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Research Article

Cordial Labeling of the Line Graph of Bistar

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Abstract. A binary vertex labeling f of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called a cordial graph. In this paper, the necessary and sufficient conditions for the line graph of bistar $B_{n,p}$ to be cordial when $p = n + 4m$ and $p = n + 4m + 2$ where $m \in \mathbb{N} \cup \{0\}$ are discussed.

Keywords. Line graph, Bistar, Cordial labeling

Mathematics Subject Classification (2020). 05C78

1. Introduction

Graph theory in discrete mathematics studies mathematical structures, specifically graphs, used to model pairwise relations between objects. A graph consists of a set of vertices (nodes) and edges (lines) connecting them, widely applied to analyze networks, paths, and relationships in computer science, biology, and optimization. A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling was first introduced in the mid 1960s. Cahit [3] introduced cordial labeling of graphs and derived various results on cordial graphs. Cordial labeling is explore by many researchers, e.g., Bosmia and Kanani [1], Cahit [3], Kuo *et al.* [7], and Vaidya and Shah [9]. The name line graph was first used by Harary and Norman [6] in 1960. Throughout this work every graph is consider as finite, undirected, simple graph. For any undefined notations and terminology, Gross and Yellen [5] is followed while for number theory, Burton [2] is followed. In this paper, cordial labeling of line graph of bistar is discussed.

2. Preliminaries

Definition 2.1 ([5]). If the vertices or edges or both are assigned numbers subject to certain condition(s) then it is known as *graph labeling*.

Definition 2.2 ([5]). A mapping $f : V(G) \to \{0,1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the label of the vertex v of G under f.

Notations 2.1 ([3]). If for an edge $e = uv$, the induced edge labeling $f^* : E(G) \to \{0,1\}$ is given by

$$f^*(e = uv) = |f(u) - f(v)|.$$

Then

$v_f(i)$ = number of vertices of G having label i under f,

$e_f(i)$ = number of edges of G having label i under f^*.

Definition 2.3 ([5]). A binary vertex labeling f of a graph G is called a *cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph which admits cordial labeling is called a *cordial graph*.

The concept of cordial labeling was introduced by Cahit [3] as a weaker version of graceful and harmonious labeling in 1987. In the same paper, Cahit has proved the following results:

- All trees are cordial.
- The complete graph K_n is cordial if and only if $n \leq 3$.
- The complete bipartite graph $K_{m,n}$ is cordial.

Vaidya and Shah [9] have proved the following results:

- The shadow graph $D_2(B_{n,n})$ of bistar $B_{n,n}$ is a cordial graph.
- The splitting graph $S'(B_{n,n})$ of bistar $B_{n,n}$ is a cordial graph.
- The degree splitting graph $DS(B_{n,n})$ of bistar $B_{n,n}$ is a cordial graph.

Definition 2.4 ([5]). *Bistar* $B_{n,p}$ is the graph obtained by joining the center (apex) vertices of $K_{1,n}$ and $K_{1,p}$ by an edge.

Definition 2.5 ([5]). The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G.

Illustration 2.2 ([1]). Bistar $B_{7,11}$ and its line graph $L(B_{7,11})$ are shown in Figure 1 and Figure 2, respectively.

Some remarkable observations about line graph $L(B_{n,p})$ of bistar $B_{n,p}$ are:

- $L(B_{n,p})$ is isomorphic to $(K_n \cup K_p) + K_1$.

- Here, the vertex e_0 in $L(B_{n,p})$ is the apex vertex with degree $d(e_0) = n + p$.
- If the apex vertex e_0 in $L(B_{n,p})$ is removed, one gets two complete graphs of order n and p as two components of a vertex deleted subgraph.
- Bistar graphs $B_{n,p}$ and $B_{p,n}$ are isomorphic. So, $L(B_{n,p})$ and $L(B_{p,n})$ are isomorphic graphs.

