



Strong Zero-divisor Graph of p.q.-Baer *-Rings

Nana Kumbhar^{*1} , Anil Khairnar²  and B. N. Waphare³ 

¹Department of Mathematics and Statistics, BVDU, Yashwantrao Mohite College (affiliated to Bharati Vidyapeeth (Deemed to be University)), Pune 411038, Maharashtra, India

²Department of Mathematics, MES Abasaheb Garware College (affiliated to Savitribai Phule Pune University), Pune 411004, Maharashtra, India

³Department of Mathematics, Savitribai Phule Pune University, Pune 411007, Maharashtra, India

*Corresponding author: kumbharnana03@gmail.com

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Abstract. In this paper, we study the strong zero-divisor graph of a p.q.-Baer *-ring and establish conditions, based on the smallest central projection in the lattice of central projections, under which the graph contains a cut vertex. We prove that the set of cut vertices forms a complete subgraph. Furthermore, we show that the complement of this graph is connected if and only if the *-ring contains at least six central projections. The diameter and girth of the complement are determined, and we characterize p.q.-Baer *-rings for which strong zero-divisor graph is complemented.

Keywords. *-Ring, p.q.-Baer *-ring, Central projections, Zero-divisor graph, Complement of the graph

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1. Introduction

A *-ring R is a ring equipped with an involution $x \rightarrow x^*$, that is, an additive anti-automorphism of the period at most two. An element e of a *-ring is a *projection* if it is idempotent (i.e., $e^2 = e$) and self adjoint (i.e., $e^* = e$). We write $r_R(B) = \{x \in R \mid bx = 0, \text{ for all } b \in B\}$ and is called the *right annihilator* of B in R , and $l_R(B) = \{x \in R \mid xb = 0, \text{ for all } b \in B\}$ and is called the *left annihilator* of B in R . A *-ring R is called a *Baer *-ring*, if for every $B \subset R$, $r_R(B) = eR$, for suitable projection e in R . A *-ring R is called a *Rickart *-ring*, if for every $x \in R$, $r_R(x) = eR$, for suitable projection e in R . For every element a in a Rickart *-ring, there exists a unique

projection e such that $ae = a$ and $ax = 0$ if and only if $ex = 0$, called a *right projection* of a denoted by $RP(a)$. Similarly, left projection $LP(a)$ is defined for every element a in a Rickart *-ring R . In fact, $r_R(a) = (1 - RP(a))R$. A subring B of a *-ring R is called a **-subring* if $x \in B$ then, $x^* \in B$. A projection e in a *-ring R is called a *central projection* if it is element of center of the ring. For more details about Baer *-rings and Rickart *-rings (see Berberian [9]).

A *-ring R is called a *quasi Baer *-ring* (Birkenmeier et al. [11]), if the right annihilator of every ideal of R is generated by a projection in R . A *-ring R is called a *p.q.-Baer *-ring* (Birkenmeier et al. [10]), if right annihilator of every principal right ideal aR of R , is generated by a projection e in R . Thakare and Waphare [21], established that $M_n(\mathbb{Z}_m)$ is a Baer *-ring, for $n \geq 2$ if and only if $n = 2$ and m is a square-free integer whose prime factor is of the form $4k + 3$. Hence, in general $M_n(R)$ need not be Baer *-ring even when R it self a Baer *-ring. It is observed that, R is a quasi-Baer *-ring if and only if $M_n(R)$, the $n \times n$ matrix ring with *-transpose as an involution, is a quasi-Baer *-ring (hence a p.q.-Baer *-ring) for all $n \geq 1$ (Birkenmeier et al. [11, Proposition 2.6]). Hence, the class of p.q.-Baer *-rings is much larger than the class of Baer *-rings. Let R be a *-ring, $a \in R$, we say that a has a *central cover* if there exists a smallest central projection h such that $ha = a$. The central cover of a is denoted by $C(a)$. In [15], Khairnar and Waphare proved that, every p.q.-Baer *-ring has a unity, involution $*$ is semi-proper, and every element possesses a central cover. In [14], Khairnar and Waphare investigates Conrad's partial order on p.q.-Baer *-rings. They show that rings form pseudo lattices under this order and identify conditions for completeness. Using central covers, the partial order is characterized, leading to a rank-like subtractive function.

A graph G that contains neither loops nor parallel edges is called a *simple graph*. A graph G is called *complete* if there is an edge between any two distinct vertices. If any two distinct vertices are joined by a path, then G is called a *connected graph*. If x and y are two vertices of G , $d(x, y)$ denotes the length of the shortest path joining x and y . The *diameter* of G is defined as $\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. The *girth* of G is the length of the shortest cycle of G , and it is denoted by $gr(G)$. Vertex a of a graph G is called a *cut vertex* if the removal of a together with all edges incident to a increases the number of connected components in the graph G . We say that a graph G is *split* into two subgraphs X and Y via a vertex a if $|V(X)| \geq 2$ and $|V(Y)| \geq 2$, $X \cup Y = G$, $V(X) \cap V(Y) = \{a\}$, the vertex a is a cut vertex of G , and no vertex $x \in V(X) \setminus \{a\}$ is adjacent to any vertex $y \in V(Y) \setminus \{a\}$ (for more about graph theoretic concepts, see, West [23]).

Relating one structure to another and studying the interplay between them is an interesting area of research. Researchers are attracted to this area as one can solve problems in one structure using another. Abrams and Pion [1] were introduced the Leavitt path algebra of a graph as an algebraic counterpart of the graph C^* -algebra and as generalizations of Leavitt algebras. The ideas or process in commutative rings becomes quite difficult for non-commutative rings, in particular *-rings. Giving examples in *-rings with certain properties is a difficult task. Hazrat and Vaš [12] characterize Leavitt path algebras, which are Rickart, Baer and Baer *-rings in terms of the properties of the underlying graph.

Researchers relate a graph to an algebraic structure. The zero-divisor graph of a commutative ring was introduced by Beck in [8]. The *chromatic number* $\chi(G)$ of a graph G is defined as the minimum number of colors required to color the vertices of G so that no two adjacent vertices receive the same color. A subset S of the vertex set $V(G)$ of G is called a *clique* if every pair of distinct vertices in S is adjacent. The clique number $\omega(G)$ is the maximum cardinality of a clique in G . Beck conjectured that for a commutative ring R , $\chi(\Gamma(R)) = \omega(\Gamma(R))$, if $\omega(\Gamma(R)) < \infty$.

