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Research Article

On Some Extended Results in Functional $q(t)$-Calculus: Theoretical and Numerical Applications

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Abstract. This paper provides further applications of a functional approach to quantum calculus, towards functional $q(t)$-integral, functional $q(t)$-Taylor's formula difference equations with variable coefficients together with functional quantum congruence theory.

Keywords. q-Calculus, q-Identities, Hypergeometric functions, Quantum calculus

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1. Introduction

The q-*Calculus* or quantum calculus, provides an alternative framework to infinitesimal calculus that avoids the classical limit concept. Instead, it uses a q-derivative defined through finite differences, leading to its characterization as *calculus without limits*. In this work, we further extend the results of Jafari and Manjarekar [8] and demonstrate their usefulness through applications in various fields.

The q-Calculus provides a mathematical framework instrumental in connecting mathematics and physics (Ernst [7]), with important applications including quantum groups, quantum field theory, and general relativity (Podleś and Müller [12]). This calculus, defined using q-difference operators instead of classical limits, was formalized through the work of key mathematicians such as Jackson as Jackson Integral (Kac and Cheung [9]), Ernst [4, 7], and Kac and Cheung [9], and references cited therein.

The paper explores with further definitions, properties and applications in various fields of the newly defined functional $q(t)$-Calculus (Jafari and Manjarekar [8]) which specially includes $q(t)$-integral, extension of Taylors formula and extension of differential equations [9] to the variable order $q(t)$ together with some solved problems.

The paper is organized into the following sections: in Section 2, we have mainly deals with basic definitions and properties. In Section 3, we have presented some theorems based on the new definitions and properties, while in Section 4, we have given some numerical applications. The paper concludes in Section 5 with a summary of the main findings.

2. Preliminaries

I. q-Derivative

Given an arbitrary function $f(x)$ then its q-derivative is denoted by $D_q f(x)$ and is given by Kac and Cheung [9],

$$D_q f(x) = \frac{f(qx)-f(x)}{qx-x} = \frac{f(qx)-f(x)}{(q-1)x}. \tag{2.1}$$

II. Functional Analogue of $q(t)$-Derivative

Given an arbitrary function $f(x)$ then its $q(t)$-derivative is defined as $D_{q(t)}f(x)$ and is given by Jafari and Manjarekar [8],

$$D_{q(t)}f(x) = \frac{f(q(t)x)-f(x)}{q(t)x-x} = \frac{f(q(t)x)-f(x)}{(q(t)-1)x}, \tag{2.2}$$

where $Q = \{q(t) \mid q(t) \to 1 \text{ as } t \to 0,\ t \in \mathbb{R}\}$.

The above set is non-empty, i.e., $Q \neq \emptyset$ and are well-defined. For example, if $q(t) = 1+t$, $t \in \mathbb{R}$ then $q(t) \in Q$.

III. $q(t)$-Analogue of n

Given any positive integer 'n' then its $q(t)$-analogue is denoted by $[n]_{q(t)}$ and is defined as ([8]),

$$[n]_{q(t)} = \frac{[q(t)]^n - 1}{q(t)-1} = 1 + q(t) + [q(t)]^2 + \cdots + [q(t)]^{n-1}. \tag{2.3}$$

IV. $q(t)$-Analogue of $n!$

Given any positive integer 'n' then its $q(t)$-analogue is denoted by $[n!]_{q(t)}$ and is defined as ([8]),

$$[n!]_{q(t)} = [n]_{q(t)} \times [n-1]_{q(t)} \times \ldots \times [3]_{q(t)} \times [2]_{q(t)} \times [1]_{q(t)}.$$

Thus, depending upon the choice of $q(t)$. The value of $[n!]_{q(t)}$ and $[n]_{q(t)}$ varies. In other words, if $q_1(t) \to 1$ as $t \to 0$ much faster than $q_2(t) \to 1$ as $t \to 0$. Then $[n!]_{q_1(t)}$ and $[n]_{q_1(t)}$ tends much faster than $[n!]_{q_2(t)}$ and $[n]_{q_2(t)}$ to the value of $n!$ and n as $t \to 0$.

V. Functional Quantum Congruence

If $b \equiv a \pmod{m}$ then $[b]_{q(t)} \equiv [a]_{q(t)} \pmod{m}$ ([8]) if there exist some $q(t) \in Q$ such that $|[b-a]_{q(t)} - km| \to 0$, where b, a, k and m are positive integers.