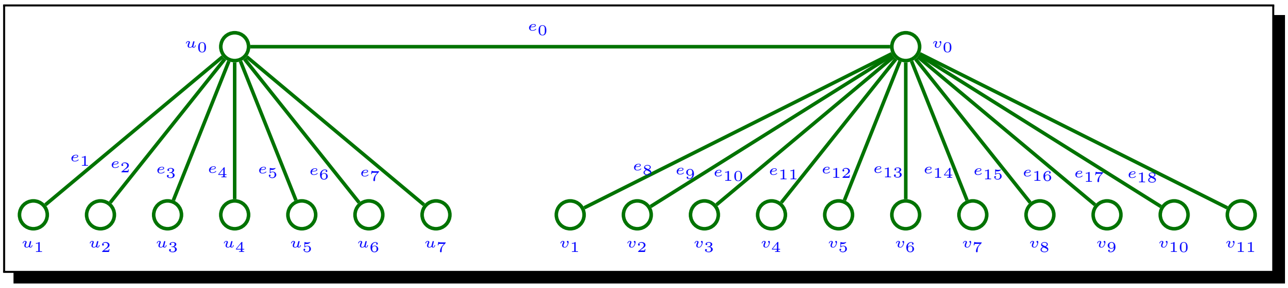

Figure 1. Bistar $B_{7,11}$

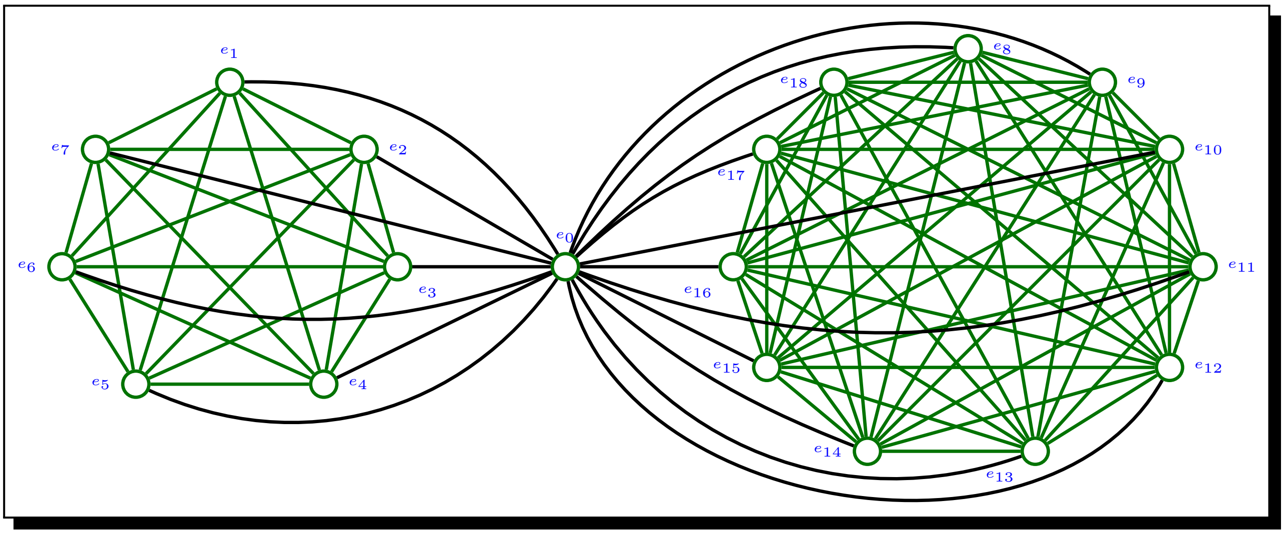

Figure 2. $L(B_{7,11})$ Line graph of bistar $B_{7,11}$

For any integers n and p, there are two possibilities $n = p$ or $n \neq p$.

Bosmia and Kanani [1] have proved that $L(B_{n,n})$ is cordial if and only if $n = t^2$ or $n = (t+1)^2 - 1$ for $t \in \mathbb{N}$.

If $n \neq p$, there are two possibilities $n < p$ or $n > p$. If $n < p$ then for some $m \in \mathbb{N} \cup \{0\}$, $p = n + 4m$, $p = n + 4m + 1$, $p = n + 4m + 2$ and $p = n + 4m + 3$.

As $L(B_{n,p})$ and $L(B_{p,n})$ are isomorphic graphs, without loss of generality cordial labeling of $L(B_{n,p})$ only for $n < p$ is to be discussed. To discuss cordial labeling of $L(B_{n,p})$ there are four cases to be considered $p = n + 4m$, $p = n + 4m + 1$, $p = n + 4m + 2$ and $p = n + 4m + 3$, where $m \in \mathbb{N} \cup \{0\}$.

In the next section, cordial labeling of $L(B_{n,p})$ for $p = n + 4m$ and $p = n + 4m + 2$, where $m \in \mathbb{N} \cup \{0\}$ is discussed. Cordial labeling of $L(B_{n,p})$ for $p = n + 4m + 1$ and $p = n + 4m + 3$, where $m \in \mathbb{N} \cup \{0\}$ are under process in a research project.

3. Cordial Labeling of Line Graph of Bistar

Theorem 3.1. *$L(B_{n,n+4m})$ is cordial if and only if either $n+2m$ or $n+2m+1$ is a perfect square number, where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.*

Proof. Let $B_{n,n+4m}$ be the bistar with vertex set $\{u_0, v_0, u_i, v_r : 1 \le i \le n, 1 \le r \le n+4m\}$, where u_0, v_0 are apex vertices and u_i, v_r are pendant vertices for all $1 \le i \le n$, $1 \le r \le n+4m$. Let $\{e_0 = u_0v_0, e_i = u_0u_i, e_{n+r} = v_0v_r : 1 \le i \le n,\ 1 \le r \le n+4m\}$ be the edge set of $B_{n,n+4m}$. Then $V(L(B_{n,n+4m})) = \{e_0, e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n+4m}\}$. Hence, $|V(L(B_{n,n+4m}))| = 2n+4m+1$ and $|E(L(B_{n,n+4m}))| = \frac{n(n-1)+(n+4m)(n+4m-1)}{2} + 2n + 4m = n^2 + n + 4mn + 8m^2 + 2m$ is an even integer.

