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Research Article

Total Vertex-Edge Domination Number of Some Graphs

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Abstract. For a graph $G=(V,E)$, a vertex u in G vertex-edge dominates (or simply ve-dominates) an edge e in G if u is incident to e, or u is incident to an edge that is adjacent to e. A set $S\subseteq V$ is a *vertex-edge dominating set* if for all edges $e\in E$, there exists a vertex $v\in S$ that ve-dominates e. The minimum cardinality of a ve-dominating set of graph G is called the *vertex-edge domination number*, and is denoted by $\gamma_{ve}(G)$. A ve-dominating set is said to be total if its induced subgraph has no isolated vertices, that is, every vertex of S has a neighbor in S. The *total vertex-edge domination number*, denoted by $\gamma^t_{ve}(G)$, of G is the minimum cardinality of a total ve-dominating set. This study focuses on the total vertex-edge domination number in graphs. The authors determined its exact value for the cycle graph C_n, as well as for the join and corona of two arbitrary graphs G and H. A complete characterization of graphs with total vertex-edge domination number equal to two was also established. Furthermore, a general bound for the total vertex-edge domination number of the Cartesian product $G\times K_n$, where $n\geq 4$, was derived.

Keywords. Vertex-edge domination, Total vertex-edge domination, Join, Corona, Cartesian product

Mathematics Subject Classification (2020). 05C69, 05C76

1. Introduction

Domination has been a central topic of study in graph theory, attracting considerable research attention. Its formal development began in the late 1950s and 1960s, initiated by Berge [1] in 1958, who introduced the concept of the 'coefficient of external stability', later recognized as the domination number of a graph. Subsequently, Ore [10], in his 1962 graph theory text, formally defined the notions of a dominating set and the domination number. About a decade later, Cockayne and Hedetniemi [5] advanced the study further through a survey paper, where the notation $\gamma(G)$ was first adopted to represent the domination number of a graph G.

In 1986, Peters [11] introduced the concepts of vertex-edge domination and edge-vertex domination. Later, Lewis *et al*. [9] provided an informal description of vertex-edge domination, stating that a vertex v dominates not only the edges incident to v but also those adjacent to such incident edges. He then examined the notion of a vertex-edge dominating set, defined as follows: a vertex $u \in V(G)$ is said to ve-dominate an edge $vw \in E(G)$ if $u = v$ or $u = w$ (u is incident to vw), or uv or uw is an edge in G (u is incident to an edge that is adjacent to vw). A set $S \subseteq V(G)$ is a vertex-edge dominating set if for all edges $e \in E(G)$, there exists a vertex $v \in S$ that vertex-edge dominates e. The minimum cardinality of the ve-dominating sets is called the ve-domination number and is denoted by $\gamma_{ve}(G)$.

The notion of total domination was initially introduced by Cockayne, Dawes, and Hedetniemi [4]. A subset $S \subseteq V(G)$ is defined as a total dominating set of G if the subgraph induced by S, denoted by $\langle S \rangle$, contains no isolated vertices; equivalently, every vertex in G must be adjacent to at least one vertex of S. The minimum size of such a set is called the total domination number of G, denoted by $\gamma_t(G)$.

Boutrig and Chellali [2] introduced the concept of total vertex-edge domination in graphs. A subset $S \subseteq V(G)$ is called a total vertex-edge dominating set (or simply a total ve-dominating set) of a graph G if it is a ve-dominating set and the subgraph induced by S contains no isolated vertices, meaning that each vertex in S has at least one neighbor within S. The total vertex-edge domination number of G, denoted by $\gamma_{ve}^t(G)$, is defined as the minimum cardinality among all such total ve-dominating sets.

This paper presents new findings on the total vertex-edge domination number for several graph classes. Specifically, the results cover cycle graphs, graphs with radius two, and the join and corona of graphs. In addition, bounds are established for the total vertex-edge domination number of the Cartesian product of a graph G with the complete graph K_n, for $n \geq 4$.

2. Preliminaries

This section presents the basic concepts and results necessary for this study.

Definition 2.1 ([6])**.** The *open neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v) = \{x \in V(G) \mid vx \in E(G)\}$. The *closed neighborhood* of a vertex v, denoted by $N[v]$, is simply the set $\{v\} \cup N(v)$, that is, $N[v] = \{y \in V(G) \mid yv \in V(G)\} \cup \{v\} = N(v) \cup \{v\}$. For a set of vertices $S \subseteq V(G)$, the *open neighborhood* of S denoted by $N(S)$, is defined to be $\bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S, denoted by $N[S]$ is given by $N[S] = N(S) \cup S$.