VI. $q(t)$-Shifting Operator

The $q(t)$-shifting operator (Sparavigna [14]) defined as

$$\Phi^a_{q(t)}(m) = q(t)m + (1-q(t))a, \tag{2.4}$$

where

$$D_{\Phi^a_{q(t)}(m)} f(x) = \frac{f(\Phi^a_{q(t)}(m)x) - f(x)}{(\Phi^a_{q(t)}(m)-1)x}. \tag{2.5}$$

From the definition (2.4) and (2.5) one can easily see that

(i) $D_{\Phi^0_{q(t)}(1)} f(x) = D_{mq(t)} f(x)$;

(ii) $D_{\Phi^a_{q(t)}(a)} f(x) = D_a f(x)$, for $a > 0$;

(iii) $D_{\Phi^m_{q(t)}(m)} f(x) = D_m f(x)$, for $m > 0$;

(iv) $\Phi^a_{q(t)}(m) \to {}_a\Phi_q(m)$ as $t \to 0$.

VII. Linear Operator

The definition (2.4) satisfies the property that

$$\Phi^a_{q_1(t)+q_2(t)}(m) = \{q_1(t)+q_2(t)\}m + (1-\{q_1(t)+q_2(t)\})a$$

$$\Longrightarrow \quad \Phi^a_{q_1(t)+q_2(t)}(m) = \{q_1(t)\}m + \{q_2(t)\}m + (1-q_1(t))a + (1-q_2(t))a$$

$$\Longrightarrow \quad \Phi^a_{q_1(t)+q_2(t)}(m) = \Phi^a_{q_1(t)}(m) + \Phi^a_{q_2(t)}(m) \tag{2.6}$$

which means, the operator $\Phi^a_{q(t)}(m)$ is a linear operator.

3. Theoretical Applications

Definition 3.1 (Functional $q(t)$-Integral)**.** For a continuous function $f(x)$, the $q(t)$-integral of $f(x)$ using q-integral (Kac and Cheung [9]) is defined as:

$$\int_0^x f(s)d_{q(t)}s = (1-q(t))x \sum_{k=0}^{\infty} q(t)^k f(q(t)^k x),$$

where $q(t) \in Q$.

Definition 3.2 (Fundamental Theorem of $q(t)$-Calculus)**.** If $F(x)$ is a $q(t)$-antiderivative of $f(x)$ under the $q(t)$-derivative $D_{q(t)}$, which means $D_{q(t)}F(x) = f(x)$, and if the $\lim_{s\to 0} F(s)$ exists, then for $0 < a < b$,

$$\int_a^b D_{q(t)}F(s)d_{q(t)}s = F(b) - F(a),$$

where $0 < q(t) < 1$.

Proof. We know that $q(t)$-derivative $F(x)$ is defined as ([8]):

$$D_{q(t)}F(x) = \frac{F(q(t)x) - F(x)}{(q(t)-1)x}, \quad \text{for } x \neq 0.$$

We assume $a > 0$. The $q(t)$-integral of a function $g(s)$ from a to b is defined as:

$$\int_a^b g(s)d_{q(t)}s = (1-q(t)) \sum_{k=0}^{\infty} q(t)^k b g(q(t)^k b) - (1-q(t)) \sum_{k=0}^{\infty} q(t)^k a g(q(t)^k a),$$

where $0 < q(t) < 1$ for the series to converge, which is the Jackson q-integral generalized for $q(t)$ which depends on a parameter t.

Step I. Given that $F(x)$ is an antiderivative of $f(x)$ under $D_{q(t)}$, we have $D_{q(t)}F(s) = f(s)$.
We have to find the value of the integral:

$$I = \int_a^b D_{q(t)}F(s)d_{q(t)}s.$$

Substituting $g(s) = D_{q(t)}F(s)$ into the $q(t)$-integral definition:

$$\begin{aligned} I &= \int_a^b D_{q(t)}F(s)d_{q(t)}s \\ &= (1-q(t))\sum_{k=0}^{\infty} q(t)^k b D_{q(t)}F(q(t)^k b) - (1-q(t))\sum_{k=0}^{\infty} q(t)^k a D_{q(t)}F(q(t)^k a). \end{aligned}$$

Step II. We now evaluate each term separately.
Consider the expression for $D_{q(t)}F(q(t)^k b)$,

$$\begin{aligned} D_{q(t)}F(q(t)^k b) &= \frac{F(q(t)q(t)^k b) - F(q(t)^k b)}{(q(t)-1)q(t)^k b} \\ &= \frac{F(q(t)^{k+1} b) - F(q(t)^k b)}{(q(t)-1)q(t)^k b}. \end{aligned}$$

Substitute this into the first summation:

$$(1-q(t))\sum_{k=0}^{\infty} q(t)^k b D_{q(t)}F(q(t)^k b) = (1-q(t))\sum_{k=0}^{\infty} q(t)^k b \frac{F(q(t)^{k+1} b) - F(q(t)^k b)}{(q(t)-1)q(t)^k b}. \tag{3.1}$$

