Communications in Mathematics and Applications
Vol. 16, No. 4, pp. 1107–1117, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i4.3246

http://www.rgnpublications.com

Research Article

The Square Graph of a Cycle Graph

Nabajit Saha* and Manmohan Das

Department of Mathematics, Bhattadev University, Pathsala, Bajali 781325, Assam India
***Corresponding author:** mat.nabajit@bhattadevuniversity.ac.in

Received: June 25, 2025 **Revised:** August 30, 2025 **Accepted:** September 8, 2025

Abstract. The square graph of a cycle graph C_n is a graph consisting of the same set of vertex as in C_n and two different vertices are considered adjacent if their distance in C_n is at most two. We denote the square graph of a cycle graph C_n by C_n^2. In this article, we first show that the square graph C_n^2 is a 4-regular graph for $n \geq 5$ and then discuss various graphical properties such as the spectrum, Laplacian spectrum of C_n^2, tree number, energy and, finally, the vertex coloring of C_n^2.

Keywords. Square graph, Spectrum, Tree number, Energy of a graph, Vertex coloring

Mathematics Subject Classification (2020). 05C50, 05C15, 05C92

1. Introduction

The square graph C_n^2 of a cycle graph C_n is a graph consisting of the same set of vertex as of C_n and two distinct vertices are adjacent if their distance is at most two in C_n. There are many papers in which the characteristic of the square graphs of a graph is introduced and the properties discussed. In particular, Ross and Harary [17] characterized the squares of trees, demonstrating that the square roots of trees, if they exist, are unique up to isomorphism. The characterization of graphs that have square roots was given by Mukhopadhyay [14], however, this is not a satisfactory characterization because it does not provide a brief certificate in the case that a graph does have a square root. Raychaudhuri [16] demonstrated different properties of the power of the chordal and circular arc graphs. In this paper, we explore some different properties of the square of the cycle graph such as the spectrum, the Laplacian spectrum of C_n^2, the tree number, the energy and lastly the coloring of the vertex of C_n^2. In particular, our main result are based on the circulant matrix. Elspas and Turner [8] discussed the circulant matrix and the circulant graph. For more information on the circulant graph, we refer to Parsons [15].

All graphs considered in this paper are simple and undirected graphs. A graph G is an order pair (V,E) with a set of vertex $V(G) = \{v_1, v_2, \dots, v_n\}$ and an edge set is $E(G)$. The cardinality of $V(G)$ is called the order, and the cardinality of $E(G)$ is called the size of G. The degree of a vertex v_i denoted $d(v_i)$, is the number of edges incident to v_i. A k-regular graph is a graph in which the degree of each vertex is k. The adjacency matrix of a graph G is a matrix $A(G) = (a_{ij})$ in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ elsewhere. The spectrum of $A(G)$ is denoted by

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{n-1} \\ m(\lambda_0) & m(\lambda_1) & \cdots & m(\lambda_{n-1}) \end{pmatrix}.$$

The first row indicates the eigenvalues of $A(G)$ and the second row indicates their corresponding multiplicities. Here, since $A(G)$ is a symmetric matrix, all eigenvalues are real. For more information on the adjacency matrix and its eigenvalues, we may refer to Biggs [4], Harary [10, 11], and Mohar [13]. The formula for a graph's Laplacian matrix, represented by $L(G)$, is $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix whose diagonal entries represent the degrees of G's vertices. Since $L(G)$ is a real, symmetric and positive semi-definite matrix, all its eigenvalues are real and non-negative. The eigenvalues of $L(G)$ are called Laplacian eigenvalues and are denoted by $\mu_0, \mu_1, \dots, \mu_{n-1}$. We might consult Anderson and Morley [1], Biggs [4] and Grone and Merris [9] for more details on the Laplacian matrix and its eigenvalues. One of the most interesting problems in algebraic graph theory is the computation of the tree-number. The tree number of a graph is denoted by $\kappa(G)$, which is the number of spanning trees of G. For more information on tree number, we may refer to Biggs [4]. The energy of a graph is another important application in algebraic graph theory. Assume that G is a graph and that its adjacency matrix is $A(G)$. If the eigenvalues of a graph G are $\lambda_1, \lambda_2, \dots, \lambda_n$, then the energy of a graph is denoted by $\varepsilon(G)$ and defined as