Definition 2.2 ([12])**.** A *dominating set* S is a set of vertices such that each vertex of graph G is either in S or has at least one neighbor in S, that is, v dominates every vertex in its closed neighborhood $N[v]$. A dominating set S is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set. The minimum cardinality of all dominating sets is called the domination number of G and is denoted by $\gamma(G)$, and the corresponding dominating set is called the γ-set of G.

Definition 2.3 ([4])**.** A dominating set S of V is called a *total dominating set* of G if the induced subgraph $\langle S\rangle$ has no isolated vertices, or equivalently, if every vertex in V is adjacent to at least one vertex in S or simply $N(S)=V(G)$. A total dominating set S is a *minimal total dominating set* if S has no proper subset that is a total dominating set. The minimum cardinality of a total dominating set of G is called *total domination number* of G and is denoted by $\gamma_t(G)$.

Definition 2.4 ([9])**.** For a graph $G=(V,E)$, a vertex $u\in V(G)$ vertex-edge dominates an edge $vw\in E(G)$ if:

(i) $u=v$ or $u=w$ (u is incident to ve); or

(ii) uv or uw is an edge in graph G (u is incident to an edge that is adjacent to vw).

A set $S\subseteq V(G)$ is a *vertex-edge dominating set* (or simply ve-dominating set) if for all edges $e\in E(G)$, there exists a vertex $v\in S$ that ve-dominates e. The minimum cardinality of a ve-dominating set of graph G is called the *vertex-edge domination number* (or simply ve-domination number), and is denoted by $\gamma_{ve}(G)$.

Definition 2.5 ([2])**.** A subset $S\subseteq V$ is a *total vertex-edge dominating set* (or simply total ve-dominating set) of G if S is a ve-dominating set and the subgraph induced by S has no isolated vertices, that is, every vertex of S has a neighbor in S. The *total vertex-edge domination number*, $\gamma^t_{ve}(G)$, of G is the minimum cardinality of a total ve-dominating set.

Theorem 2.6 ([2])**.** *For every nontrivial connected graph G, $\gamma_{ve}(G)\le\gamma^t_{ve}(G)\le\gamma_t(G)$.*

Proposition 2.7 ([2])**.** *For every nontrivial connected graph G, $\gamma^t_{ve}(G)\le 2\gamma_{ve}(G)$.*

Proposition 2.8 ([8])**.** *For any graph G of order n, $\gamma_{ve}(G)=1$ if an only if there exists a vertex $x\in V(G)$ such that every vertex of G is within distance two of x and if $Y=\{y\in V(G):d(x,y)=2\}$ then Y is an independent set of vertices.*

Definition 2.9 ([3])**.** For a connected graph G, the *distance* $d(u,v)$ between two vertices u and v is the minimum length of the $u-v$ paths of G. The *eccentricity* of a vertex v of a connected graph, denoted by $e(v)$, is the number $\max\limits_{u\in V(G)} d(u,v)$. The *radius*, denoted by $\operatorname{rad}(G)$, of G is the minimum eccentricity among the vertices of G, while the *diameter*, $\operatorname{diam}(G)$, of G is the maximum eccentricity.

Definition 2.10 ([3])**.** The *join* of two graphs G_1 and G_2, denoted by G_1+G_2, is the graph with vertex set $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and the edge set $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{uv:u\in V(G_1)$ and $v\in V(G_2)\}$.

Definition 2.11 ([6])**.** The *corona* of two graphs G and H, denoted by $G\circ H$, is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the ith vertex of G to every vertex in the ith copy of H.

Let $u \in V(G)$. Denote by H^u the copy of graph H connected to u. Hence, $G \circ H$ is composed of the subgraphs $H^u + \{u\}$ joint together by the edges of G. Also,

$$V(G \circ H) = \bigcup_{u \in V(G)} V(H^u + u).$$

Definition 2.12 ([7])**.** The *Cartesian Product* of two graphs G and H, denoted by $G \times H$, is the graph whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ precisely when either $u_1 = u_2$ and $v_1 v_2 \in E(H)$, or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

3. Main Results

3.1 Cycle Graphs

Before presenting the results, we first establish a basic property of total vertex-edge dominating sets that will be useful throughout this section.