To define vertex labeling $f : V(L(B_{n,n+4m})) \to \{0,1\}$ following two cases are considered:

Case 1: When $f(e_0) = 1$.
In order to satisfy vertex condition $|v_f(0) - v_f(1)| \le 1$ it must be either $v_f(0) = n+2m$ and $v_f(1) = n+2m+1$ or $v_f(0) = n+2m+1$ and $v_f(1) = n+2m$.

Subcase 1.1: When $v_f(0) = n+2m$ and $v_f(1) = n+2m+1$.
Define $f : V(L(B_{n,n+4m})) \to \{0,1\}$ as follows:

$$f(e_0) = 1.$$

Now, $n+2m$ vertices must be labeled with 0 and remaining $n+2m$ with 1. To consider all different possibility of labeling the variable j is defined as

j = Number of vertices having label 1 from the vertices $e_1, e_2, \dots, e_n$.

It is noted that $0 \le j \le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i) = \begin{cases} 1, & 1 \le i \le j, \\ 0, & j+1 \le i \le n, \\ 0, & n+1 \le i \le n+2m+j, \\ 1, & n+2m+j+1 \le i \le 2n+4m. \end{cases}$$

In view of above define labeling pattern

$$e_f(1) = j(n-j) + (n+2m-j)(2m+j) + n + 2m = 2nj - 2j^2 + 2mn + 4m^2 + n + 2m.$$

$L(B_{n,n+4m})$ is cordial if and only if

- $|e_f(0) - e_f(1)| \le 1$;
- $e_f(0) = e_f(1) = \frac{n^2+n+4mn+8m^2+2m}{2}$ because $|E(L(B_{n,n}))| = n^2 + n + 4mn + 8m^2 + 2m$;
- $2nj - 2j^2 + 2mn + 4m^2 + n + 2m = \frac{n^2+n+4mn+8m^2+2m}{2}$;
- $n^2 + (-4j-1)n + 4j^2 - 2m = 0$.

Now, discriminant of the equation $n^2 + (-4j-1)n + 4j^2 - 2m = 0$ is $8(j+m)+1$.
$L(B_{n,n+4m})$ is cordial if and only if $8(j+m)+1$ is a perfect square number.
It is known that the integer $j+m$ is a triangular number if and only if $8(j+m)+1$ is a perfect square and a number is triangular if and only if it is of the form $\frac{k(k+1)}{2}$, for some $k \in \mathbb{N}$.

$L(B_{n,n+4m})$ is cordial if and only if

- $j+m=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$;
- $n=k^2-2m$ or $(k+1)^2-2m$ for $k\in\mathbb{N}$;
- $n+2m=t^2$ for $t\in\mathbb{N}$;
- $n+2m$ is a perfect square number.

$L(B_{n,n+4m})$ is cordial if and only if $n+2m$ is a perfect square number.

Subcase 1.2: When $v_f(0)=n+2m+1$ and $v_f(1)=n+2m$.
Define $f: V(L(B_{n,n+4m}))\to\{0,1\}$ as follows:

$$f(e_0)=1.$$

Now, $n+2m+1$ vertices must be labeled with 0 and remaining $n+2m-1$ with 1. To consider all different possibility of labeling the variable j is defined as

$$j=\text{Number of vertices having label 1 from the vertices } e_1,e_2,\dots,e_n.$$

Observe that $0\le j\le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i)=\begin{cases}1, & 1\le i\le j,\\ 0, & j+1\le i\le n,\\ 0, & n+1\le i\le n+2m+j+1,\\ 1, & n+2m+j+2\le i\le 2n+4m.\end{cases}$$

In view of above define labeling pattern

$$\begin{aligned}e_f(1)&=j(n-j)+(n+2m-j-1)(2m+j+1)+n+2m+1\\ &=2nj-2j^2+2mn+4m^2+2n+2m-2j.\end{aligned}$$

$L(B_{n,n+4m})$ is cordial if and only if

- $|e_f(0)-e_f(1)|\le 1$;
- $e_f(0)=e_f(1)=\frac{n^2+n+4mn+8m^2+2m}{2}$ because $|E(L(B_{n,n}))|=n^2+n+4mn+8m^2+2m$;
- $2nj-2j^2+2mn+4m^2+2n+2m-2j=\frac{n^2+n+4mn+8m^2+2m}{2}$;
- $n^2+(-4j-3)n+4j^2+4j-2m=0$.