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Please refer to Bapat [2] and Ivan [12] for further details concerning the energy of a graph. One of the oldest and most interesting problems in graph theory is the vertex-coloring problem. If we partition the vertex set V of a graph G into subsets $\{V_i\}$; $i = 1, 2, \dots, s$ in such a way that each V_i does not contain two adjacent vertices, then such a partition is called a color partition and the subsets $\{V_i\}$, $i = 1, 2, \dots, s$ are called color classes. On the other hand, the induced subgraph generated by $\{V_i\}$, $i = 1, 2, \dots, s$ has no edges. The chromatic number of a graph G is denoted by $\nu(G)$ and is defined by the least natural number l for which such a partition is possible. For more information on the problem of vertex coloration of a graph, we may refer to Behzad [3], Biggs [5], Cvetković [7], Whitney [19], and Wilf[20].

The content of the paper is as follows: In Section 2, we give some known results that will be used in our subsequent results. In Section 3, we prove that the square graph of a cycle graph is 4-regular graph and further we find its eigenvalues. In addition, we investigate some new graphs whose least eigenvalue is -2. In Section 4, we find Laplacian eigenvalues, tree numbers and energy of a square graph. In the last section, we find vertex-coloring and give some examples of vertex-coloring of a square graph.

2. Preliminaries

In this section, we list some previously known results that will be used in our subsequent results.

Theorem 2.1 ([4]). *Let $\mu_o \le \mu_1 \le \mu_2 \le \dots \le \mu_{n-1}$ be the eigenvalues of the Laplacian matrix $L(G)$. Then*

(a) *$\mu_o = 0$ with eigenvector $[1,1,\dots,1]$;*

(b) *if G is connected then $\mu_1 > 0$;*

(c) *if G is a regular graph of degree k, then $\mu_i = k - \lambda_i$, where λ_i are the (ordinary) eigenvalues of G in weakly decreasing order.*

Theorem 2.2 ([4]). *Let $0 \le \mu_1 \le \mu_2 \le \dots \le \mu_{n-1}$ be the Laplacian eigenvalues of a graph G with n-vertices. Then the tree number of G is*

$$\kappa(G) = \frac{\mu_1\mu_2\dots\mu_{n-1}}{n}.$$

Theorem 2.3 ([7]). *The chromatic number of the general graph G with n vertices is*

$$\nu(G) \ge \frac{n}{n-\lambda_{\max}}.$$

3. Main Results

Our main result is an application of the power of graph. In this section, we first show that the square graph C_n^2 of a cycle graph C_n for $n \ge 5$ is a 4-regular graph. Then find its eigenvalues using the concept of circulant matrix.

Theorem 3.1. *The square graph C_n^2 of a cycle graph C_n for $n \ge 5$ is a 4-regular graph.*

Proof. Let C_n be a cycle graph whose vertex set is $\{v_1, v_2, \dots, v_n\}$, where $n \ge 5$. Thus, C_n is the cycle $v_1v_2\dots v_nv_1$. In C_n, there are four paths, v_1v_2, v_1v_n, $v_1v_2v_3$, $v_1v_nv_{n-1}$ whose length is at most two from v_1. Therefore, the vertices v_2, v_3, v_n, v_{n-1} are adjacent to v_1 in C_n^2. Consequently, the degree of v_1 is 4 in C_n^2. In C_n, there are four paths, v_2v_1, v_2v_3, $v_2v_1v_n$, $v_2v_3v_4$ whose length is at most two from v_2. Therefore, the vertices v_1, v_3, v_4, v_n are adjacent to v_2 in C_n^2. As a result, the degree of v_2 is 4 in C_n^2. In C_n, there are four paths, v_nv_1, v_nv_{n-1}, $v_nv_1v_2$, $v_nv_{n-1}v_{n-2}$ whose length is at most two from v_n. Therefore, the vertices v_1, v_2, v_{n-1}, v_{n-2} are adjacent to v_n in C_n^2. Thus, the degree of v_n is 4 in C_n^2. In C_n, there are four paths, $v_{n-1}v_n$, $v_{n-1}v_{n-2}$, $v_{n-1}v_nv_1$, $v_{n-1}v_{n-2}v_{n-3}$ whose length is at most two from v_{n-1}. Therefore, the vertices v_n, v_1, v_{n-2}, v_{n-3} are adjacent to v_{n-1} in C_n^2. For this reason, the degree of v_{n-1} is 4 in C_n^2. In C_n, there are four paths, v_jv_{j-1}, v_jv_{j+1}, $v_jv_{j-1}v_{j-2}$, $v_jv_{j+1}v_{j+2}$ whose length is at most two from v_j for $j = 3,4,\dots,n-2$. Therefore, the vertices v_{j-1}, v_{j-2}, v_{j+1}, v_{j+2} are adjacent to v_j in C_n^2 for $j = 3,4,\dots,n-2$. So, the degree of v_j is 4 in C_n^2 for $j = 3,4,\dots,n-2$. Therefore, the degree of each vertex is 4 in C_n^2. Hence C_n^2 is a 4-regular graph. □