Simplify the expression inside the summation:

$$(1-q(t))\frac{F(q(t)^{k+1} b) - F(q(t)^k b)}{(q(t)-1)}\frac{q(t)^k b}{q(t)^k b} = (1-q(t))\frac{F(q(t)^{k+1} b) - F(q(t)^k b)}{(q(t)-1)}.$$

Put $q(t)-1 = -(1-q(t))$, we get

$$(1-q(t))\frac{F(q(t)^{k+1} b) - F(q(t)^k b)}{-(1-q(t))} = -(F(q(t)^{k+1} b) - F(q(t)^k b)).$$

Step III. Summation in equation (3.1) simplifies to

$$\begin{aligned} (1-q(t))\sum_{k=0}^{\infty} q(t)^k b D_{q(t)}F(q(t)^k b) &= \sum_{k=0}^{\infty} -(F(q(t)^{k+1} b) - F(q(t)^k b)) \\ &= -\sum_{k=0}^{\infty} (F(q(t)^{k+1} b) - F(q(t)^k b)). \end{aligned}$$

Similarly, for the second summation involving a:

$$\begin{aligned} (1-q(t))\sum_{k=0}^{\infty} q(t)^k a D_{q(t)}F(q(t)^k a) &= \sum_{k=0}^{\infty} -(F(q(t)^{k+1} a) - F(q(t)^k a)) \\ &= -\sum_{k=0}^{\infty} (F(q(t)^{k+1} a) - F(q(t)^k a)). \end{aligned}$$

Now, substitute these back into the integral expression:

$$\int_a^b D_{q(t)}F(s)d_{q(t)}s = \left[-\sum_{k=0}^{\infty} (F(q(t)^{k+1} b) - F(q(t)^k b))\right] - \left[-\sum_{k=0}^{\infty} (F(q(t)^{k+1} a) - F(q(t)^k a))\right]$$

$$= -\sum_{k=0}^{\infty}(F(q(t)^{k+1}b) - F(q(t)^k b)) + \sum_{k=0}^{\infty}(F(q(t)^{k+1}a) - F(q(t)^k a)).$$

Step IV. For the part involving b:

$$-\sum_{k=0}^{\infty}(F(q(t)^{k+1}b) - F(q(t)^k b)) = -\sum_{k=0}^{\infty}F(q(t)^{k+1}b) + \sum_{k=0}^{\infty}F(q(t)^k b).$$

Re-indexing the first sum by letting $j = k+1$. When $k = 0$, $j = 1$, thus

$$\begin{aligned}-\sum_{j=1}^{\infty}F(q(t)^j b) + \sum_{k=0}^{\infty}F(q(t)^k b) &= -\sum_{j=1}^{\infty}F(q(t)^j b) + \left(F(q(t)^0 b) + \sum_{j=1}^{\infty}F(q(t)^j b)\right)\\ &= -\sum_{j=1}^{\infty}F(q(t)^j b) + F(b) + \sum_{j=1}^{\infty}F(q(t)^j b).\end{aligned}$$

The sums $-\sum_{j=1}^{\infty}F(q(t)^j b)$ and $\sum_{j=1}^{\infty}F(q(t)^j b)$ cancel gives

$$F(b) - \lim_{k\to\infty}F(q(t)^k b).$$

Step V. The limit arises because the difference $\sum_{k=0}^{\infty}F(q(t)^k b) - \sum_{j=1}^{\infty}F(q(t)^j b)$ implicitly includes the behavior as $k \to \infty$.

As, $0 < q(t) < 1$ and b is finite it gives

$$\lim_{k\to\infty} q(t)^k b = 0.$$

Now, $\lim_{s\to 0}F(s) = L$ exists

$$\Longrightarrow \quad \lim_{k\to\infty}F(q(t)^k b) = L.$$

Thus, the b-part simplifies to

$$F(b) - L.$$

Similarly, for the part involving a:

$$\begin{aligned}\sum_{k=0}^{\infty}(F(q(t)^{k+1}a) - F(q(t)^k a)) &= \sum_{k=0}^{\infty}F(q(t)^{k+1}a) - \sum_{k=0}^{\infty}F(q(t)^k a)\\ &= \sum_{j=1}^{\infty}F(q(t)^j a) - \left(F(q(t)^0 a) + \sum_{j=1}^{\infty}F(q(t)^j a)\right)\\ &= \sum_{j=1}^{\infty}F(q(t)^j a) - F(a) - \sum_{j=1}^{\infty}F(q(t)^j a).\end{aligned}$$

This simplifies to

$$-F(a) + \lim_{k\to\infty}F(q(t)^k a).$$

Using the same limit assumption, $\lim_{k\to\infty}F(q(t)^k a) = L$, so, the a-part is $-F(a) + L$.