Now, discriminant of the equation

$$n^2+(-4j-3)n+4j^2+4j-2m=0 \text{ is } 8(j+m+1)+1.$$

$L(B_{n,n+4m})$ is cordial if and only if $8(j+m+1)+1$ is a perfect square number.
$L(B_{n,n+4m})$ is cordial if and only if

- $j+m+1=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$;
- $n=k^2-2m-1$ or $(k+1)^2-2m-1$ for $k\in\mathbb{N}$;
- $n+2m+1=t^2$ for $t\in\mathbb{N}$;
- $n+2m+1$ is a perfect square number.

$L(B_{n,n+4m})$ is cordial if and only if $n+2m+1$ is a perfect square number.

Case 2: When $f(e_0) = 0$.
In order to satisfy vertex condition $|v_f(0) - v_f(1)| \le 1$ it must be either $v_f(0) = n + 2m$ and $v_f(1) = n + 2m + 1$ or $v_f(0) = n + 2m + 1$ and $v_f(1) = n + 2m$.

Subcase 2.1: When $v_f(0) = n + 2m$ and $v_f(1) = n + 2m + 1$.
Define $f : V(L(B_{n,n+4m})) \to \{0,1\}$ as follows:

$$f(e_0) = 0.$$

Now, $n + 2m - 1$ vertices must be labeled with 0 and remaining $n + 2m + 1$ with 1. To consider all different possibility of labeling the variable j is defined as

j = Number of vertices having label 0 from the vertices $e_1, e_2, \dots, e_n$.

Clearly, $0 \le j \le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i) = \begin{cases} 0, & 1 \le i \le j, \\ 1, & j+1 \le i \le n, \\ 1, & n+1 \le i \le n+2m+j+1, \\ 0, & n+2m+j+2 \le i \le 2n+4m. \end{cases}$$

In view of above define labeling pattern

$$\begin{aligned} e_f(1) &= j(n-j) + (n+2m-j-1)(2m+j+1) + n + 2m + 1 \\ &= 2nj - 2j^2 + 2mn + 4m^2 + 2n + 2m - 2j. \end{aligned}$$

$L(B_{n,n+4m})$ is cordial if and only if

- $|e_f(0) - e_f(1)| \le 1$;
- $e_f(0) = e_f(1) = \frac{n^2+n+4mn+8m^2+2m}{2}$ because $|E(L(B_{n,n}))| = n^2 + n + 4mn + 8m^2 + 2m$;
- $2nj - 2j^2 + 2mn + 4m^2 + 2n + 2m - 2j = \frac{n^2+n+4mn+8m^2+2m}{2}$;
- $n^2 + (-4j-3)n + 4j^2 + 4j - 2m = 0$.

Now, discriminant of the equation $n^2 + (-4j-3)n + 4j^2 + 4j - 2m = 0$ is $8(j+m+1)+1$.
$L(B_{n,n+4m})$ is cordial if and only if $8(j+m+1)+1$ is a perfect square number.
$L(B_{n,n+4m})$ is cordial if and only if

- $j + m + 1 = \frac{k(k+1)}{2}$ for some $k \in \mathbb{N}$;
- $n = k^2 - 2m - 1$ or $(k+1)^2 - 2m - 1$ for $k \in \mathbb{N}$;
- $n + 2m + 1 = t^2$ for $t \in \mathbb{N}$;
- $n + 2m + 1$ is a perfect square number.

$L(B_{n,n+4m})$ is cordial if and only if $n + 2m + 1$ is a perfect square number.

Subcase 2.2: When $v_f(0) = n + 2m + 1$ and $v_f(1) = n + 2m$.
Define $f : V(L(B_{n,n+4m})) \to \{0,1\}$ as follows:

$$f(e_0) = 0.$$

Now, $n+2m$ vertices must be labeled with 0 and remaining $n+2m$ with 1. To consider all different possibility of labeling the variable j is defined as

$$j = \text{Number of vertices having label 0 from the vertices } e_1, e_2, \dots, e_n,$$

where $0 \le j \le n$.

Without loss of generality $f(e_i)$ is defined as

$$f(e_i) = \begin{cases} 0, & 1 \le i \le j, \\ 1, & j+1 \le i \le n, \\ 1, & n+1 \le i \le n+2m+j, \\ 0, & n+2m+j+1 \le i \le 2n+4m. \end{cases}$$

In view of above define labeling pattern

$$\begin{aligned} e_f(1) &= j(n-j)+(n+2m-j)(2m+j)+n+2m \\ &= 2nj-2j^2+2mn+4m^2+n+2m. \end{aligned}$$

$L(B_{n,n+4m})$ is cordial if and only if

- $|e_f(0)-e_f(1)| \le 1$;
- $e_f(0)=e_f(1)=\frac{n^2+n+4mn+8m^2+2m}{2}$ because $|E(L(B_{n,n}))| = n^2+n+4mn+8m^2+2m$;
- $2nj-2j^2+2mn+4m^2+n+2m = \frac{n^2+n+4mn+8m^2+2m}{2}$;
- $n^2+(-4j-1)n+4j^2-2m=0$.