Proposition 3.1. *Let G be a connected graph and let $S \subseteq V(G)$ be a total vertex-edge dominating set of G. Then, for every $v \in S$, $N(v) \cap S \neq \varnothing$.*

Proof. Let $S \subseteq V(G)$ be a total vertex-edge dominating set of G. Suppose, on the contrary, that there exists $v \in S$ such that $N(v) \cap S = \varnothing$. Then v is an isolated vertex in $\langle S \rangle$, a contradiction. □

Intuitively, Proposition 3.1 tells us that in a total vertex-edge dominating set, no vertex can 'stand alone'-every chosen vertex must be adjacent to at least one other in the set to ensure all vertices and edges remain dominated. This idea underpins our constructive approach in the following theorem.

Theorem 3.2. *Let C_n be a cycle graph of order $n \geq 3$. The total vertex-edge domination number of C_n is given by*

$$\gamma_{ve}^t(C_n) = \begin{cases} \frac{2n+3}{5}, & n \equiv 1 \mod 5 \\ 2\lceil \frac{n}{5} \rceil, & \textit{otherwise}. \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, v_3, \dots, v_n\}$ and$E(C_n) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_1v_n\}$. To construct a total vertex-edge dominating set $S \subseteq V(C_n)$, choose an arbitrary vertex v_i for some $i = 1, 2, 3, \dots, n$ to be the first element of S. Note that $N(v_i) = \{v_{i-1}, v_{i+1}\}$ for each $i = 1, 2, 3, \dots, n$, where i is taken modulo n. By Proposition 3.1, either $v_{i-1} \in S$ or $v_{i+1} \in S$. Without loss of generality, we choose v_{i+1} as the second element of S. Note that v_{i+1} ve-dominates edges $v_i v_{i+1}$, $v_{i+1} v_{i+2}$ and $v_{i+2} v_{i+3}$. Now, the edge $v_{i+3} v_{i+4}$ was not ve-dominated by $v_{i+1} \in S$. We can choose either vertex v_{i+4} or v_{i+5} which can ve-dominates edge $v_{i+3} v_{i+4}$. However, vertex v_{i+5} ve-dominates more edges compared to v_{i+4}. Thus, we let $v_{i+5} \in S$. Since $N(v_{i+5}) = \{v_{i+4}, v_{i+6}\}$, either $v_{i+4} \in S$ or $v_{i+6} \in S$ by Proposition 3.1. Choose $v_{i+6} \in S$. Then repeat the same argument. Now, consider the following cases:

Case 1. Suppose $n \equiv 1 \mod 5$. Observe that, edge $v_{i-2} v_{i-3}$ was not ve-dominated by $v_i \in S$. Since $N(v_i) = \{v_{i+1}, v_{i-1}\}$ and $v_{i+1} \in S$. However, v_{i+1} does not ve-dominates edge $v_{i-2} v_{i-3}$. So, choose $v_{i-1} \in S$. By construction, S is a minimum total vertex-edge dominating set of C_n and

$S = \bigcup_{j=1}^{\frac{n-1}{5}} \{v_{i+5(j-1)}, v_{i+1+5(j-1)}\} \cup \{v_{i-1}\}$. Thus, $\gamma_{ve}^t(C_n) = |S| = 2\left(\frac{n-1}{5}\right) + 1 = \frac{2n+3}{5}$.

Case 2. Suppose $n \not\equiv 1 \bmod 5$. Consider the following subcases:

Subcase 2.1. Let $n \equiv 0 \bmod 5$. Applying the same argument as in *Case* 1 until $v_{i-4} \in S$.

Subcase 2.2. Let $n \equiv 2 \bmod 5$. Applying the same argument as in *Case* 1 until $v_{i-1} \in S$.

Subcase 2.3. Let $n \equiv 3 \bmod 5$. Applying the same argument as in *Case* 1 until $v_{i-2} \in S$.

Subcase 2.4. Let $n \equiv 4 \bmod 5$. Applying the same argument as in *Case* 1 until $v_{i-3} \in S$.

By construction, S is a minimum total vertex-edge dominating set of C_n and

$$S = \bigcup_{j=1}^{\lceil \frac{n}{5} \rceil} \{v_{i+5(j-1)}, v_{i+1+5(j-1)}\}.$$

Hence, $\gamma_{ve}^t(C_n) = |S| = 2\left\lceil \frac{n}{5} \right\rceil$. Consequently,

$$\gamma_{ve}^t(C_n) = \begin{cases} \frac{2n+3}{5}, & n \equiv 1 \bmod 5, \\ 2\left\lceil \frac{n}{5} \right\rceil, & \text{otherwise.} \end{cases}$$

□

The pattern every five vertices reflects how each chosen pair v_i and v_{i+1} collectively covers nearby edges efficiently. The periodicity of five arises from the interplay between vertex coverage and edge adjacency in a cycle.