In [4], Anderson and Naseer provided a counterexample of a commutative ring for which the conjecture fails. Anderson and Livingston [5] redefined the zero-divisor graph, and proved such a graph is connected and has diameter less than or equal to 3. Properties of $\Gamma(R)$, such as connectedness, diameter, and girth, have been investigated by various authors, see Akbari and Mohammadian [2, 3], and Anderson and Livingston [5]. In [20], Redmond generalized the concept of the zero-divisor graph to non-commutative rings. Associate directed graph $\Gamma(G)$ for a non-commutative ring R , its vertices are all non-zero zero-divisors of R , in which for any two distinct vertices x and y , $x \rightarrow y$ is an edge if and only if $xy = 0$. In [3], Akbari and Mohammadin associate a simple undirected graph $\bar{\Gamma}(R)$ for a non-commutative ring R , in which two vertices x and y are adjacent if and only if $x \neq y$ and either $xy = 0$ or $yx = 0$.

In [18], Patil and Waphare associated a simple undirected graph to a *-ring R whose vertices are the nonzero left zero-divisors, that is, $\{a \in R \mid a \neq 0, ab = 0, \text{ for some nonzero } b \in R\}$. Two distinct vertices a and b are adjacent if and only if $ab^* = 0$. This graph is called the *zero-divisor graph* of the *-ring R and is denoted by $\Gamma^*(R)$. They investigated properties of $\Gamma^*(R)$, including connectedness, diameter, and girth. Also, they proved Beck's conjecture is true for $\Gamma^*(R)$. In [17], Patil and Waphare proved that the set of cut vertices of $\Gamma^*(R)$ forms a complete subgraph. They also characterize those Rickart *-rings for which the complement of $\Gamma^*(R)$ is connected. Axtell *et al.* [6] determine when the zero-divisors form an ideal in a finite commutative ring with identity. Axtell *et al.* [7] and Redmond [19] examined and characterized the existence of cut vertices in the zero-divisor graph of a commutative ring. In particular, Redmond [19] proved that the set of cut vertices in the zero-divisor graph of a commutative ring forms a complete subgraph. Patil and Waphare [17], proved analogous results for zero-divisor graph of Rickart *-rings. Vishweswaran [22], studied the complement and connectedness of the zero-divisor graph of commutative rings.

Kumbhar *et al.* [16] introduced the *strong zero-divisor graph* of a *-ring R , this is the simple undirected graph associated to R whose vertex set is $V(\Gamma_s^*(R)) = \{a \in R \mid a \neq 0, r_R(aR) \neq \{0\}\}$. Two distinct vertices a and b are adjacent if and only if $aRb^* = 0$. This graph is called the *strong zero-divisor graph* of R and is denoted by $\Gamma_s^*(R)$. In general, $\Gamma_s^*(R)$ is not connected. In the same paper, the authors studied the girth, diameter, and connectedness of $\Gamma_s^*(R)$. Also, proved that, the graph $\Gamma_s^*(R)$ is a uniquely complemented, for a p.q.-Baer *-ring R .

In this paper, we continue the study of the strong zero-divisor graph of rings with involution. Axtell *et al.* [6] examined the cut vertex in the sense of zero-divisor graphs. In Section 2, we study the existence of a cut vertex in the $\Gamma_s^*(R)$, when R is a *-ring. This works extends the results of Axtell *et al.* [7] to *-ring. In Section 3, we prove that, for a p.q.-Baer *-ring R , the set of cut vertices of $\Gamma_s^*(R)$ forms a complete subgraph. In Section 4, we study the complement of the strong zero-divisor graph of a p.q.-Baer *-ring. We prove that the complement of this graph is connected if and only if the ring contains at least six central projections. Furthermore, we characterize the diameter and girth of the complement of the strong zero-divisor graph of p.q.-Baer *-rings.

2. Strong Zero-divisor Graph of *-Rings

The notion of strong zero-divisors in a *-ring was introduced by Kumbhar *et al.* [16]. In this section, we examine the cut vertices of $\Gamma_s^*(R)$. Henceforth, we assume that $V(\Gamma_s^*(R)) \neq \emptyset$, refer to $\Gamma_s^*(R)$ as the *strong zero-divisor graph* of a *-ring R , and use the notation $x \sim y$ to indicate that x and y are adjacent.

Definition 2.1 ([16, Definition 2.1]). Let R be a $*$ -ring. We associate a simple undirected graph $\Gamma_s^*(R)$ to R whose vertex set is $V(\Gamma_s^*(R)) = \{a \in R \mid a \neq 0, r_R(aR) \neq \{0\}\}$ and two distinct vertices a and b are adjacent if and only if $aRb^* = 0$.

Example 2.2. Let $R = M_2(\mathbb{Z}_6)$ be a $*$ -ring with the transpose as an involution. Consider the elements $a = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \in R$. Observe that $aRc^* = 0$ and $bRc^* = 0$, i.e., $a, b \in V(\Gamma_s^*(R))$. On the other hand $ab^* = 0$, so $a, b \in V(\Gamma^*(R))$ but $aRb^* \neq 0$, and hence, the vertex a and b are adjacent in $\Gamma^*(R)$ but not in $\Gamma_s^*(R)$.

Example 2.3. Let $R = M_2(\mathbb{Z}_6)$, be a $*$ -ring with the transpose as an involution. Consider the elements $a = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \in R$. Clearly, $aRc^* = 0$ and $bRc^* = 0$, i.e., $a, b \in V(\Gamma_s^*(R))$. On the other hand $ab = 0$, so $a, b \in V(\bar{\Gamma}(R))$ but $aRb^* \neq 0$. Therefore, the vertex a and b are adjacent in $\bar{\Gamma}(R)$ but not in $\Gamma_s^*(R)$.

From the above examples, it is clear that the general graph $\Gamma_s^*(R)$ is not an induced subgraph of $\bar{\Gamma}(R)$ and $\Gamma^*(R)$.

Theorem 2.1 ([16, Theorem 2.1]). Let R_1, R_2 be two $*$ -rings with $V(\Gamma_s^*(R_1)) \neq \emptyset$ and $V(\Gamma_s^*(R_2)) \neq \emptyset$. Then, $A = R_1 \oplus R_2$ equipped with the componentwise involution, is a $*$ -ring such that $\Gamma_s^*(A)$ is connected and $\text{diam}(\Gamma_s^*(A)) \leq 4$.