Now, discriminant of the equation $n^2+(-4j-1)n+4j^2-2m=0$ is $8(j+m)+1$.

$L(B_{n,n+4m})$ is cordial if and only if $8(j+m)+1$ is a perfect square number.

$L(B_{n,n+4m})$ is cordial if and only if

- $j+m = \frac{k(k+1)}{2}$ for some $k \in \mathbb{N}$;
- $n = k^2-2m$ or $(k+1)^2-2m$ for $k \in \mathbb{N}$;
- $n+2m = t^2$ for $t \in \mathbb{N}$;
- $n+2m$ is a perfect square number.

$L(B_{n,n+4m})$ is cordial if and only if $n+2m$ is a perfect square number.

Hence, $L(B_{n,n+4m})$ is cordial if and only if either $n+2m$ or $n+2m+1$ is a perfect square number. □

Illustration 3.2. Cordial labeling of $L(B_{7,11})$ is shown in Figure 3.

Theorem 3.3. *$L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m$ or $n+2m+1$ or $n+2m+2$ or $n+2m+3$ is a perfect square number, where $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.*

Proof. Let $B_{n,n+4m+2}$ be the bistar with vertex set $\{u_0, v_0, u_i, v_r : 1 \le i \le n, 1 \le r \le n+4m+2\}$, where u_0, v_0 are apex vertices and u_i, v_r are pendant vertices for all $1 \le i \le n$, $1 \le r \le n+4m+2$. Let $\{e_0 = u_0v_0, e_i = u_0u_i, e_{n+r} = v_0v_r : 1 \le i \le n,\ 1 \le r \le n+4m+2\}$ be the edge set of $B_{n,n+4m+2}$. Then $V(L(B_{n,n+4m+2})) = \{e_0, e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n+4m+2}\}$. Hence, $|V(L(B_{n,n+4m+2}))| = 2n+4m+3$ and $|E(L(B_{n,n+4m+2}))| = \frac{n(n-1)+(n+4m+2)(n+4m+1)}{2}+2n+4m+2 = n^2+3n+4mn+8m^2+10m+3$ is an odd integer.

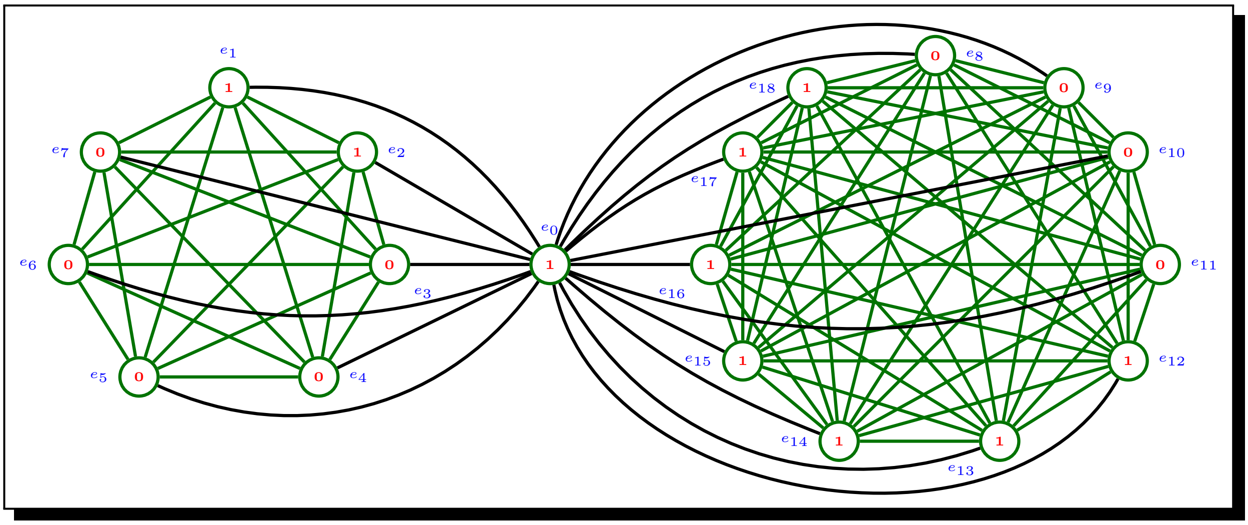

Figure 3. Cordial labeling of $L(B_{7,11})$

To define vertex labeling $f:V(L(B_{n,n+4m+2}))\to\{0,1\}$ following two cases are considered:

Case 1: When $f(e_0)=1$.
In order to satisfy vertex condition $|v_f(0)-v_f(1)|\le 1$ it must be either $v_f(0)=n+2m+1$ and $v_f(1)=n+2m+2$ or $v_f(0)=n+2m+2$ and $v_f(1)=n+2m+1$.