From Theorem 3.1, we see that, $C_5^2, C_6^2, C_7^2, C_8^2, C_9^2, C_{10}^2, \dots$ are 4-regular graphs with vertices $5,6,7,8,9,10,\dots$, respectively. Therefore, the square graphs C_n^2 is an example of class of 4-regular graphs with vertices $n \ge 5$.

Theorem 3.2. *The eigenvalues of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are*

$$\lambda_r = 2\cos\frac{2\pi r}{n} + 2\cos\frac{4\pi r}{n}, \quad r = 0,1,2,3,\dots,n-1.$$

Proof. Because its adjacency matrix is a circulant matrix, the square graph C_n^2 of a cycle graph C_n is also a circulant graph. Since $[0,1,1,0,\dots,0,1,1]$ is the first row on the adjacency matrix, as a result the eigenvalues of C_n^2 are

$$\lambda_r = \sum_{j=2}^{n} s_j \omega^{(j-1)r}, \quad r = 0,1,2,3,\dots,n-1,$$

where

$$\omega = \exp\left(\frac{2\pi i}{n}\right) \text{ and } s_j \text{ is the } j\text{th entry of } [0,1,1,0,\dots,0,1,1],$$

$$\begin{aligned}\lambda_r &= \omega^r + \omega^{2r} + \omega^{(n-2)r} + \omega^{(n-1)r}, \quad r = 0,1,2,3,\dots,n-1,\\ &= 2\cos\frac{2\pi r}{n} + 2\cos\frac{4\pi r}{n}, \quad r = 0,1,2,3,\dots,n-1.\end{aligned}$$

□

Corollary 3.3. *The spectrum of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are*

$$\begin{pmatrix} 4 & 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n} & \dots & 2\cos\frac{(n-1)\pi}{n} + 2\cos\frac{2(n-1)\pi}{n} \\ 1 & 2 & \dots & 2 \end{pmatrix} \quad \textit{when } n \textit{ is odd}$$

and

$$\begin{pmatrix} 4 & 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n} & \dots & 2\cos\frac{(n-2)\pi}{n} + 2\cos\frac{2(n-2)\pi}{n} & 0 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix} \quad \textit{when } n \textit{ is even.}$$

Proof. For $r = 0$, $\lambda_r = 4$.

Case 1: When n is odd. Let $n = 2m+1$. Then $r = 0,1,2,\dots,2m$.

Now

$$\begin{aligned}\lambda_{2m} &= 2\cos\frac{2\pi 2m}{n} + 2\cos\frac{4\pi 2m}{n}\\ &= 2\cos\frac{2\pi(n-1)}{n} + 2\cos\frac{4\pi(n-1)}{n}\\ &= 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n}\\ &= \lambda_1.\end{aligned}$$

Therefore, λ_1 coincides with λ_{2m}. Hence, the multiplicity of λ_1 is 2. Thus, the multiplicities of the other eigenvalues are also 2. Therefore, the spectrums of C_n^2 are

$$\begin{pmatrix} 4 & 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n} & \dots & 2\cos\frac{(n-1)\pi}{n} + 2\cos\frac{2(n-1)\pi}{n} \\ 1 & 2 & \dots & 2 \end{pmatrix} \quad \text{when } n \text{ is odd.}$$

Case 2: When n is even. Let $n = 2m+2$. Then $r = 0,1,2,\dots,2m+1$.

Now

$$\begin{aligned}\lambda_{2m+1} &= 2\cos\frac{2\pi(2m+1)}{n} + 2\cos\frac{4\pi(2m+1)}{n}\\ &= 2\cos\frac{2\pi(n-1)}{n} + 2\cos\frac{4\pi(n-1)}{n}\end{aligned}$$

$$= 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n}$$
$$= \lambda_1 .$$

Therefore, λ_1 is coincide with λ_{2m+1}. So, multiplicity of λ_1 is 2. Thus, multiplicity of other eigenvalues is also 2 except λ_o and λ_{m+1}.

