x}+3(1-\cos y)}\}=\frac{1}{(p-2)(q-3)}.$$

Example 6.5. Using Theorem 3.5 and as an application of (5.3) and (5.4), we have

$$\mathcal{L}^2_{\Psi,\xi}\{\cos(2x+3y)\}=\mathcal{L}^2_{\Psi,\xi}\{\cos(2\sqrt{x}+3\sin y)\}=\frac{pq-6}{(p^2+4)(q^2+9)}.$$

Example 6.6. Using Theorem 3.5 and as an application of (5.3) and (5.5), we have

$$\mathcal{L}^2_{\Psi,\xi}\{\sin(3x+2y)\}=\mathcal{L}^2_{\Psi,\xi}\{\sin(3\sqrt{x}+2\ln(1+y))\}=\frac{2p+3q}{(p^2+9)(q^2+4)}.$$

Example 6.7. Using Theorem 3.6 and as an application of (5.9), for $\Psi(x)=x$, $\xi(y)=y$ or $\Psi(x)=\sqrt{x}$, $\xi(y)=\sqrt{y}$, we have

$$\mathcal{L}^2_{\Psi,\xi}\{(\Psi(x))^{1/2}(\xi(y))^{3/2}\}=\frac{\Gamma(3/2)}{p^{3/2}}\cdot\frac{\Gamma(5/2)}{q^{5/2}}.$$

Example 6.8. Using Theorem 3.7 and as an application of (5.2), we have

$$\mathcal{L}^2_{\Psi,\xi}\{e^{2\ln(1+x)+3\sqrt{y}}\}=\frac{1}{(p-2)(q-3)}.$$

Example 6.9. Using the Convolution Theorem 4.2 and as an application of (5.5) and (5.8), we have

$$\mathcal{L}^2_{\Psi,\xi}\{\sin(x+2y)**\,x^2y^3\}=\frac{12(2p+q)}{p^3q^4(p^2+1)(q^2+4)}.$$

Example 6.10. Using Theorem 3.3 and Theorem 3.7, and as an application of (5.2) and (5.7), we have

$$\mathcal{L}^2_{\Psi,\xi}\{2e^{2\sqrt{x}}+3(1+y)^3\}=\frac{2}{p-2}+\frac{18}{q^4}.$$

Remark 6.11. It should be noted that the generalized double Laplace transform depends critically on the choice of the kernel functions Ψ and ξ. By varying these functions, one can recover many well-known transforms as special cases. For example, when $\Psi(x) = x$ and $\xi(y) = y$, the transform reduces to the classical double Laplace transform. Choosing $\Psi(x) = \sqrt{x}$, $\xi(y) = \sqrt{y}$ or $\Psi(x) = \ln(1+x)$, $\xi(y) = y$ generates entirely new families of transforms, while still retaining the linearity, shifting, and convolution properties. Thus, flexibility in selecting Ψ and ξ not only broadens the scope of applications, but also provides a unified framework to study a wide variety of functions in a single transform domain.

7. Concluding Remark

This paper presents a generalized double Laplace transform and established its fundamental properties, including linearity, the shifting property, and change-of-scale property and convolution. Furthermore, we have derived its relationship with the classical double Laplace transform, thereby demonstrating that the proposed framework encompasses existing results as special cases. Moreover, the generalized double Laplace transform provides a versatile tool for practical applications, including the analytical and numerical solutions of classical, fractional, and conformable partial differential equations. Its formulation allows independent scaling of multiple variables, which can be leveraged in modeling complex physical processes, engineering systems, and signal analysis. Future computational implementations of this framework could significantly enhance efficiency in solving higher dimensional problems, thus bridging theoretical development and practical applicability. The developed formulation opens avenues for further theoretical extensions and the exploration of diverse applications, which will be addressed in future research.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. S. Aboodh, The new integral transform "Aboodh Transform", *Global Journal of Pure and Applied Mathematics* **9**(1) (2013), 35 – 43.

[2] V. K. Borawake and A. P Hiwarekar, Modified double Laplace transform of partial derivatives and its applications, *Gulf Journal of Mathematics* **16**(2) (2024), 353 – 363, DOI: 10.56947/gjom.v16i2.1892.