Subcase 1.1: When $v_f(0)=n+2m+1$ and $v_f(1)=n+2m+2$.
Define $f:V(L(B_{n,n+4m+2}))\to\{0,1\}$ as follows:

$$f(e_0)=1.$$

Now, $n+2m+1$ vertices must be labeled with 0 and remaining $n+2m+1$ with 1. To consider all different possibility of labeling the variable j is defined as

$$j=\text{Number of vertices having label 1 from the vertices } e_1,e_2,\dots,e_n,$$

where $0\le j\le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i)=\begin{cases}1, & 1\le i\le j,\\ 0, & j+1\le i\le n,\\ 0, & n+1\le i\le n+2m+j+1,\\ 1, & n+2m+j+2\le i\le 2n+4m+2.\end{cases}$$

In view of above define labeling pattern we have

$$\begin{aligned}e_f(1)&=j(n-j)+(n+2m-j+1)(2m+j+1)+n+2m+1\\ &=2nj-2j^2+2mn+4m^2+2n+6m+2.\end{aligned}$$

$L(B_{n,n+4m+2})$ is cordial if and only if

- $|e_f(0)-e_f(1)|\le 1$,
- $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$
  because $|E(L(B_{n,n}))|=n^2+3n+4mn+8m^2+10m+3$;

- $2nj-2j^2+2mn+4m^2+2n+6m+2=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $2nj-2j^2+2mn+4m^2+2n+6m+2=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$;
- $n^2+(-4j-1)n+4j^2-2m=0$ or $n^2+(-4j-1)n+4j^2-2m-2=0$.

Now, discriminant of the equations $n^2+(-4j-1)n+4j^2-2m=0$ and $n^2+(-4j-1)n+4j^2-2m-2=0$ are $8(j+m)+1$ and $8(j+m+1)+1$, respectively.
$L(B_{n,n+4m+2})$ is cordial if and only if $8(j+m)+1$ or $8(j+m+1)+1$ is a perfect square number.
$L(B_{n,n+4m+2})$ is cordial if and only if

- $j+m=\frac{k(k+1)}{2}$ or $j+m+1=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$;
- $n=k^2-2m$ or $n=(k+1)^2-2m$ or $n=k^2-2m-2$ or $n=(k+1)^2-2m-2$ for $k\in\mathbb{N}$;
- $n+2m=t^2$ or $n+2m+2=t^2$ for $t\in\mathbb{N}$;
- $n+2m$ or $n+2m+2$ is a perfect square number.

$L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m$ or $n+2m+2$ is a perfect square number.

Subcase 1.2: When $v_f(0)=n+2m+2$ and $v_f(1)=n+2m+1$.
Define $f:V(L(B_{n,n+4m+2}))\to\{0,1\}$ as follows:

$$f(e_0)=1.$$

Now, $n+2m+2$ vertices must be labeled with 0 and remaining $n+2m$ with 1. To consider all different possibility of labeling the variable j is defined as

$$j=\text{Number of vertices having label 1 from the vertices } e_1,e_2,\dots,e_n.$$

Observe that $0\le j\le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i)=\begin{cases}1, & 1\le i\le j,\\ 0, & j+1\le i\le n,\\ 0, & n+1\le i\le n+2m+j+2,\\ 1, & n+2m+j+3\le i\le 2n+4m+2.\end{cases}$$

In view of above define labeling pattern, we have

$$\begin{aligned}e_f(1)&=j(n-j)+(n+2m-j)(2m+j+2)+n+2m+2\\&=2nj-2j^2+2mn+4m^2+3n+6m-2j+2.\end{aligned}$$

$L(B_{n,n+4m+2})$ is cordial if and only if

- $|e_f(0)-e_f(1)|\le 1$;
- $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$
  because $|E(L(B_{n,n}))|=n^2+3n+4mn+8m^2+10m+3$;
- $2nj-2j^2+2mn+4m^2+3n+6m-2j+2=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $2nj-2j^2+2mn+4m^2+3n+6m-2j+2=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$;
- $n^2+(-4j-3)n+4j^2+4j-2m=0$ or $n^2+(-4j-3)n+4j^2+4j-2m-2=0$.