Combining both parts

$$\begin{aligned}\int_a^b D_{q(t)}F(s)d_{q(t)}s &= (F(b) - L) + (-F(a) + L)\\ &= F(b) - F(a).\end{aligned}$$

This completes the proof of the fundamental theorem of $q(t)$-Calculus. □

Theorem 3.1 ($q(t)$-Taylor's formula)**.** *For an analytic function $f(x)$,*

$$f(x)=\sum_{n=0}^{\infty}\frac{(x-a)^n_{q(t)}}{[n]_{q(t)}!}D^n_{q(t)}f(a),$$

where $(x-a)^n_{q(t)}=\prod_{k=0}^{n-1}(x-\Phi^a_{q(t)}(x))$ and $\Phi^a_{q(t)}$ is the shift operator (definition (2.6)) and

- $(x-a)^n_{q(t)}=\prod_{k=0}^{n-1}(x-\Phi^a_{q(t)}(x))$,
- *$\Phi^a_{q(t)}$ is the shift operator defined by $\Phi^a_{q(t)}(x)=a+q(t)(x-a)$,*
- *$[n]_{q(t)}=\frac{1-q(t)^n}{1-q(t)}$ is the $q(t)$-number* ([8]),
- *$[n]_{q(t)}!=[1]_{q(t)}[2]_{q(t)}\ldots[n]_{q(t)}$ is the $q(t)$-factorial* ([8]), *with* $[0]_{q(t)}!=1$.

Proof. We will prove this by induction on n. We use properties of the $q(t)$-derivative and shift operator.

Step 1. $n=0$
For $n=0$, the partial sum consists of the 0th term,

$$\sum_{k=0}^{0}\frac{(x-a)^k_{q(t)}}{[k]_{q(t)}!}D^k_{q(t)}f(a)=\frac{(x-a)^0_{q(t)}}{[0]_{q(t)}!}D^0_{q(t)}f(a).$$

Now, $(x-a)^0_{q(t)}=1$, $[0]_{q(t)}!=1$ and $D^0_{q(t)}f(a)=f(a)$.
Thus,

$$f(x)=f(a)+R_0(x),$$

where $R_0(x)=f(x)-f(a)$.
This is consistent with the theorem for $n=0$.

Step 2. Assumption
Assume that the result hold for some $n\ge 0$,

$$f(x)=\sum_{k=0}^{n}\frac{(x-a)^k_{q(t)}}{[k]_{q(t)}!}D^k_{q(t)}f(a)+R_n(x) \tag{3.2}$$

where $R_n(x)$ is the remainder term.
Apply the $q(t)$-derivative $D_{q(t)}$ to eq. (3.2). Using linearity property defined in (2.6),

$$D_{q(t)}f(x)=\sum_{k=0}^{n}\frac{D^k_{q(t)}f(a)}{[k]_{q(t)}!}D_{q(t)}((x-a)^k_{q(t)})+D_{q(t)}R_n(x). \tag{3.3}$$

Step 3. Compute $D_{q(t)}((x-a)^k_{q(t)})$
The $q(t)$-power basis satisfies

$$D_{q(t)}((x-a)^k_{q(t)})=[k]_{q(t)}(x-a)^{k-1}_{q(t)},\quad \text{for } k\ge 1.$$

For $k=0$, $(x-a)^0_{q(t)}=1$ (constant), so $D_{q(t)}((x-a)^0_{q(t)})=0$. Thus, eq. (3.3) simplifies to

$$D_{q(t)}f(x)=\sum_{k=1}^{n}\frac{D^k_{q(t)}f(a)}{[k]_{q(t)}!}[k]_{q(t)}(x-a)^{k-1}_{q(t)}+D_{q(t)}R_n(x). \tag{3.4}$$

Since $\frac{[k]_{q(t)}}{[k]_{q(t)}!} = \frac{1}{[k-1]_{q(t)}!}$ (because $[k]_{q(t)}! = [k]_{q(t)}[k-1]_{q(t)}!$), we rewrite eq. (3.4) as

$$D_{q(t)}f(x) = \sum_{k=1}^{n} \frac{(x-a)_{q(t)}^{k-1}}{[k-1]_{q(t)}!} D_{q(t)}^{k} f(a) + D_{q(t)} R_n(x). \tag{3.5}$$

Shifting the index $j = k-1$ (so $k = j+1$),

$$D_{q(t)}f(x) = \sum_{j=0}^{n-1} \frac{(x-a)_{q(t)}^{j}}{[j]_{q(t)}!} D_{q(t)}^{j+1} f(a) + D_{q(t)} R_n(x). \tag{3.6}$$

Step 4. Apply Inductive Hypothesis to $D_{q(t)}f(x)$

Define, $g(x) = D_{q(t)}f(x)$. Since $f(x)$ is analytic $\Longrightarrow g(x)$ is analytic and using inductive hypothesis to $g(x)$ up to order $n-1$,

$$g(x) = \sum_{j=0}^{n-1} \frac{(x-a)_{q(t)}^{j}}{[j]_{q(t)}!} D_{q(t)}^{j} g(a) + \tilde{R}_{n-1}(x), \tag{3.7}$$

where $\tilde{R}_{n-1}(x)$ is the remainder for $g(x)$.