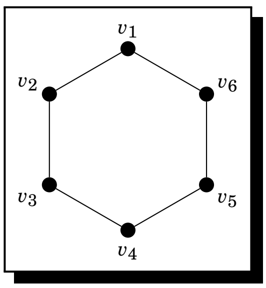
Now for $r = m+1$,

$$\begin{aligned}\lambda_{m+1} &= 2\cos\frac{2\pi(m+1)}{n} + 2\cos\frac{4\pi(m+1)}{n}\\ &= 2\cos 2\pi + 2\cos 4\pi\\ &= 0.\end{aligned}$$

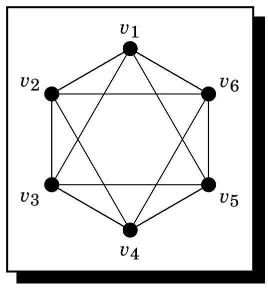
Therefore, the spectrum's of C_n^2 are

$$\begin{pmatrix} 4 & 2\cos\frac{2\pi}{n} + 2\cos\frac{4\pi}{n} & \dots & 2\cos\frac{(n-2)\pi}{n} + 2\cos\frac{2(n-2)\pi}{n} & 0 \\ 1 & 2 & \dots & 2 & 1 \end{pmatrix} \quad \text{when } n \text{ is even.} \qquad \square$$

At this point, we give some particular examples of cycle graphs and its square graphs as follows



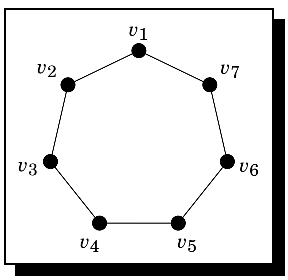
(a) The cycle graph C_6



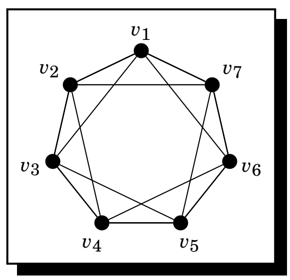
(b) The square graph C_6^2

Figure 1. The cycle graph C_6 and its square graph C_6^2

The eigenvalues of the square graph C_6^2 are $\lambda_0 = 4$, $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = 0$, $\lambda_4 = -2$, $\lambda_5 = 0$.



(a) The cycle graph C_7



(b) The square graph C_7^2

Figure 2. The cycle graph C_7 and its square graph C_7^2

The eigenvalues of the square graph C_7^2 are $\lambda_0 = 4$, $\lambda_1 = 0.801937736$, $\lambda_2 = -2.2469796$, $\lambda_3 = -0.554958132$, $\lambda_4 = -0.554958132$, $\lambda_5 = -2.2469796$, $\lambda_6 = 0.801937736$.

(a) The cycle graph C_8 **(b)** The square graph C_8^2

Figure 3. The cycle graph C_8 and its square graph C_8^2

The eigenvalues of the square graph C_8^2 are $\lambda_0 = 4$, $\lambda_1 = 1.41421356$, $\lambda_2 = -2$, $\lambda_3 = -1.41421356$, $\lambda_4 = 0$, $\lambda_5 = -1.41421356$, $\lambda_6 = -2$, $\lambda_7 = 1.41421356$.

(a) The cycle graph C_9 **(b)** The square graph C_9^2

Figure 4. The cycle graph C_9 and its square graph C_9^2

The eigenvalues of the square graph C_9^2 are $\lambda_0 = 4$, $\lambda_1 = 1.87938524$, $\lambda_2 = -1.53208889$, $\lambda_3 = -2$, $\lambda_4 = -0.347296356$, $\lambda_5 = -0.347296356$, $\lambda_6 = -2$, $\lambda_7 = -1.53208889$, $\lambda_8 = 1.87938524$.

(a) The cycle graph C_{10} **(b)** The square graph C_{10}^2

Figure 5. The cycle graph C_{10} and its square graph C_{10}^2

The eigenvalues of the square graph C_{10}^2 are $\lambda_0 = 4$, $\lambda_1 = 2.23606798$, $\lambda_2 = -1$, $\lambda_3 = -2.23606798$, $\lambda_4 = -1$, $\lambda_5 = 0$, $\lambda_6 = -1$, $\lambda_7 = -2.23606798$, $\lambda_8 = -1$, $\lambda_9 = 2.23606798$.