The above theorem shows that the direct sum of two $*$ -rings with nonempty strong zero-divisor graphs always yields a connected strong zero-divisor graph. This behavior naturally extends to finite direct sums of more than two $*$ -rings.

Corollary 2.2. If A_1, A_2, \dots, A_n are $*$ -subrings of a $*$ ring A such that $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$, then $\Gamma_s^*(A)$ is connected.

Corollary 2.3. Let R be a $*$ -ring. If R admits a nontrivial central projection e , then R decomposes as $R = eR \oplus (1-e)R$. Consequently, the strong zero-divisor graph $\Gamma_s^*(R)$ is connected.

Axtell *et al.* [7] and Redmond [19] investigated the zero-divisor graph of a commutative ring and provided characterizations for the existence of cut vertices. Axtell *et al.* [6, Theorem 4.4], proved that $\{0, a\}$ is an ideal of a finite commutative ring R if a is a cut vertex in $V(\Gamma(R))$. We prove its analogous for $\Gamma_s^*(R)$ in the following theorem.

Theorem 2.4. Let R be a $*$ -ring. If $\Gamma_s^*(R)$ splits into two subgraphs X and Y via vertex a , then the set $\{0, a\}$ forms an ideal of R .

Proof. Clearly, $a+a \neq a$ and from $aRx^* = 0$ it follows that $(a+a)Rx^* = 0$. We claim that $a+a = 0$. Suppose, on the contrary, that $a+a \neq 0$. Then, $a+a \in V(\Gamma_s^*(R))$. Assume that $a+a \in V(X) \setminus \{a\}$. Let $b \in V(Y) \setminus \{a\}$ be such that $aRb^* = 0$. This implies $(a+a)Rb^* = 0$, which contradicts the assumption that a is a cut vertex. Therefore, we must have $a+a = 0 \in \{0, a\}$. Let $r \in R$. If $aRc^* = 0$ for some $c \in R$, then $raRc^* = 0$ and $arRc^* = 0$. Suppose $ra \neq 0$ and $ra \neq a$. Then, every vertex adjacent to a would also be adjacent to ra , which contradicts the fact that a is a cut vertex. Hence, $ra \in \{0, a\}$. Similarly, $ar \in \{0, a\}$. Thus, $\{0, a\}$ is an ideal of R . \square

Theorem 2.5. Let R be a *-ring and $\Gamma_s^*(R)$ splits into two subgraphs X and Y via a such that X is complete subgraph, then $V(X) \cup \{0\}$ is an ideal of R .

Proof. Let $b \in X \setminus \{a\}$ such that a and b are adjacent. As $X \setminus \{a\}$ is complete, for any $x, y \in V(X) \cup \{0\}$, we have $bRx^* = 0$ and $bRy^* = 0$. Consequently, $bR(x+y)^* = bR(x^* + y^*) = bRx^* + bRy^* = 0$ which implies $x + y \in V(X) \cup \{0\}$. Furthermore, for any $r \in R$ then $bR(rx)^* = bR(x^*r^*) = 0$, which gives $rx \in V(X) \cup \{0\}$. Also, if $r \in R$ then $bR(xr)^* = bR(r^*x^*) = 0$, which gives $xr \in V(X) \cup \{0\}$. Hence, $V(X) \cup \{0\}$ is an ideal of R . \square

The following example show that the converse of Theorem 2.5, is not true.

Example 2.4. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_4$ be a *-ring with identity mapping as an involution. Here, $X = \{(1,0), (0,2), (1,2)\}$, $Y = \{(0,1), (1,0), (0,3)\}$ and $(1,0)$ is a cut vertex and $V(X) \cup \{(0,0)\} = \{(0,0), (1,0), (0,2), (1,2)\}$ is an ideal of R but X is not complete subgraph of $\Gamma_s^*(R)$.

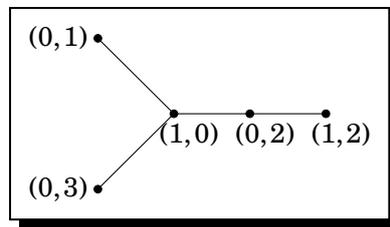


Figure 1. $\Gamma_s^*(\mathbb{Z}_2 \times \mathbb{Z}_4)$

Proposition 2.6. Let R be a *-ring. If $\Gamma_s^*(R)$ contains a cut vertex, then for all $x, y \in R \setminus (V(\Gamma_s^*(R)) \cup \{0\})$, we have $x + y \in V(\Gamma_s^*(R)) \cup \{0\}$.

Proof. Let a be a cut vertex of $\Gamma_s^*(R)$. By Theorem 2.4, the set $\{0, a\}$ is a left ideal in R . Hence, $Ra = I = \{0, a\}$, and for any $x \in R$, we have $xRa = xI = \{0, xa\} \subseteq I$. If $x \in R \setminus V(\Gamma_s^*(R))$, then $xRa \neq 0$, hence $xRa \subseteq I$. Similarly $y \notin V(\Gamma_s^*(R))$ implies $-y \notin V(\Gamma_s^*(R))$ implies $-yRa \subseteq I$. This gives $xRa = -yRa = I$, implies $(x + y)Ra = 0$. Therefore, $x + y \in V(\Gamma_s^*(R)) \cup \{0\}$. \square

The converse of the above statement is not true.

Example 2.5. Let $R = \mathbb{Z}_4$ with identity mapping as an involution. Here $V(\Gamma_s^*(R)) = \{2\}$ and for any $x, y \notin V(\Gamma_s^*(R)) \cup \{0\}$, $x + y \in V(\Gamma_s^*(R)) \cup \{0\}$ but $\Gamma_s^*(R)$ does not have a cut vertex.

Definition 2.6. Let R be a ring and $a \in R \setminus \{0\}$. The right annihilator of the principal right ideal aR , denoted $r_R(aR)$, is said to be properly maximal if for every $b \in R \setminus \{0, a\}$ we have $r_R(aR) \not\subseteq r_R(bR)$.

Proposition 2.7. Let R be a *-ring. If a is a cut vertex of $\Gamma_s^*(R)$, then $r_R(aR)$ is properly maximal.