Now, discriminant of the equations $n^2+(-4j-3)n+4j^2+4j-2m=0$ and $n^2+(-4j-3)n+4j^2+4j-2m-2=0$ are $8(j+m+1)+1$ and $8(j+m+2)+1$, respectively.
$L(B_{n,n+4m+2})$ is cordial if and only if $8(j+m+1)+1$ or $8(j+m+2)+1$ is a perfect square number.
$L(B_{n,n+4m+2})$ is cordial if and only if

- $j+m+1=\frac{k(k+1)}{2}$ or $j+m+2=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$;
- $n=k^2-2m-1$ or $n=(k+1)^2-2m-1$ or $n=k^2-2m-3$ or $n=(k+1)^2-2m-3$ for $k\in\mathbb{N}$;
- $n+2m+1=t^2$ or $n+2m+3=t^2$ for $t\in\mathbb{N}$;
- $n+2m+1$ or $n+2m+3$ is a perfect square number.

$L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m+1$ or $n+2m+3$ is a perfect square number.

Case 2: When $f(e_0)=0$.
In order to satisfy vertex condition $|v_f(0)-v_f(1)|\le 1$ it must be either $v_f(0)=n+2m+1$ and $v_f(1)=n+2m+2$ or $v_f(0)=n+2m+2$ and $v_f(1)=n+2m+1$.

Subcase 2.1: When $v_f(0)=n+2m+1$ and $v_f(1)=n+2m+2$.
Define $f:V(L(B_{n,n+4m+2}))\to\{0,1\}$ as follows:

$f(e_0)=0.$

Now, $n+2m$ vertices must be labeled with 0 and remaining $n+2m+2$ with 1. To consider all different possibility of labeling the variable j is defined as

$j=$ Number of vertices having label 0 from the vertices $e_1,e_2,\dots,e_n$.

It is noted that $0\le j\le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i)=\begin{cases}0, & 1\le i\le j,\\ 1, & j+1\le i\le n,\\ 1, & n+1\le i\le n+2m+j+2,\\ 0, & n+2m+j+3\le i\le 2n+4m+2.\end{cases}$$

In view of above define labeling pattern

$$\begin{aligned}e_f(1)&=j(n-j)+(n+2m-j)(2m+j+2)+n+2m+2\\&=2nj-2j^2+2mn+4m^2+3n+6m-2j+2.\end{aligned}$$

$L(B_{n,n+4m+2})$ is cordial if and only if

- $|e_f(0)-e_f(1)|\le 1$;
- $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$
  because $|E(L(B_{n,n}))|=n^2+3n+4mn+8m^2+10m+3$;
- $2nj-2j^2+2mn+4m^2+3n+6m-2j+2=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $2nj-2j^2+2mn+4m^2+3n+6m-2j+2=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$;
- $n^2+(-4j-3)n+4j^2+4j-2m=0$ or $n^2+(-4j-3)n+4j^2+4j-2m-2=0$.

Now, discriminant of the equations $n^2+(-4j-3)n+4j^2+4j-2m=0$ and $n^2+(-4j-3)n+4j^2+4j-2m-2=0$ are $8(j+m+1)+1$ and $8(j+m+2)+1$, respectively.
$L(B_{n,n+4m+2})$ is cordial if and only if $8(j+m+1)+1$ or $8(j+m+2)+1$ is a perfect square number.
$L(B_{n,n+4m+2})$ is cordial if and only if

- $j+m+1=\frac{k(k+1)}{2}$ or $j+m+2=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$;
- $n=k^2-2m-1$ or $n=(k+1)^2-2m-1$ or $n=k^2-2m-3$ or $n=(k+1)^2-2m-3$ for $k\in\mathbb{N}$;
- $n+2m+1=t^2$ or $n+2m+3=t^2$ for $t\in\mathbb{N}$;
- $n+2m+1$ or $n+2m+3$ is a perfect square number.

$L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m+1$ or $n+2m+3$ is a perfect square number.

Subcase 2.2: When $v_f(0)=n+2m+2$ and $v_f(1)=n+2m+1$.
Define $f:V(L(B_{n,n+4m+2}))\to\{0,1\}$ as follows:

$$f(e_0)=0.$$

Now, $n+2m+1$ vertices must be labeled with 0 and remaining $n+2m+1$ with 1. To consider all different possibility of labeling the variable j is defined as

$j=$ Number of vertices having label 0 from the vertices $e_1,e_2,\dots,e_n$.