Because $g(a) = D_{q(t)}f(a)$ and $D_{q(t)}^{j} g(a) = D_{q(t)}^{j+1} f(a)$, eq. (3.7) becomes

$$D_{q(t)}f(x) = \sum_{j=0}^{n-1} \frac{(x-a)_{q(t)}^{j}}{[j]_{q(t)}!} D_{q(t)}^{j+1} f(a) + \tilde{R}_{n-1}(x). \tag{3.8}$$

Compare eq. (3.6) and eq. (3.8),

$$\sum_{j=0}^{n-1} \frac{(x-a)_{q(t)}^{j}}{[j]_{q(t)}!} D_{q(t)}^{j+1} f(a) + D_{q(t)} R_n(x) = \sum_{j=0}^{n-1} \frac{(x-a)_{q(t)}^{j}}{[j]_{q(t)}!} D_{q(t)}^{j+1} f(a) + \tilde{R}_{n-1}(x).$$

Thus,

$$D_{q(t)} R_n(x) = \tilde{R}_{n-1}(x). \tag{3.9}$$

Step 5. Express $R_{n(x)}$ Using the Fundamental Theorem

$$R_n(x) = \int_a^x D_{q(t)} R_n(s) d_{q(t)} s + C,$$

where C is a constant. From eq. (3.9),

$$R_n(x) = \int_a^x \tilde{R}_{n-1}(s) d_{q(t)} s + C. \tag{3.10}$$

Evaluate at $x = a$:

$$R_n(a) = \int_a^a \tilde{R}_{n-1}(s) d_{q(t)} s + C = 0 + C.$$

From eq. (3.2), at $x = a$,

$$f(a) = \sum_{k=0}^{n} \frac{(a-a)_{q(t)}^{k}}{[k]_{q(t)}!} D_{q(t)}^{k} f(a) + R_n(a) = f(a) + R_n(a),$$

so $R_n(a) = 0$. Thus, $C = 0$, and

$$R_n(x) = \int_a^x \tilde{R}_{n-1}(s) d_{q(t)} s. \tag{3.11}$$

Step 6. Express $\tilde{R}_{n-1}(s)$ and Integrate
By the inductive hypothesis for $g(x) = D_{q(t)}f(x)$,

$$\tilde{R}_{n-1}(s) = g(s) - \sum_{j=0}^{n-1} \frac{(s-a)^j_{q(t)}}{[j]_{q(t)}!} D^{j+1}_{q(t)} f(a) = \frac{(s-a)^n_{q(t)}}{[n]_{q(t)}!} D^n_{q(t)} g(a) + \tilde{\tilde{R}}_n(s),$$

where $\tilde{\tilde{R}}_n(s)$ is the next remainder for $g(s)$.
Because

$$D^n_{q(t)} g(a) = D^{n+1}_{q(t)} f(a) \Longrightarrow \tilde{R}_{n-1}(s) = \frac{(s-a)^n_{q(t)}}{[n]_{q(t)}!} D^{n+1}_{q(t)} f(a) + \tilde{\tilde{R}}_n(s). \tag{3.12}$$

Substitute eq. (3.12) into eq. (3.11),

$$R_n(x) = \int_a^x \left(\frac{(s-a)^n_{q(t)}}{[n]_{q(t)}!} D^{n+1}_{q(t)} f(a) + \tilde{\tilde{R}}_n(s) \right) d_{q(t)} s. \tag{3.13}$$

Split the integral:

$$R_n(x) = \frac{D^{n+1}_{q(t)} f(a)}{[n]_{q(t)}!} \int_a^x (s-a)^n_{q(t)} d_{q(t)} s + \int_a^x \tilde{\tilde{R}}_n(s) d_{q(t)} s. \tag{3.14}$$

By the Fundamental Theorem of $q(t)$-Calculus,

$$\int_a^x (s-a)^n_{q(t)} d_{q(t)} s = \frac{(x-a)^{n+1}_{q(t)}}{[n+1]_{q(t)}}, \tag{3.15}$$

Because $D_{q(t)}((x-a)^{n+1}_{q(t)}) = [n+1]_{q(t)} (x-a)^n_{q(t)}$. Thus,

$$R_n(x) = \frac{D^{n+1}_{q(t)} f(a)}{[n]_{q(t)}!} \frac{(x-a)^{n+1}_{q(t)}}{[n+1]_{q(t)}} + \int_a^x \tilde{\tilde{R}}_n(s) d_{q(t)} s. \tag{3.16}$$