Seidel *et al*. [18] showed that the following graphs are neither line graph nor hyperoctahedral graph whose least eigenvalue is -2:

(a) the Petersen graph;
(b) a 5-regular graph with 16 vertices;
(c) a 16-regular graph with 27 vertices;
(d) the exceptional graphs.

Also, Cameron *et al.* [6] showed that all graphs with least eigenvalue greater than equal to -2 fall into three classes:

(a) the line graphs of bipartite graphs;
(b) the generalized line graphs;
(c) a finite class of graphs arising from the root systems E_6, E_7, E_8,

where we observed that the square graphs C_6^2, C_8^2 and C_9^2 of the cycle graphs C_6, C_8 and C_9 respectively does not belong to the above category of graphs although its least eigenvalue is -2.

Theorem 3.4. *The Laplacian eigenvalues of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are $\mu_o = 0$ and $\mu_r = 4 - 2\left(\cos\frac{2\pi r}{n} + \cos\frac{4\pi r}{n}\right)$, $r = 1,2,3,\dots,n-1$.*

Proof. In Theorem 3.2 we find out that the eigenvalues of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are $\lambda_o = 4$ and $\lambda_r = 2\left(\cos\frac{2\pi r}{n} + \cos\frac{4\pi r}{n}\right)$, $r = 1,2,3,\dots,n-1$. In Theorem 3.1, we also observed that the square graph C_n^2 of a cycle graph C_n for $n \geq 5$ is a 4-regular graph. In Theorem 2.1, we see, if G is a regular graph of degree k, then the Laplacian eigenvalues are $\mu_i = k - \lambda_i$, where λ_i are the (ordinary) eigenvalues of the graph G. Therefore, the Laplacian eigenvalues of C_n^2 are

$$\begin{aligned}\mu_o &= 4 - \lambda_o \\ &= 4 - 4 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\mu_r &= 4 - \lambda_r, \quad r = 1,2,3,\dots,n-1 \\ &= 4 - 2\left(\cos\frac{2\pi r}{n} + \cos\frac{4\pi r}{n}\right), \quad r = 1,2,3,\dots,n-1.\end{aligned}$$ □

4. Tree Number and Energy of a Square Graph

In this section, we find the tree number of the square graph C_n^2 of a cycle graph C_n for $n \geq 5$ using the formula

$$\kappa(G) = \frac{\mu_1 \mu_2 \dots \mu_{n-1}}{n},$$

where $\mu_1, \mu_2, \dots, \mu_{n-1}$ are the Laplacian eigenvalues of the graph G.
Also, find the energy of the square graph C_n^2 of a cycle graph C_n for $n \geq 5$ using the formula

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Theorem 4.1. *The tree number of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are*

$$\kappa(C_n^2) = \frac{2^{n-1}}{n} \prod_{r=1}^{n-1} \left(2 - \cos\frac{2\pi r}{n} - \cos\frac{4\pi r}{n}\right).$$

Proof. From Theorem 3.4, we observed that the Laplacian eigenvalues of a square graph C_n^2 are $\mu_o = 0$ and $\mu_r = 4-2\left(\cos\frac{2\pi r}{n}+\cos\frac{4\pi r}{n}\right)$; $r = 1,2,3,\dots,n-1$. Also in Theorem 2.2, we see that if $\mu_1,\mu_2,\dots,\mu_{n-1}$ are the Laplacian eigenvalues of the graph G, then the tree number of the G is

$$\kappa(G) = \frac{\mu_1\mu_2\dots\mu_{n-1}}{n}.$$

Therefore, the tree number of a square graph C_n^2 is

$$\begin{aligned}\kappa(C_n^2) &= \frac{1}{n}\prod_{r=1}^{n-1}\left(4-2\cos\frac{2\pi r}{n}-2\cos\frac{4\pi r}{n}\right)\\ &= \frac{1}{n}\prod_{r=1}^{n-1}2\left(2-\cos\frac{2\pi r}{n}-\cos\frac{4\pi r}{n}\right)\\ &= \frac{2^{n-1}}{n}\prod_{r=1}^{n-1}\left(2-\cos\frac{2\pi r}{n}-\cos\frac{4\pi r}{n}\right).\end{aligned}$$