[3] R. Bracewell, *The Fourier Transform and its Applications*, MacGraw Hill, (1963).

[4] L. Debnath, The double Laplace transforms and their properties with applications to functional, integral and partial differential equations, *International Journal of Applied and Computational Mathematics* **2** (2016), 223 – 241, DOI: 10.1007/s40819-015-0057-3.

[5] L. Debnath and B. Bhatta, *Integral Transform and Their Applications*, Third edition, CRC Press, Boca Raton, xxvi + 777 pages (2015).

[6] R. R. Dhunde and G. L. Waghmare, Double Laplace transform method in mathematical physics, *International Journal of Theoretical and Mathematics Physics* **7**(1) (2017), 14 – 20, URL: http://article.sapub.org/10.5923.j.ijtmp.20170701.04.html.

[7] T. M. Elzaki, The new integral transform Elzaki Transform, *Global Journal of Pure and Applied Mathematics* **7**(1) (2011), 57 – 64.

[8] H. M. Fahad, M. ur Rehman and A. Fernandez, On Laplace transforms with respect to functions and their applications to fractional differential equations, *Mathematical Methods in the Applied Sciences* **46**(7) (2021), 8304 – 8323, DOI: 10.1002/mma.7772.

[9] K. A. S. Hassan, The new Integral transform "Kamal Transform", *Advances in Theoretical and Applied Mathematics* **11**(4) (2016), 451 – 458.

[10] A. P. Hiwarekar, Triple Laplace transforms and its properties, *Advances and Applications in Mathematical Sciences* **20**(11) (2021), 2843 – 2851.

[11] M. I. Idrees, Z. Ahmed, M. Awais and Z. Perveen, On the convergence of Double Elzaki transform, *International Journal of Advanced and Applied Sciences* **5**(6) (2018), 19 – 24, DOI: 10.21833/ijaas.2018.06.003.

[12] F. Jarad and T. Abdeljawad, Generalized fractional derivatives and Laplace transform, *Discrete and Continuous Dynamical Systems - S* **13**(3) (2020), 709 – 722, DOI: 10.3934/dcdss.2020039.

[13] M. Kazim, M. Abbas, S. Hussain and M. A. Abbas, Fractional modelling of heat transfer through porous media for incompressible MHD fluid flow with laplace transform approach, *Partial Differential Equations in Applied Mathematics* **15** (2025), 101242, DOI: 10.1016/j.padiff.2025.101242.

[14] M. R. Lemnaouar and I. El Hakki, On the double Laplace transform with respect to another function, *Chaos, Solitons & Fractals* **194** (2025), 116237, DOI: 10.1016/j.chaos.2025.116237.

[15] O. Ozkan and A. Kurt, Conformable fractional double laplace transform and its applications to fractional partial integro-differential equations, *Journal of Fractional Calculus and Applications* **11**(1) (2020), 70 – 81.

[16] T. Pradhan, S. Jena and S. R. Mishra, Time-dependent free convection of MHD Newtonian fluid over a vertical oscillating plate for the interaction of the heat source and the chemical reaction: The Laplace transformation technique, *Heat Transfer* (2025), DOI: 10.1002/htj.23359.

[17] R. Saadeh, A. Qazza and A. Burqan, A new integral transform: ARA transform and its properties and applications, *Symmetry* **12**(6) (2020), 925, DOI: 10.3390/sym12060925.

[18] M. Saif, F. Khan, K. S. Nisar and S. Araci, Modified Laplace transform and its properties, *Journal of Mathematics and Computer Science* **21**(2) (2020), 127 – 135, DOI: 10.22436/jmcs.021.02.04.

[19] R. Sharma and H. K. Meena, Emerging trends in EEG signal processing: A systematic review, *SN Computer Science* **5** (2024), article number 415, DOI: 10.1007/s42979-024-02773-w.

[20] G. K. Watugala, Sumudu transform: A new integral transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in Science and Technology* **24**(1) (1993), 35 – 43, DOI: 10.1080/0020739930240105.

[21] G. K. Watugala, The Sumudu transform for functions of two variables, *Mathematics in Engineering, Science and Aerospace* **8**(4) (2002), 293 – 302.