3.2 Graphs with Total Vertex-Edge Domination Number Equal to Two

Intuitively, graphs whose total vertex-edge domination number equals two exhibit strong local connectivity — two appropriately placed vertices suffice to dominate all vertices and edges. The next results formalize this observation.

Remark 3.3. Let G be a nontrivial connected graph.

(i) A vertex u ve-dominates an edge $e = vw$ in G if and only if $d(u,v) = 1$ and $d(u,w) \le 2$, or $d(u,v) \le 2$ and $d(u,w) = 1$.

(ii) $\gamma_{ve}^t(G) \ge 2$.

The result below follows directly from Proposition 2.8.

Corollary 3.4. *For any graph G of order n, $\gamma_{ve}(G) = 1$ if and only if* $\mathrm{rad}(G) \le 2$.

We now characterize the graphs whose total ve-domination number is equal to two.

Theorem 3.5. *For every nontrivial connected graph G, $\gamma_{ve}^t(G) = 2$ if and only if* $\mathrm{rad}(G) \le 2$.

Proof. Suppose that $\gamma_{ve}^t(G) = 2$. Let $S = \{x, y\}$ be a total ve-dominating set of G. Then for every adjacent vertices u and v in G, either x or y ve-dominates uv, say x. Then $d(x,u) \le 2$ and $d(x,v) = 1$, or $d(x,u) = 1$ and $d(x,v) \le 2$ by Remark 3.3(i). These imply that $e(x) \le 2$. Similarly, $e(y) \le 2$. Hence, $\mathrm{rad}(G) \le 2$.

Conversely, suppose that $\mathrm{rad}(G) \le 2$. By Corollary 3.4 and Proposition 2.7, $\gamma_{ve}^t(G) \le 2$. By Remark 3.3(ii), $\gamma_{ve}^t(G) = 2$. □

Corollary 3.6. *Let G be a bipartite graph. Then $\gamma^t_{ve}(G) = 2$.*

Corollary 3.7. (i) *For the complete graph K_n, $\gamma^t_{ve}(K_n) = 2$.*
(ii) *For the complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$, $\gamma^t_{ve}(K_{n_1,n_2,\ldots,n_r}) = 2$.*
(iii) *For the join of graphs G and H, $\gamma^t_{ve}(G+H) = 2$.*
(iv) *For the wheel graph $W_n = K_1 + C_n$, $\gamma^t_{ve}(W_n) = 2$.*
(v) *For the generalized wheel graph $W_{m,n} = \bar{K}_m + C_n$, $\gamma^t_{ve}(W_{m,n}) = 2$.*
(vi) *For the fan graph $F_n = K_1 + P_n$, $\gamma^t_{ve}(F_n) = 2$.*
(vii) *For the generalized fan graph $F_{m,n} = \bar{K}_m + P_n, \gamma^t_{ve}(F_{m,n}) = 2$.*
(viii) *For the Cartesian product of complete graphs K_n and K_2, $\gamma^t_{ve}(K_n \times K_2) = 2$.*

3.3 Total Vertex-Edge Domination Number of $G \circ H$

Intuitively, domination within each copy behaves independently, allowing a reduction of the global problem to local domination in each $v + H^v$ subgraph.

Theorem 3.8. *Let G be a connected graph of order m and let H be a connected graph of order $n \geq 2$. Then, $C \subseteq V(G \circ H)$ is a ve-dominating set in $G \circ H$ if and only if $V(v + H^v) \cap C$ is a ve-dominating set of $v + H^v$ for every $v \in V(G)$.*

Proof. Let C be a ve-dominating set in $G \circ H$ and let $v \in V(G)$. If $v \in C$, then $\{v\}$ is a ve-dominating set of $v + H^v$. It follows that $V(v + H^v) \cap C$ is a ve-dominating set of $v + H^v$. Suppose $v \notin C$ and let $e = uw \in E(v + H^v)$ with $u \neq v$ and $w \neq v$. Since C is a ve-dominating set of $G \circ H$, there exists $y \in C$ such that y ve-dominates e. This implies that $d(y,u) \leq 2$ and $d(y,w) = 1$, or $d(y,w) \leq 2$ and $d(y,u) = 1$ by Remark 3.3(i). This means that $y \in V(H^v) \cap C = V(v + H^v) \cap C$. This proves that $V(v + H^v) \cap C$ is a ve-dominating set of $v + H^v$.