□

Here, we find the tree number of the square graph C_6^2 as illustrative purpose. The eigenvalues of the square graph C_6^2 are $\lambda_0 = 4$, $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = 0$, $\lambda_4 = -2$, $\lambda_5 = 0$. Since the Laplacian eigenvalues of the square graph C_6^2 are

$$\mu_r = 4-\lambda_r, \quad r = 0,1,2,3,4,5.$$

Therefore, $\mu_0 = 0$, $\mu_1 = 4$, $\mu_2 = 6$, $\mu_3 = 4$, $\mu_4 = 6$, $\mu_5 = 4$. Then, the tree number of C_6^2 are

$$\begin{aligned}\kappa(C_6^2) &= \frac{1}{6}\prod_{r=1}^{5}\mu_r\\ &= \frac{1}{6}\times 4\times 6\times 4\times 6\times 4\\ &= 384.\end{aligned}$$

Therefore, the tree number of the square graph C_6^2 are 384.

Theorem 4.2. *The energy of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ is at most $2m$, where m is the number of edges of C_n^2.*

Proof. We observed in Theorem 3.2 that the eigenvalues of a square graph C_n^2 of a cycle graph C_n for $n \geq 5$ are

$$\lambda_r = 2\cos\frac{2\pi r}{n}+2\cos\frac{4\pi r}{n}, \quad r = 0,1,2,3,\dots,n-1.$$

Also, we know that, if $\lambda_1,\lambda_2,\dots,\lambda_n$ are the eigenvalues of a graph G, then the energy of G is

$$\varepsilon(G) = \sum_{i=1}^{n}|\lambda_i|.$$

Therefore, the energy of C_n^2 is

$$\begin{aligned}\varepsilon(C_n^2) &= \sum_{r=0}^{n-1}|\lambda_r|\\ &= 4+\sum_{r=1}^{n-1}\left|2\cos\frac{2\pi r}{n}+2\cos\frac{4\pi r}{n}\right|\\ &\leq 4+2\sum_{r=1}^{n-1}\left(\left|\cos\frac{2\pi r}{n}\right|+\left|\cos\frac{4\pi r}{n}\right|\right)\end{aligned}$$

$$\leq 4+2\sum_{r=1}^{n-1}(1+1)$$
$$= 4+4(n-1)$$
$$= 4n$$
$$= 2m\,.$$ □

From Theorem 4.2, we see that the upper bounds of the square graph C_n^2 of a cycle graph C_n for $n \geq 5$ is twice the number of its edges.

5. Vertex Coloring of a Square Graph

In this section, we find the chromatic number of the square graph C_n^2 of cycle graph C_n. Actually, we find the lower bounds of the chromatic number using the formula

$$\nu(G) \geq \frac{n}{n-\lambda_{\max}},$$

where $\lambda_{\max}$ is the maximum eigenvalues of the graph G.

Theorem 5.1. *The chromatic number of the square graph C_n^2 of cycle graph C_n is*

$$\nu(C_n^2) \geq \frac{n}{n-4}.$$

Proof. We observed in Theorem 2.3 that the chromatic number of the general graph G with n vertices is

$$\nu(G) \geq \frac{n}{n-\lambda_{\max}}.$$

Since C_n^2 is a 4-regular graph with n-vertices and maximum eigenvalue $\lambda_{\max} = 4$. Then, the chromatic number of C_n^2 is

$$\nu(C_n^2) \geq \frac{n}{n-\lambda_{\max}}$$
$$= \frac{n}{n-4}.$$ □

At this place, in particular, we determine the lower bound of the chromatic number of the graphs C_6 and C_6^2 as follows:

$$\nu(C_6) \geq \frac{6}{6-2}$$
$$= 1.5$$

and

$$\nu(C_6^2) \geq \frac{6}{6-4}$$
$$= 3$$

Therefore, there are at least two different colors for vertex-coloring of the graph C_6 and there are at least three different colors for vertex-coloring of the graph C_6^2.

6. Conclusion

In this paper, we determined the eigenvalues, Laplacian eigenvalues, tree numbers and vertex coloring of the square graph of a cycle graph also investigate a new class of 4-regular graphs with vertices $n \geq 5$. We find new upper bound of the energy of a square graph $\varepsilon(C_n^2)$ also investigate some new square graphs whose least eigenvalue is -2. One can find higher power of cycle graphs and check its regularity, other graphical parameters and topological indices.

Acknowledgment

The authors would like to thank Dr. Ankur Sharmah for inspiring discussions and for motivating this work.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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