Proof. Suppose that, there exists $b \in R \setminus \{0, a\}$ such that $r_R(aR) \subseteq r_R(bR)$. Then, every vertex adjacent to a in $\Gamma_s^*(R)$ is also adjacent to b . Hence, a is not a cut vertex, a contradiction. Therefore, no such b exists, and $r_R(aR)$ is properly maximal. \square

The following example shows that the converse of Proposition 2.7 is not true.

Example 2.7. Let $R = \mathbb{Z}_6$ with identity mapping as an involution. Consider $a = 4 \in R$. We have $r_R(4) = \{0, 3\}$. Clearly, $r_R(4)$ is properly maximal. However, 4 is not a cut vertex of $\Gamma_s^*(R)$. Thus, the converse of Proposition 2.7 does not hold.

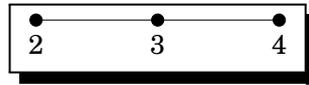


Figure 2. $\Gamma_s^*(\mathbb{Z}_6)$

3. Strong Zero-divisor Graph of p.q.-Baer *-Rings

Birkenmeier *et al.* [10] introduced the class of *principally quasi-Baer* (p.q.-Baer) *-rings. Let $CP(R)$ denote the set of all central projections in a *-ring R . An involution $*$ on R is said to be *semiproper* if for any $a \in R$, the condition $aRa^* = \{0\}$ implies $a = 0$. The class of p.q.-Baer *-rings is larger than the class of Baer *-rings and Rickart *-rings (see, Birkenmeier *et al.* [10, Exercise 10.2.24.4 and 10.2.24.5], and Khairnar and Waphare [13, Example 2.3 and 2.6]). In a p.q.-Baer *-ring, $aRb^* = 0$ holds if and only if $C(a)C(b) = 0$. Throughout this work, we use this equivalent criterion to describe adjacency in $\Gamma_s^*(R)$. It is important to observe that, for a p.q.-Baer *-ring R , the vertex set of a graph is empty whenever $CP(R) = \{0, 1\}$. Henceforth, we assume that a p.q.-Baer *-ring R contains at least four central projections, i.e., $|CP(R)| \geq 4$.

In [6], Axtell proved the following result.

Theorem 3.1 ([6, Theorem 4.3]). *Let R be a commutative ring with identity such that $\Gamma(R)$ is partitioned into two subgraphs X and Y with cut vertex a and $|X| > 2$. If X is a complete subgraph, then every vertex of X is looped.*

Theorem 3.1 shows that in the commutative case, completeness of a component forces every vertex to be looped. The following theorem is analogous to Theorem 3.1.

Theorem 3.2. *Let R be a p.q.-Baer *-ring, and suppose that $\Gamma_s^*(R)$ splits into two subgraphs X and Y via a cut vertex a with $|V(X)| > 2$. If X is a complete subgraph, then $b^2 \in \{0, b\}$, for every $b \in V(X) \setminus \{a\}$.*

Proof. Let $b \in V(X) \setminus \{a\}$, with $b^2 \notin \{0, b\}$. Since X is a complete subgraph, we have $cRb^* = 0$, for all $c \in V(X) \setminus \{a, b\}$. Hence, $cR(b^*)^2 = cR(b^2)^* = 0$ this gives $b^2 \in V(X)$ and $cR(b^2 + b)^* = 0$. If $b^2 + b = 0$, then $b^2 = -b$, $b^2 \neq b$ therefore b^2 is adjacent to b , implies $0 = b^2Rb^* = -bRb^* = bRb^*$. Since involution in a p.q.-Baer *-ring is semiproper, we have $b = 0$, a contradiction. Hence, $b^2 + b \neq 0$. Thus, $b^2 + b \in V(X)$. If $b^2 + b = b$ then $b^2 = 0 \in \{0, b\}$, a contradiction. If $b^2 + b \notin \{0, b\}$, then $0 = (b^2 + b)R(b)^* = bRb^*$, again a contradiction. Therefore, $b^2 \in \{0, b\}$, for all $b \in V(X) \setminus \{a\}$. □

Corollary 3.3. *Let R be a p.q.-Baer *-ring such that $\Gamma_s^*(R)$ splits into two subgraphs X and Y via a cut vertex a with $|V(X)| > 2$. Then, the following statements hold:*

- (i) *If there exists $b \in V(X) \setminus \{a\}$ with $b^2 \notin \{0, b\}$, then X cannot be complete.*
- (ii) *If X is a complete subgraph, then $V(X) \cup \{0\}$ forms an ideal of R .*

Proof. (a): This follows directly from Theorem 3.2.

(b): Let X be complete. Then, for any $b, c \in V(X)$ with $b \neq c$, we have $bRc^* = 0$ which gives $(b + b)Rc^* = 0$. If $b + b \neq 0$, then $b + b \in V(X) \subseteq V(X) \cup \{0\}$. Let $x, y \in V(X) \cup \{0\}$, if $x = 0$ or $y = 0$, then $x + y \in V(X) \cup \{0\}$. Let $x = b, y \neq b$, then $(b + y)Rc^* = 0$, for any $c \in V(X) \cup \{0\}$. Hence, $x + y \in V(X) \cup \{0\}$. Let $x \neq b$, and $y \neq b$. Then, $xRb^* = 0 = yRb^*$ gives $(x + y)Rb^* = 0$. Therefore, $x + y \in V(X) \cup \{0\}$. Moreover, if $r \in R$ and $x \in V(X)$, then $xRc^* = 0$ implies $(rx)Rc^* = 0 = (xr)Rc^*$, so $rx, xr \in V(X) \cup \{0\}$. Thus, $V(X) \cup \{0\}$ is an ideal of R . \square

Set of all central projections in a p.q.-Baer *-ring forms a lattice under the partial order, $e \leq f$ if and only if $e = ef = fe$ and $e \wedge f = ef$ and $e \vee f = e + f - ef$, it is denoted by $L(CP(R))$. An element e in a lattice is an atom if $0 \leq f \leq e$ implies either $f = 0$ or $f = e$.

Theorem 3.4. *Let R be a p.q.-Baer *-ring. If a is a cut vertex of $\Gamma_s^*(R)$, then a is an atom in the lattice of central projections $CP(R)$.*

Proof. Suppose $a \in V(\Gamma_s^*(R))$ is a cut vertex. Let $e = C(a)$, we have $r_R(aR) = r_R(eR)$. If $a \neq e$, then $e \in R \setminus \{0, a\}$, such that $r_R(aR) \subseteq r_R(eR)$, a contradiction. Hence, $a = C(a) = e$, i.e., a is a central projection. Let f be a central projection in R , with $f \leq a$, i.e., $f = af = fa$. Then, $r_R(aR) \subseteq r_R(fR)$. Since, $r_R(aR)$ is properly maximal, we get either $f = 0$ or $f = a$. Hence, a is an atom in the lattice of central projections in R . \square

The converse of the above theorem is not true. The following examples demonstrate that not every atom of $CP(R)$ is necessarily a cut vertex of $\Gamma_s^*(R)$.