Simplify using $[n]_{q(t)}! [n+1]_{q(t)} = [n+1]_{q(t)}!$, we get

$$R_n(x) = \frac{(x-a)^{n+1}_{q(t)}}{[n+1]_{q(t)}!} D^{n+1}_{q(t)} f(a) + R_{n+1}(x), \tag{3.17}$$

where $R_{n+1}(x) = \int_a^x \tilde{\tilde{R}}_n(s) d_{q(t)} s$.
Substitute eq. (3.17) into eq. (3.2):

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k_{q(t)}}{[k]_{q(t)}!} D^k_{q(t)} f(a) + \frac{(x-a)^{n+1}_{q(t)}}{[n+1]_{q(t)}!} D^{n+1}_{q(t)} f(a) + R_{n+1}(x).$$

This is the expansion up to order $n+1$. Since f is analytic, the remainder $R_{n+1}(x) \to 0$ as $n \to \infty$ for x in a neighborhood of a. Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n_{q(t)}}{[n]_{q(t)}!} D^n_{q(t)} f(a).$$

Step 8. Limit as $t \to 0$
As $t \to 0$, $q(t) \to 1$:

$$(x-a)^n_{q(t)} = \prod_{k=0}^{n-1} (1 - q(t)^k)(x-a) \quad \text{and} \quad \frac{\prod_{k=0}^{n-1} (1 - q(t)^k)}{[n]_{q(t)}!} \to \frac{(x-a)^n}{n!}.$$

Since $[n]_{q(t)} \to n$ and $\prod_{k=0}^{n-1}(1-q(t)^k) \sim \text{const}(1-q(t))^n$, but the ratio converges to $\frac{(x-a)^n}{n!}$. The $q(t)$-derivative $D_{q(t)}f(x) \to f'(x)$. Thus, the series converges to the classical Taylor's series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Theorem 3.2 ($q(t)$-Analogue Ratio Test)**.** *If* $\sum_{n=0}^{\infty} c_n$ *is a series of real valued numbers and if* $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| < 1$ *with* $|q(t)| < 1$ *then the series converges.*

Proof. Step 1. Asymptotic Behavior of $q(t)$-Numbers
As $n \to \infty$, since $|q(t)| < 1$, we have $(q(t))^n \to 0$ and $(q(t))^{n+1} \to 0$. Thus,

$$[n]_{q(t)} = \frac{1-(q(t))^n}{1-q(t)} \sim \frac{1}{1-q(t)}, \quad [n+1]_{q(t)} = \frac{1-(q(t))^{n+1}}{1-q(t)} \sim \frac{1}{1-q(t)}.$$

It gives us,

$$\frac{[n]_{q(t)}}{[n+1]_{q(t)}} = \frac{1-(q(t))^n}{1-(q(t))^{n+1}} \to \frac{1-0}{1-0} = 1 \text{ as } n \to \infty.$$

This convergence is uniform in n for $|q(t)| < 1$.

Step 2. Ratio Test
Define the sequence, $d_n = c_n[n]_{q(t)}!$, where $[n]_{q(t)}! = [1]_{q(t)}[2]_{q(t)}\dots[n]_{q(t)}$ is the $q(t)$-factorial [8]. Consider the ratio:

$$\left|\frac{d_{n+1}}{d_n}\right| = \left|\frac{c_{n+1}}{c_n}\frac{[n+1]_{q(t)}!}{[n]_{q(t)}!}\right| = \left|\frac{c_{n+1}}{c_n}[n+1]_{q(t)}\right|.$$

Now,

$$[n+1]_{q(t)} = \frac{[n+1]_{q(t)}}{[n]_{q(t)}}[n]_{q(t)},$$

$$\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| = \left|\frac{c_{n+1}}{c_n}[n+1]_{q(t)}\right| \frac{1}{|[n+1]_{q(t)}|^2}.$$

Because $\frac{[n]_{q(t)}}{[n+1]_{q(t)}} \to 1$, we get

$$\limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}[n+1]_{q(t)}\right| = \limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| |[n+1]_{q(t)}|^2.$$

Instead, directly observe

$$\left|\frac{d_{n+1}}{d_n}\right| = \left|\frac{c_{n+1}}{c_n}[n+1]_{q(t)}\right| = \left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| |[n+1]_{q(t)}|^2 \text{ as } n \to \infty.$$

As $n \to \infty$, $|[n+1]_{q(t)}|^2 \to \left|\frac{1}{1-q(t)}\right|^2$.
Let $L = \limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| < 1$.
By the limit superior property, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right| < L + \epsilon.$$

Define,

$$\epsilon = \frac{1-L}{2} > 0$$

$$\implies L+\epsilon = \frac{L+1}{2} < 1$$

$$\implies \left|\frac{d_{n+1}}{d_n}\right| < \left(\frac{L+1}{2}\right) |[n+1]_{q(t)}|^2$$

for $n \geq N$.