For the converse, suppose that $V(v + H^v) \cap C$ is a ve-dominating set of $v + H^v$ for every $v \in V(G)$. Then, clearly, C is a ve-dominating set of $G \circ H$. □

Corollary 3.9. *Let G be a connected graph of order m and let H be a connected graph of order $n \geq 2$. Then, $\gamma_{ve}(G \circ H) = m$.*

Proof. Let $C = V(G)$. Then $V(v + H^v) \cap C = \{v\}$ is a ve-dominating set of $v + H^v$ for every $v \in V(G)$. By Theorem 3.8, C is a ve-dominating set of $G \circ H$. Hence, $\gamma_{ve}(G \circ H) \leq m$.

Next, let C^* be a minimum ve-dominating set of $G \circ H$. Then, by Theorem 3.8, $V(v + H^v) \cap C^*$ is a ve-dominating set of $v + H^v$, for every $v \in V(G)$. It follows that $\gamma_{ve}(G \circ H) = |C^*| \geq m$. Therefore, $\gamma_{ve}(G \circ H) = m$. □

Theorem 3.10. *Let G be a connected graph of order $m \geq 2$ and let H be a connected graph of order $n \geq 2$. Then $\gamma^t_{ve}(G \circ H) = m$.*

Proof. Let $C = V(G)$. Then, $V(v + H^v) \cap C = \{v\}$ is a ve-dominating set of $v + H^v$ for every $v \in V(G)$. By Theorem 3.8, C is a ve-dominating set of $G \circ H$. Since G is a connected graph, $\langle C \rangle$ has no isolated vertices. Hence C is a total ve-dominating set of $G \circ H$. This implies that, $\gamma^t_{ve}(G \circ H) \leq m$. By Theorems 3.8 and Theorem 2.6, $m = \gamma_{ve}(G \circ H) \leq \gamma^t_{ve}(G \circ H) \leq m$. Therefore, $\gamma^t_{ve}(G \circ H) = m$. □

The equality $\gamma_{ve}^t(G \circ H) = m$ shows that the corona preserves a simple linear relationship between the domination parameter and the base graph order, reflecting the independence of copies.

3.4 Upper Bound for the Total Vertex-Edge Domination number of $G \times K_n$

Finally, we examine the Cartesian product with a complete graph. Intuitively, each layer K_n^i behaves as a highly connected module, allowing us to bound the total domination number in terms of $|V(G)|$.

Let G be a connected graph of order m and let $V(G \times K_n) = \{(u_i, v_j) \mid i = 1,2,3,\dots,m$, and $j = 1,2,3,\dots,n\}$. For $i = 1,2,3,\dots,m$, denote by K_n^i the subgraph of $G \times K_n$ with vertex set $V(K_n^i) = \{(u_i, v_j) \mid j = 1,2,3,\dots,n\}$.

Theorem 3.11. *Let G be a connected graph of order m and K_n be a complete graph of order $n \geq 4$. Let $S \subseteq V(G \times K_n)$. If*

(i) *$V(K_n^i) \cap S$ is a ve-dominating set of K_n^i for all $i = 1,2,3,\dots,m$, and*

(ii) *if $(u_i, v_j) \in S$, then $N(u_i, v_j) \cap S \neq \emptyset$,*

then S is a total ve-dominating set in $G \times K_n$.

Proof. Let $S \subseteq V(G \times K_n)$. Suppose condition (i) holds. Then, $T = \bigcup_{i=1}^{m} (V(K_n^i) \cap S)$ is a ve-dominating set of $G \times K_n$. Since $T \subseteq S$, S is also a ve-dominating set of $G \times K_n$. Condition (ii) implies that $\langle S \rangle$ has no isolated vertices. Thus, S is a total ve-dominating set in $G \times K_n$. □

Corollary 3.12. *Let G be a connected graph of order m and let K_n be a complete graph of order $n \geq 4$. Then, $\gamma_{ve}^t(G \times K_n) \leq m$.*