Example 3.1. Let $R = M_2(\mathbb{Z}_6)$ with identity mapping as an involution. Then, R is a p.q.-Baer *-ring such that

$$CP(R) = \left\{ 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, e = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, f = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right\}$$

are the central projections in R . Here, $\Gamma_s^*(R) \cong K_{80,15}$ and $e = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is an atom in $L(CP(R))$ but e is not cut vertex in $\Gamma_s^*(R)$.

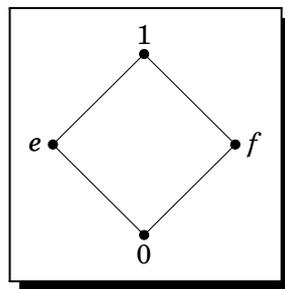


Figure 3. $L(CP(R))$

Vertex of degree 1 is called a pendant vertex. Note that, if a is pendant vertex in $\Gamma_s^*(R)$ then $a = c(a)$.

Here, we establish a necessary and sufficient condition for the existence of a cut vertex in $\Gamma_s^*(R)$.

Theorem 3.5. *Let R be p.q.-Baer *-ring. Then, the graph $\Gamma_s^*(R)$ contains a cut vertex if and only if it contains a pendant vertex.*

Proof. Suppose that a is a cut vertex in $\Gamma_s^*(R)$. By Theorem 3.4, a is an atom in $L(CP(R))$. Moreover, a is adjacent to $1 - a$. Let b be a vertex adjacent to $1 - a$. Then, $(1 - a)RC(b) = 0$, which implies $(1 - a)C(b) = 0$, i.e., $C(b) \leq a$ in $L(CP(R))$. Since a is an atom and $C(b) \neq 0$, we obtain $C(b) = a$. As $r_R(bR) = r_R(C(b)R)$, it follows that $r_R(bR) = r_R(aR)$. By Proposition 2.7, $r_R(aR)$ properly maximal this implies $a = b$. Therefore, a is the only vertex adjacent to $1 - a$, showing that $1 - a$ is a pendant vertex.

Conversely, assume that a is a pendant vertex in $\Gamma_s^*(R)$, so $C(a)$ is non-trivial by Corollary 2.3, $\Gamma_s^*(R)$ is connected. Clearly, a and $1 - C(a)$ are adjacent in $\Gamma_s^*(R)$. Since a is a pendant vertex, it follows that $1 - C(a)$ is a cut vertex. \square

Corollary 3.6. *Let R be a p.q.-Baer *-ring. Then, $\Gamma_s^*(R)$ contains a cut vertex if and only if there exists an element $a \in R$ such that $|r_R(a)| = 2$.*

Proof. Suppose that a is a cut vertex in $\Gamma_s^*(R)$. By Theorem 3.5, $1 - a$ is a pendant vertex, which implies $|r_R(1 - aR)| = 2$. Conversely, assume there exists $a \in R$ such that $|r_R(aR)| = 2$. Clearly, a is a pendant vertex. By Theorem 3.5, $\Gamma_s^*(R)$ contains a cut vertex. \square

The following result is an analogue of results by Redmond [19, Corollary 6] and Patil *et al.* [17, Theorem 3.9].

Theorem 3.7. *Let R be a p.q.-Baer *-ring. Then, the set of cut vertices of $\Gamma_s^*(R)$ forms a complete subgraph of $\Gamma_s^*(R)$.*

Proof. If $\Gamma_s^*(R)$ contains only one cut vertex, then the statement is trivial. Suppose that $\Gamma_s^*(R)$ has at least two cut vertices, say e and f are central projections in R , and $1 - e$ and $1 - f$ are pendant vertices in $\Gamma_s^*(R)$. Since $\Gamma_s^*(R)$ is connected, there exists a path joining $1 - e$ and $1 - f$, which must pass through e and f . If $ef \neq 0$, then this vertex would be adjacent to both $1 - e$ and $1 - f$, contradicting to the fact that they are pendant vertices. Hence, e and f are adjacent. Thus, the set of cut vertices forms a complete subgraph of $\Gamma_s^*(R)$. \square

4. Complement of Strong Zero-Divisor Graph of a p.q.-Baer *-Ring

In this section, we characterize all p.q.-Baer *-rings for which $\Gamma_s^*(R)^c$ is connected. We recall the following standard properties of simple graphs:

Lemma 4.1. *Let G be a simple graph. Then, the following statements hold:*

- (1) *The complement G^c is connected if and only if $\text{diam}(G) \geq 3$.*
- (2) *Either G or G^c is connected.*
- (3) *The complement of a complete graph has no edges.*
- (4) *The complement of a complete bipartite graph is disconnected.*

Example 4.1. Let $R_1 = \mathbb{Z}_6$ and $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ be *-rings with identity mapping as an involution. Note that, R_1 and R_2 are p.q.-Baer *-rings. Observe that, $V(\Gamma_s^*(\mathbb{Z}_6)) = \{2, 3, 4\}$ and $V(\Gamma_s^*(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = \{a = (1, 0, 0), b = (0, 1, 0), c = (0, 0, 1), d = (1, 1, 0), e = (1, 0, 1), f = (0, 1, 1)\}$. See Figure 4, the complement of $\Gamma_s^*(R_1)^c$ is disconnected and complement of $\Gamma_s^*(R_2)^c$ is connected.