Because $|[n+1]_{q(t)}|^2$ converges to a constant, it is bounded.

Define, $M = \sup_{n\geq N} |[n+1]_{q(t)}|^2 < \infty$.

Then

$$\left|\frac{d_{n+1}}{d_n}\right| < \frac{(L+1)M}{2}, \quad \text{for } n \geq N.$$

As $\frac{(L+1)M}{2}$ is a finite constant, this does not directly give convergence. Instead, we use the original limit condition.

Step 3. Apply Classical Ratio Test

Now we observe that:

$$\limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\frac{[n+1]_{q(t)}}{[n]_{q(t)}}\right|.$$

Because

$$\frac{[n+1]_{q(t)}}{[n]_{q(t)}} = \frac{1-q(t)^{n+1}}{1-q(t)^n} \xrightarrow{n\to\infty} 1,$$

and the limit superior of a product is bounded by the product of limit superiors. It gives us,

$$\limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \leq \underbrace{\limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\frac{[n]_{q(t)}}{[n+1]_{q(t)}}\right|}_{L} \underbrace{\limsup_{n\to\infty}\left|\frac{[n+1]_{q(t)}}{[n]_{q(t)}}\right|}_{1} = L < 1.$$

By the classical ratio test, $\limsup_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| < 1$ implies absolute convergence of $\sum_{n=0}^{\infty} c_n$. □

4. Numerical Applications

Example 4.1. Solve the equation $D_{q(t)}y(x) - p(x)y(x) = 0$ with $|q(t)| < 1$, where $\exp_{q(t)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{q(t)}!}$ and $D_{q(t)}$ is the $q(t)$-derivative ([8]).

Solution. *Step* 1. $q(t)$-derivative

The $q(t)$-derivative of $y(x)$ is defined for $x \neq 0$ as:

$$D_{q(t)}y(x) = \frac{y(q(t)x) - y(x)}{(q(t)-1)x}.$$

Substitute this into the equation $D_{q(t)}y(x) - p(x)y(x) = 0$ with $|q(t)| < 1$. It gives us

$$\frac{y(q(t)x) - y(x)}{(q(t)-1)x} - p(x)y(x) = 0.$$

Step 2. Solve for $y(q(t)x)$

Rearranging the equation:

$$y(q(t)x)-y(x)=p(x)y(x)(q(t)-1)x.$$

To solve for $y(q(t)x)$:

$$y(q(t)x)=y(x)[1+(q(t)-1)xp(x)].$$

Let $a(x)=1+(q(t)-1)xp(x)\Longrightarrow y(q(t)x)=y(x)a(x)$.

Step 3. Iterate the Functional Equation

Apply the relation iteratively to express $y(x)$ at scaled points:

- At x: $y(q(t)x)=y(x)a(x)$.
- At $q(t)x$: $y(q(t)^2x)=y(q(t)x)a(q(t)x)=y(x)a(x)a(q(t)x)$.
- At $q(t)^2x$: $y(q(t)^3x)=y(q(t)^2x)a(q(t)^2x)=y(x)a(x)a(q(t)x)a(q(t)^2x)$.

For n iterations:

$$y(q(t)^n x)=y(x)\prod_{k=0}^{n-1}a(q(t)^k x),$$

where $y(x)=\frac{y(q(t)^n x)}{\prod\limits_{k=0}^{n-1}a(q(t)^k x)}$.

Step 4. Taking the Limit as $n\to\infty$

Assume $|q(t)|<1$ (so $(q(t))^n x\to 0$ as $n\to\infty$) and that y is continuous at 0 with $\lim\limits_{s\to 0}y(s)=y(0)$. Then

$$\lim_{n\to\infty}y(q(t)^n x)=y(0).$$

Thus,

$$y(x)=\frac{y(0)}{\prod\limits_{k=0}^{\infty}a(q(t)^k x)}.$$

Substitute $a(s)=1+(q(t)-1)sp(s)$,

$$y(x)=y(0)\prod_{k=0}^{\infty}\frac{1}{1+(q(t)-1)(q(t)^k x)p(q(t)^k x)}.$$

Step 5. Write the general solution

The general solution is

$$y(x)=C\prod_{k=0}^{\infty}\frac{1}{1+(q(t)-1)q(t)^k xp(q(t)^k x)},$$

where $C=y(0)$ is an arbitrary constant.