Clearly $0\le j\le n$.
Without loss of generality $f(e_i)$ is defined as

$$f(e_i)=\begin{cases}0, & 1\le i\le j,\\ 1, & j+1\le i\le n,\\ 1, & n+1\le i\le n+2m+j+1,\\ 0, & n+2m+j+2\le i\le 2n+4m+2.\end{cases}$$

In view of above define labeling pattern

$$\begin{aligned}e_f(1)&=j(n-j)+(n+2m-j+1)(2m+j+1)+n+2m+1\\ &=2nj-2j^2+2mn+4m^2+2n+6m+2.\end{aligned}$$

$L(B_{n,n+4m+2})$ is cordial if and only if

- $|e_f(0)-e_f(1)|\le 1$,
- $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $e_f(0)=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$, $e_f(1)=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$
  because $|E(L(B_{n,n}))|=n^2+3n+4mn+8m^2+10m+3$,
- $2nj-2j^2+2mn+4m^2+2n+6m+2=\frac{n^2+3n+4mn+8m^2+10m+4}{2}$
  or $2nj-2j^2+2mn+4m^2+2n+6m+2=\frac{n^2+3n+4mn+8m^2+10m+2}{2}$,
- $n^2+(-4j-1)n+4j^2-2m=0$ or $n^2+(-4j-1)n+4j^2-2m-2=0$.

Now, discriminant of the equations $n^2+(-4j-1)n+4j^2-2m=0$ and $n^2+(-4j-1)n+4j^2-2m-2=0$ are $8(j+m)+1$ and $8(j+m+1)+1$, respectively.
$L(B_{n,n+4m+2})$ is cordial if and only if $8(j+m)+1$ or $8(j+m+1)+1$ is a perfect square number.

$L(B_{n,n+4m+2})$ is cordial if and only if

- $j+m=\frac{k(k+1)}{2}$ or $j+m+1=\frac{k(k+1)}{2}$ for some $k\in\mathbb{N}$,
- $n=k^2-2m$ or $n=(k+1)^2-2m$
  or $n=k^2-2m-2$ or $n=(k+1)^2-2m-2$ for $k\in\mathbb{N}$,
- $n+2m=t^2$ or $n+2m+2=t^2$ for $t\in\mathbb{N}$,
- $n+2m$ or $n+2m+2$ is a perfect square number.

$L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m$ or $n+2m+2$ is a perfect square number. Hence, $L(B_{n,n+4m+2})$ is cordial if and only if either $n+2m$ or $n+2m+1$ or $n+2m+2$ or $n+2m+3$ is a perfect square number. □

Illustration 3.4. Cordial labeling of $L(B_{6,12})$ is shown in Figure 4.

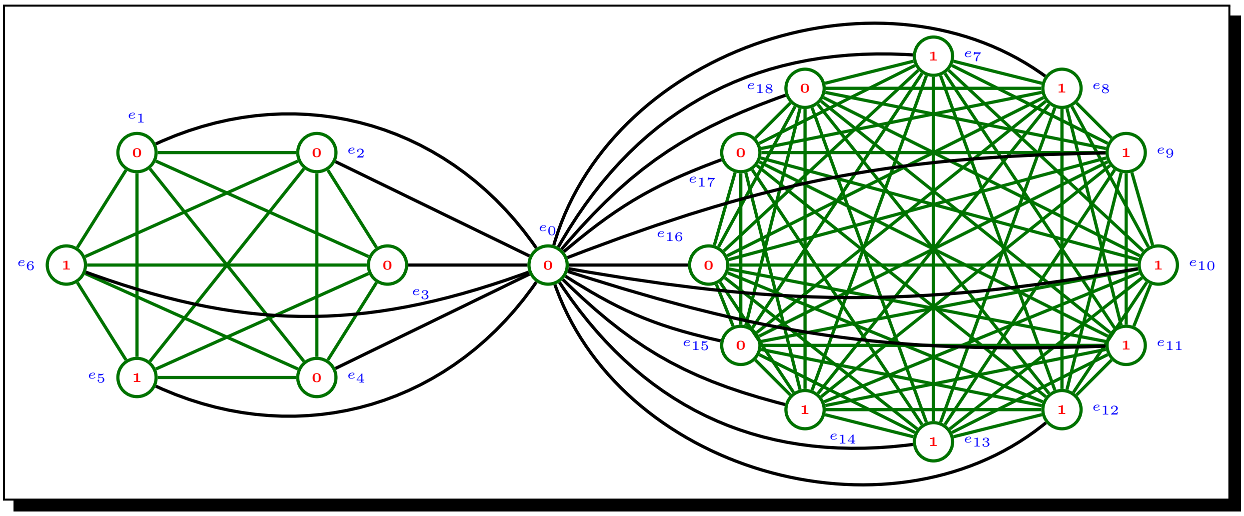

Figure 4. Cordial labeling of $L(B_{6,12})$

4. Concluding Remarks

The necessary and sufficient conditions for the line graph of bistar $B_{n,p}$ to be cordial when $p=n+4m$ and $p=n+4m+2$ for $m\in\mathbb{N}\cup\{0\}$ are discussed in this research paper. The necessary and sufficient conditions for the line graph of bistar $B_{n,p}$ to be cordial when $p=n+4m+1$ and $p=n+4m+3$ for $m\in\mathbb{N}\cup\{0\}$ are currently being investigated in an ongoing research project.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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