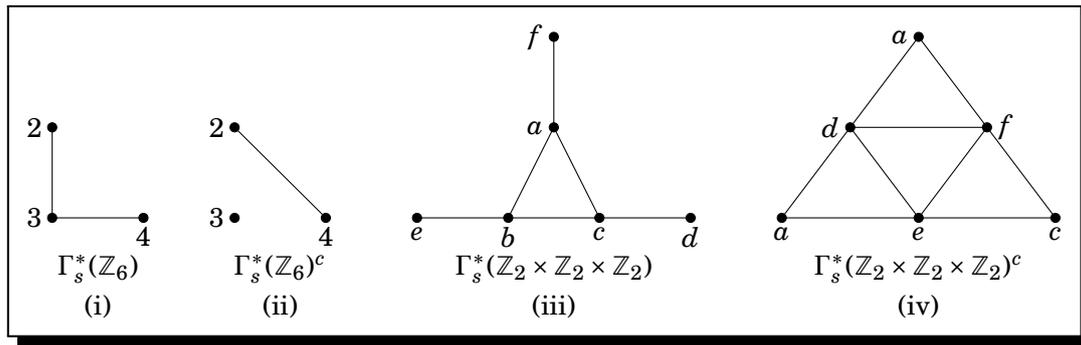


Figure 4

From the above example, it is clear that the complement of a strong zero-divisor graph of a p.q.-Baer *-ring is not connected in general.

In [22], Visweswaran determined when $\Gamma(R)^c$ is connected for a commutative ring with unity.

Theorem 4.1 ([22, Theorem 1.1]). *Let R be a ring such that $Z(R)^*$ contains at least two elements. Then, $(\Gamma(R))^c$ is connected if and only if either (i), (ii) or (iii) holds:*

- (i) R has exactly one maximal N -prime P of (0) and P is not a B -prime of (0) .
- (ii) R has exactly two maximal N -primes P_1, P_2 of (0) and $P_1 \cap P_2 \neq (0)$.
- (iii) R has at least three maximal N -primes of (0) .

In the following result, we prove that the distance between two elements in $\Gamma_s^*(R)$ is the same as the distance between their central covers. We use this result to characterize the connectedness of $\Gamma_s^*(R)^c$.

Lemma 4.2. *Let R be a p.q.-Baer *-ring, and $a, b \in V(\Gamma_s^*(R))$. Then, $d(a, b) = n$ if and only if $d(C(a), C(b)) = n$, where $C(a)$ and $C(b)$ denote the central covers of a and b , respectively.*

Proof. Suppose $d(a, b) = n$ in $V(\Gamma_s^*(R))$. Let $a = a_1 \sim a_2 \sim \dots \sim a_{n+1} = b$ be a path of length n joining a and b . Therefore, $C(x) \notin \{0, 1\}$ for $x \in \{a_1, a_2, \dots, a_{n+1}\}$ and $C(a_i)C(a_j) = 0$ if and only if $j = i + 1$, for $i \in \{1, 2, \dots, n\}$. Hence, $C(a_1) \sim C(a_2) \sim \dots \sim C(a_{n+1})$ is a path. Observe that $C(a_i) \neq C(a_j)$ for $i \neq j$, otherwise there would exist a path of length less than n connecting a and b , contradicting $d(a, b) = n$. As a and b are non-adjacent, we have $C(a)C(b) \neq 0$, so $C(a)$ and $C(b)$ are also non-adjacent. Suppose that there exists a path of length less than n connecting $C(a)$ and $C(b)$. Since a and $C(a)$ are adjacent to the same vertices, we get a path of length less than n , which connects a and b , a contradiction. Therefore, $d(C(a), C(b)) = n$.

Conversely, suppose that $d(C(a), C(b)) = n$. Let $C(a) \sim b_1 \sim b_2 \sim \dots \sim b_{n-1} \sim C(b)$ be a path of length n . Since a is adjacent to the same vertices as $C(a)$ and b is adjacent to the same vertices as $C(b)$, we obtain a corresponding path $a \sim b_1 \sim b_2 \sim \dots \sim b_{n-1} \sim b$ of length n . By the argument above, no path of length less than n connects a and b . Therefore, $d(a, b) = n$. □

For any central projection e in a p.q.-Baer *-ring R , we define $C_e = \{a \in R \mid C(a) = e\}$.

Proposition 4.2. *Let R be a p.q.-Baer *-ring. If $|CP(R)| = 4$, then $\Gamma_s^*(R)$ is a complete bipartite graph. Consequently, its complement $\Gamma_s^*(R)^c$ is disconnected.*

Proof. Suppose that, $CP(R) = \{0, 1, e, 1 - e\}$. Then, $C_e, C_{1-e} \subseteq V(\Gamma_s^*(R))$. In $\Gamma_s^*(R)$, each element of C_e is adjacent to every element of C_{1-e} , and no elements within C_e or C_{1-e} are adjacent. Therefore, $\Gamma_s^*(R)$ is a complete bipartite graph. By Lemma 4.1, $\Gamma_s^*(R)^c$ disconnected. \square

Lemma 4.3. *Let R be a p.q.-Baer *-ring, and let e, f be nontrivial central projections in R with $e + f \neq 1$. If $ef = 0$, then $d(1 - e, 1 - f) = 3$.*

Proof. Let e, f be two non-trivial central projection in a p.q.-Baer *-ring R with $e + f \neq 1$. If $ef = 0$, then e is adjacent to f . We know that e is adjacent to $1 - e$ and f is adjacent to $1 - f$. Hence, $(1 - e) \sim e \sim f \sim (1 - f)$ is a path of length 3. Clearly, $(1 - e)(1 - f) \neq 0$, if possible $(1 - e)(1 - f) = 0$, implies that $1 - f - e - ef = 0$, i.e., $e + f = 1$ which is a contradiction to $e + f \neq 1$. If $(1 - e) \sim a \sim (1 - f)$, then $(1 - e) \sim C(a) \sim (1 - f)$. Then, $c(a)(1 - e) = 0$ and $c(a)(1 - f) = 0$ which gives $c(a) = c(a)e = c(a)f$. This implies $c(a) = c(a)f = c(a)ef = 0$. So $a = c(a)a = 0$, which is contradiction to $a \in V(\Gamma_s^*(R))$. Hence, there is no path of length 2 connecting $1 - e$ and $1 - f$. Therefore, $d(1 - e, 1 - f) = 3$. \square

In the following theorem, we characterize all p.q.-Baer *-rings R for which the graph $\Gamma_s^*(R)^c$ is connected.

Theorem 4.3. *Let R be a p.q.-Baer *-ring. Then, the graph, $\Gamma_s^*(R)^c$, is connected if and only if $|CP(R)| \geq 6$.*

Proof. Suppose that $\Gamma_s^*(R)^c$ is connected. Observe that the sets $\{e, 1 - e\}_{e \in CP(R)}$ form a partition of $CP(R)$. If $|CP(R)|$ is infinite, then $|CP(R)| \geq 6$. If $|CP(R)|$ is finite, then it must be even. If $|CP(R)| = 2$, then $V(\Gamma_s^*(R)) = \emptyset$. If $|CP(R)| = 4$, by Proposition 4.2, $\Gamma_s^*(R)^c$ is disconnected, which is a contradiction. Hence, $|CP(R)| \geq 6$.