Example 4.2. For $q(t)=e^{-t}$, $[n]_{q(t)}=\frac{e^{-nt}-1}{e^{-t}-1}$. Show that the series $\sum\frac{x^n}{[n]_{q(t)!}}$ converges absolutely, for all x.

Solution. *Step* 1. $[n]_{q(t)}>0$,

$$[n]_{q(t)}=\frac{e^{-nt}-1}{e^{-t}-1}=\frac{-(1-e^{-nt})}{e^{-t}-1}.$$

This is because,

- $e^{-t}-1<0$ since $e^{-t}<1$, for $t>0$.
- $1-e^{-nt}>0$, for $t>0$.
- This mean both numerator and denominator are negative.

Hence, $[n]_{q(t)}>0$.
Also, as $n\to\infty$, $e^{-nt}\to 0$, so

$$[n]_{q(t)}\to \frac{-1}{e^{-t}-1}=\frac{1}{1-e^{-t}}=C_t>0.$$

So $[n]_{q(t)}\to C_t$ as $n\to\infty$.

Step 2. Asymptotic behaviour of $[n]_{q(t)!}$
Because $[n]_{q(t)}\to C_t$ as $n\to\infty$, this suggests that $[n]_{q(t)}!\sim C_t^n$ as $n\to\infty$.
This approximation helps us estimate the general term in the series

$$\frac{x^n}{[n]_{q(t)}!}\sim\frac{x^n}{C_t^n}=\left(\frac{x}{C_t}\right)^n.$$

Step 3. Absolute Convergence
Let $a_n=\frac{x^n}{[n]_{q(t)}!}$.
Compute:

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}}{[n+1]_{q(t)}!}\frac{[n]_{q(t)}!}{x^n}\right|=|x|\frac{1}{[n+1]_{q(t)}}.$$

As $n\to\infty$, we have $[n+1]_{q(t)}\to C_t>0$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{|x|}{C_t}.$$

Now using the Ratio Test: $[n]_{q(t)}\to C_t$, so the denominator is eventually close to C_t^n, and hence

$$\frac{x^n}{[n]_{q(t)}!}\sim\left(\frac{x}{C_t}\right)^n.$$

Thus,

$$\sum_{n=0}^{\infty}\frac{x^n}{[n]_{q(t)}!}\quad \text{converges absolutely for all } x\in\mathbb{R} \text{ or } \mathbb{C}.$$

This is because geometric series converges for all x, as long as the denominator grows at least exponentially, which it does since $[n]_{q(t)}!\sim C_t^n$ with $C_t>0$.

Example 4.3. Solve $D_{q(t)}y=y$, $y(0)=1$ with $|q(t)|<1$.

Solution. Given that, $|q(t)|<1$. Assume $y(x)=\sum\limits_{n=0}^{\infty}a_nx^n$. The $q(t)$-derivative of x^n is ([8]):

$$D_{q(t)}(x^n)=[n]_{q(t)}x^{n-1},\quad n\ge 1,$$

where $[n]_{q(t)}=\frac{1-q(t)^n}{1-q(t)}$ is the $q(t)$-function for $|q(t)|<1$, and $D_{q(t)}(x^0)=0$.
Thus,

$$D_{q(t)}y(x)=\sum_{n=1}^{\infty}a_n[n]_{q(t)}x^{n-1}.$$

By rearranging the terms of the series

$$\sum_{n=1}^{\infty} a_n [n]_{q(t)} x^{n-1} = \sum_{n=0}^{\infty} a_n x^n.$$

Shift the index on the left side by letting $m = n-1$, so $n = m+1$,

$$\sum_{m=0}^{\infty} a_{m+1} [m+1]_{q(t)} x^m = \sum_{n=0}^{\infty} a_n x^n.$$

Equating coefficients of x^m:

$$a_{m+1}[m+1]_{q(t)} = a_m, \quad m \geq 0.$$

Solving recursively,

$$a_1 = \frac{a_0}{[1]_{q(t)}},\ a_2 = \frac{a_1}{[2]_{q(t)}} = \frac{a_0}{[1]_{q(t)}[2]_{q(t)}},\ \ldots,\ a_n = \frac{a_0}{[n]_{q(t)}!},$$

where $[n]_{q(t)}! = [1]_{q(t)}[2]_{q(t)} \ldots [n]_{q(t)}$ is the $q(t)$-factorial ([8]), with $[0]_{q(t)}! = 1$. The initial condition $y(0) = 1$ gives $a_0 = 1$. Thus,

$$a_n = \frac{1}{[n]_{q(t)}!}.$$

The solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{q(t)}!}.$$

□

5. Conclusion

The paper represents the further theoretical applications of $q(t)$-Calculus which opens new avenues for research in $q(t)$-Differential Equations and Number theoretic properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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