Proof. For some fixed j, let $S = \{(u_i, v_j) \mid i = 1,2,3,\dots,m\}$. Then $V(K_n^i) \cap S = \{(u_i, v_j)\}$. Since $V(K_n^i) \cap S \subseteq V(K_n^i)$ and $\gamma_{ve}(K_n^i) = 1$, $V(K_n^i) \cap S$ is a ve-dominating set of K_n^i for all $i = 1,2,3,\dots,m$. Also, since G is connected and $\langle S \rangle \cong G$, $N(u_i, v_j) \cap S \neq \emptyset$ for every $i = 1,2,3,\dots,m$. By Theorem 3.11, S is a total ve-dominating of $G \times K_n$. Thus, $\gamma_{ve}^t(G \times K_n) \leq |S| = m$. □

4. Conclusions

This paper investigated the total vertex-edge domination number, denoted by $\gamma_{ve}^t(G)$, for various classes of graphs and fundamental graph operations. We established the exact values of $\gamma_{ve}^t(G)$ for the cycle graph C_n, characterizing it in terms of n modulo 5. Moreover, we characterized all connected graphs whose total vertex-edge domination number equals two, showing that $\gamma_{ve}^t(G) = 2$ if and only if the radius of G satisfies $\text{rad}(G) \leq 2$. This result provided immediate corollaries for several families of graphs, including complete, complete multipartite, wheel, generalized wheel, fan, and generalized fan graphs, as well as for joins and certain Cartesian products.

In addition, we derived the total vertex-edge domination number for the corona of two graphs, proving that $\gamma_{ve}^t(G \circ H) = |V(G)|$ whenever G and H are connected graphs with $|V(H)| \geq 2$. An upper bound was also established for the total vertex-edge domination number of the Cartesian product $G \times K_n$, showing that $\gamma_{ve}^t(G \times K_n) \leq |V(G)|$ for $n \geq 4$. These findings enrich the study of domination parameters by integrating vertex and edge domination under total domination constraints, offering a broader understanding of graph structures governed by mixed domination conditions.

Several promising directions may extend the present study:

- Determine the exact value or tight bounds of $\gamma_{ve}^t(G)$ for other graph products, such as the lexicographic product, strong product, and tensor product.
- Investigate the behavior of $\gamma_{ve}^t(G)$ under graph operations such as subdivision, duplication, complement, and line graph transformation.

Overall, this work provides foundational results that can serve as a basis for deeper exploration of vertex-edge domination and its applications in both theoretical and applied graph theory.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] C. Berge, *Theory of Graphs and its Applications*, Methuen & Company Limited, London, (1964).

[2] R. Boutrig and M. Chellali, Total vertex-edge domination, *International Journal of Computer Mathematics* **95**(9) (2018), 1820 – 1828, DOI: 10.1080/00207160.2017.1343469.

[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 3rd edition, Chapman & Hall/CRC, Boca Raton — London, x + 415 pages (1996).

[4] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, Total domination in graphs, *Networks* **10**(3) (1980), 211 – 219. DOI: 10.1002/net.3230100304.

[5] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7**(3) (1977), 247 – 261, DOI: 10.1002/net.3230070305.

[6] F. Harary, *Graph Theory*, Addison-Wesley Publishing Compnay, Reading — Michigan, ix + 274 pages (1969), URL: https://users.metu.edu.tr/aldoks/341/Book%201%20(Harary).pdf.

[7] B. Hartnell and R. Douglas, On dominating the cartesian product of a graph and K_2, *Discussiones Mathematicae Graph Theory* **24**(3) (2004), 389 – 402, DOI: 10.7151/dmgt.1238.

[8] J. Lewis, *Vertex-Edge and Edge-Vertex Parameters in Graphs*, Ph.D. Dissertation, Clemson University, Clemson, South Carolina, (2007), URL: https://tigerprints.clemson.edu/all_dissertations/103.

[9] J. Lewis, S. T. Hedetniemi, T. W. Haynes and G. H. Fricke, Vertex-edge domination, *Utilitas Mathematica* **81** (2010), 193 – 213.

[10] O. Ore, *Theory of Graphs*, American Mathematical Society Colloquium Publications, Vol. 38, American Mathematical Society, Providence, RI, (1962).

[11] K. W. Peters, *Theoretical and Algorithmic Results on Domination and Connectivity*, Ph.D. Dissertation, Clemson University, Clemson, South Carolina, (1986), URL: https://open.clemson.edu/arv_dissertations/1412/.

[12] J. M. Tarr, *Domination in Graphs*, Master of Arts thesis, Department of Mathematics and Statistics, University of South Florida, Tampa, Florida, (2010).