Conversely, suppose that $|CP(R)| \geq 6$. If graph $\Gamma_s^*(R)$ is disconnected, then by Lemma 4.1, $\Gamma_s^*(R)^c$ is connected. Suppose $\Gamma_s^*(R)$ is connected. Now, to show $\Gamma_s^*(R)^c$ is connected, for this, it is sufficient to show $\text{diam}(\Gamma_s^*(R)) = 3$. Let e be any atom in the lattice of central projections in R . In view of Lemma 4.3, it suffices to show that there exists a nontrivial central projection f such that $ef = 0$ with $e + f \neq 1$.

Suppose, on the contrary, that for every nontrivial central projection f , $ef = 0$ and $e + f = 1$. Let h be any projection in R such that $h \notin \{0, 1, e, 1 - e\}$ (such an h exists because $|CP(A)| \geq 6$). Since $h \neq 1 - e$, we have $eh \neq 0$. If $(1 - e)h = 0$, then, we get $h = he = eh$, i.e., $h \leq e$. Since e is an atom, this implies $h = 0$ or $h = e$, a contradiction to the choice of h . Hence, $(1 - e)h \neq 0$. Since $\Gamma_s^*(R)$ is connected, there exists a path joining e and h . Let $e \sim a_1 \sim a_2 \sim \dots \sim a_{k-1} \sim a_k = h$ is a path of length k connecting e and h . Then, $eC(a_1) = 0$. By assumption, we have $C(a_1) = 1 - e$. Also, $C(a_1)C(a_2) = 0$ which yields $(1 - e)C(a_2) = 0$. This gives $C(a_2) \leq e$. Since, e is an atom and $C(a_2) = 0$, we get $C(a_2) = e$. Continuing in this way, we get $C(a_i) = e$, if i is even and $C(a_i) = 1 - e$, if i is odd. Thus, $h = e$ or $h = 1 - e$, which leads to a contradiction in both cases. Therefore, there exists a nontrivial central projection f such that $ef = 0$ with $e + f \neq 1$. By Lemma 4.3, $d(1 - e, 1 - f) = 3$. Hence, by Lemma 4.1, $\Gamma_s^*(R)^c$ connected. \square

Corollary 4.4. *Let R be a p.q.-Baer *-ring. If $\Gamma_s^*(R)$ contains a vertex that is adjacent to all other vertices, then $|CP(R)| = 4$.*

Proof. Suppose a is a vertex adjacent to all other vertices in $\Gamma_s^*(R)$. Then, a is an isolated vertex in the complement graph $\Gamma_s^*(R)^c$, which implies that $\Gamma_s^*(R)^c$ is disconnected. By Theorem 4.3, it follows that $|CP(R)| = 4$. \square

Proposition 4.5. *Let R be a p.q.-Baer *-ring. If $\Gamma_s^*(R)^c$ is disconnected, then $gr(\Gamma_s^*(R)^c) \in \{3, \infty\}$.*

Proof. Since $\Gamma_s^*(R)^c$ is disconnected, by Theorem 4.3, we have $|CP(R)| = 4$. Let $CP(R) = \{0, 1, e, 1 - e\}$, $C_e = \{x \in R \mid C(x) = e\}$ and $C_{1-e} = \{x \in R \mid C(x) = 1 - e\}$. Clearly, C_e and C_{1-e} are two connected components of $\Gamma_s^*(R)^c$. If $|C_e| \geq 3$ or $|C_{1-e}| \geq 3$, then $gr(\Gamma_s^*(R)) = 3$. Otherwise, there are no cycles, and hence $gr(\Gamma_s^*(R)^c) = \infty$. \square

Lemma 4.4. *Let R be a p.q.-Baer *-ring such that graph $\Gamma_s^*(R)$ is triangle-free. Then, two vertices a and b are adjacent in $\Gamma_s^*(R)$ if and only if $C(a) = 1 - C(b)$.*

Proof. Suppose $a \sim b$ in $\Gamma_s^*(R)$. Then, $C(a)C(b) = 0$, which implies that $C(a) + C(b)$ is a central projection in R . If $C(a) \neq (1 - C(b))$, then $e = 1 - (C(a) + C(b))$ is a nontrivial central projection and $C(a) \sim C(b) \sim e \sim C(a)$ is a 3-cycle in $\Gamma_s^*(R)$, contradicting the assumption that the graph is triangle-free. Hence, $C(a) = 1 - C(b)$. \square

Theorem 4.6 ([16, Theorem 3.2]). *Let R be a p.q.-Baer *-ring. Then, the following statements hold:*

(A) *The following conditions are equivalent:*

- (i) $gr(\Gamma_s^*(R)) = 3$.
- (ii) R contains two nontrivial central projections e and f such that $ef = 0$ and $e + f \neq 1$.
- (iii) The lattice $L(CP(R))$ contains a 3-element chain not containing 0.

(B) $gr(\Gamma_s^*(R)) = 4$ if and only if $\Gamma_s^*(R)$ is triangle-free and there exists a nontrivial central projection e in R such that $|C_e| \geq 2$ and $|C_{1-e}| \geq 2$.

In the following result, we determine the diameter and girth of the complement of the strong zero-divisor graph, $\Gamma_s^*(R)^c$, in the case when it is connected.

Theorem 4.7. *Let R be a p.q.-Baer *-ring. If $\Gamma_s^*(R)^c$ is connected, then $\text{diam}(\Gamma_s^*(R)^c) = 2$ and $gr(\Gamma_s^*(R)^c) = 3$.*

Proof. Suppose that $\Gamma_s^*(R)^c$ is connected. By Theorem 4.3, we have $|CP(R)| \geq 6$. Let $a, b \in V(\Gamma_s^*(R)^c)$. If $C(a)C(b) \neq 0$, then a and b are adjacent in $\Gamma_s^*(R)^c$.

If $C(a)C(b) = 0$, then $g = C(a) + C(b)$ is a central projection in R . If $g < 1$, then $aRg = aR \neq 0$ and $bRg = bR \neq 0$, so g is adjacent to both a and b in $\Gamma_s^*(R)^c$, which gives $a \sim g \sim b$ is path of length 2 in $\Gamma_s^*(R)^c$.

If $g = 1$, choose a central projection $f \notin \{0, 1, C(a), C(b)\}$. If $C(a)f = 0$, then $(1 - f)C(a) \neq 0$ and $(1 - f)C(b) \neq 0$ (by selection of f). Thus $a \sim (1 - f) \sim b$ is a path of length 2 in $\Gamma_s^*(R)^c$. Similarly, if $C(b)f = 0$, a path of length 2 exists. If $C(a)f \neq 0$ and $C(b)f \neq 0$, then $a \sim f \sim b$ forms a path of length 2 in $\Gamma_s^*(R)^c$. Thus, in any case $d(a, b) = 2$. Hence, $\text{diam}(\Gamma_s^*(R)^c) = 2$.

Now we prove $gr(\Gamma_s^*(R)^c) = 3$.

Case i: If $gr(\Gamma_s^*(R)) = 3$. By Theorem 4.6, there exists two nontrivial central projections $e, f \in R$ such that $ef = 0$ and $e + f \neq 1$, which gives $e(e + f) = e + ef = e \neq 0$, $(e + f)(1 - f) = e - ef + f - f =$

$1 \neq 0$ and $(1 - f)e \neq 0$. Thus $e \sim (e + f) \sim (1 - f) \sim e$ forms a cycle of length 3 in $\Gamma_s^*(R)^c$. Therefore, $gr(\Gamma_s^*(R)^c) = 3$.

Case ii: If $gr(\Gamma_s^*(R)) \neq 3$ i.e., $\Gamma_s^*(R)$ is triangle free. By Lemma 4.4, for nontrivial central projections $e, f \in R$, we have $ef = 0$ if and only if $e + f = 1$. Let $g, h \in CP(R)$, with $g + h \neq 1$, then $gh \neq 0$. If $gR(gh)^* = gRhg = ghR \neq 0$, then $g \sim gh \sim h \sim g$ forms a cycle of length 3 in $\Gamma_s^*(R)^c$. If $gR(gh)^* = gRhg = ghR = \{0\}$, then $hg = 0$ which is a contradiction to $hg \neq 0$. Hence, in both cases $gr(\Gamma_s^*(R)^c) = 3$. □

For distinct vertices a and b of a graph G , we say that a and b are *orthogonal*, denoted $a \perp b$, if a and b are adjacent and share no common neighbors. A graph G is called *complemented* if, for every vertex $a \in V(G)$, there exists a vertex $b \in V(G)$, referred to as a *complement* of a , such that $a \perp b$.

Lemma 4.5. *Let R be a p.q.-Baer *-ring such that $\Gamma_s^*(R)^c$ is connected. Then, $\Gamma_s^*(R)^c$ is not complemented*

Proof. Suppose that $\Gamma_s^*(R)^c$ is connected. Then, R must contains at least 6 central projections. Let a, b are adjacent in $\Gamma_s^*(R)^c$. If $C(a) \neq C(b)$ then $C(a)C(b) \neq 0$ this gives $C(a)$ and $C(b)$ adjacent in $\Gamma_s^*(R)^c$. Also, $aC(a)C(b) = aC(b) \neq 0$ and $bC(a)C(b) = bC(a) \neq 0$ i.e., $C(a)C(b)$ is adjacent to both a and b in $\Gamma_s^*(R)^c$. If $C(a) = C(b) = e$, then choose central projection $f \notin \{0, 1, e, 1 - e\}$. If $ef \neq 0$ then e and f are adjacent in $\Gamma_s^*(R)^c$, i.e., $C(a)f \neq 0$ and $C(b)f \neq 0$ which gives $a \sim f \sim b$. Hence, f is adjacent to both a and b in $\Gamma_s^*(R)^c$. If $ef = 0$, then e and f are non-adjacent in $\Gamma_s^*(R)^c$, i.e., $e(1 - f) = e \neq 0$ implies $a(1 - f) \neq 0$ and $b(1 - f) \neq 0$. Therefore, $a \sim (1 - f) \sim b$ in $\Gamma_s^*(R)^c$. Hence, for any adjacent vertex a and b in $\Gamma_s^*(R)^c$, there is a vertex that is adjacent to both a and b . Thus, $\Gamma_s^*(R)^c$ is not complemented. □

Theorem 4.8. *Let R be a p.q.-Baer *-ring. Suppose there exists an element $a \in V(\Gamma_s^*(R))$ with $a \neq C(a)$. Then, $\Gamma_s^*(R)^c$ is complemented if and only if $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.*

Proof. If $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, then $\Gamma_s^*(R)$ is isomorphic to a 4-cycle, and consequently, its complement $\Gamma_s^*(R)^c$ is complemented.

Conversely, suppose that $\Gamma_s^*(R)^c$ is complemented. Since there exists $a \in V(\Gamma_s^*(R))$ with $a \neq C(a)$, it follows that a is not adjacent to $C(a)$. Hence, $\Gamma_s^*(R)$ is not complete. By Lemma 4.5, this implies that $\Gamma_s^*(R)^c$ is disconnected. Consequently, R contains exactly four central projections. Let $CP(R) = \{0, 1, e, 1 - e\}$, $C_e = \{x \in R \mid C(x) = e\}$ and $C_{1-e} = \{x \in R \mid C(x) = 1 - e\}$. If $|C_e| > 2$, then any three elements in C_e form a connected subgraph in $\Gamma_s^*(R)^c$, implying that no element of C_e can have a complement, which is a contradiction. Thus, $|C_e| \leq 2$. Similarly $|C_{1-e}| \leq 2$. If $|C_e| = |C_{1-e}| = 1$, then $\Gamma_s^*(R)$ is a complete graph on two vertices, a contradiction to the assumption. Therefore $|C_e| = |C_{1-e}| = 2$. Since e is a central projection, we can decompose, $R = eR \oplus (1 - e)R$. Where $|eR| = |(1 - e)R| = 3$ (as $C_e = eR \setminus \{0\}$ and $C_{1-e} = (1 - e)R \setminus \{0\}$). Hence, $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. □

5. Conclusion

We shows that in a p.q.-Baer *-ring, the set of cut vertices in the strong zero-divisor graph forms a complete subgraph. The complement of this graph is connected if and only if the ring has at least six central projections, with diameter 2 and girth 3. Moreover, the complement is complemented only when $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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