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Research Article

Common Fixed Point Results for Self-Mappings in Generalized Fuzzy Cone Metric Spaces

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Abstract. A number of generalized contraction results already existed in *Cone Metric Space* (CMS), *Fuzzy Metric Space* (FMS) and generalized *Fuzzy Cone Metric* (FCM) spaces. In the present manuscript, we established some *Common Fixed Point* (CFP) results in generalized FCM spaces for compatible and weakly compatible self-maps using continuity and without continuity. We establish a new class of *Fuzzy Cone Contraction* (FCC) theorems extending existing results in the literature including some related examples to these outcomes.

Keywords. Common fixed point, Contraction conditions, Fuzzy cone metric space, Weakly compatible, Self mapping

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction and Preliminaries

The theory of fuzzy sets was introduced by Zadeh [18]. The *Fuzzy Metric Space* (FMS) was defined by Kramosil and Michalek [6] and a new form of the FMS was given by George and Veeramani [2]. Jungck and Rhoads [4], introduced weakly compatible maps and proved some results in the context of metric spaces, and Pant [9] gave a short discussion of non-commuting maps. Som [16] generalized some results in FMS and proved *Common Fixed Point* (CFP) results for continuous self-maps. Pant and Chauhan [10] gave CFP theorems for two pairs of weakly compatible maps in menger spaces and FMS. Kiany and Amini-Harandi [5] gave results for set-valued fuzzy contraction maps in FMS and Sadeghi *et al*. [14] improvise the outcomes for partially ordered FMS. Huang and Zhang [3], introduced the *Cone Metric Space* (CMS), also gave topology and related FP results. Öner *et al*. [8], illustrate a new concept of *Fuzzy Cone Metric* (FCM) space, later Öner [7] establishing its topology and a *Banach Contraction Principle* (BCP) with the use of Cauchy sequences. In the past few decades, many related results have been studied in FP theory in metric space and FMS. The idea of contraction mapping was already introduced in FCM space and established a related CFP theorem in such space. Some more outcomes in the similar manner can be seen in [11, 12]. Rehman *et al*. [13], generalize the result of CFP theorems in FCM spaces. Sunganthi *et al*. [17], generalized FCM space and gave contraction theorems in FCM spaces.

In this paper, we obtain some CFP results in generalized FCM spaces for compatible and weakly compatible self-maps. Some illustrative examples and new FCC theorem in generalized FCM spaces are also being discussed, including generalized results for self-maps with using a continuity and without continuity.

Definition 1.1 ([15])**.** A continuous t-norm (t-conorm) is a binary operation $\widehat{\mathfrak{S}}:[0,1]^2 \to [0,1]$ which satisfies the following conditions for all $\eth_1,\eth_2,\eth_3,\eth_4 \in [0,1]$:

(T^1) $\widehat{\mathfrak{S}}$ is continuous, commutative and associative,

(T^2) $\widehat{\mathfrak{S}}(\eth_1,1)=\eth_1$,

(T^3) $\widehat{\mathfrak{S}}(\eth_1,\eth_2)\le\widehat{\mathfrak{S}}(\eth_3,\eth_4)$ whenever $\eth_1\le\eth_2$ and $\eth_3\le\eth_4$.

CMS was introduced when the real numbers ($\mathbb{R}$) were replaced by ordering Banach's space. In this paper, $\mathbb{B}$ represents the real Banach space and $\mathbb{N}$ the set of natural numbers, and θ is zero of $\mathbb{B}$.

Definition 1.2 ([3])**.** A subset $O\subset\mathbb{B}$ is defined as a cone if:

(C_1) O is closed, nonempty and $O\neq\{\theta\}$,

(C_2) $\eth_1,\eth_2\in[0,\infty)$ and $\varpi,w\in O$, then $\eth_1\varpi+\eth_2 w\in O$,

(C_3) both $\varpi,-\varpi\in O$ then $\varpi=\theta$.

A partial ordering on a given cone $O\subset\mathbb{B}$ is defined by $\varpi\preccurlyeq w$ iff $w-\varpi\in O$. $\varpi\prec w$ stands for $\varpi\preccurlyeq w$ and $\varpi\neq w$, while $\varpi\ll w$ stands for $w-\varpi\in \mathrm{int}(O)$. In this paper, all cones have nonempty interior.

Definition 1.3 ([2])**.** The 3-tuple $(\breve{\mathfrak{A}},\mathbb{M},\widehat{\mathfrak{S}})$ is known as FMS if $\breve{\mathfrak{A}}$ is an arbitrary set, $\widehat{\mathfrak{S}}$ is a t-conorm, $\mathbb{M}$ is a fuzzy set in $\breve{\mathfrak{A}}\times\breve{\mathfrak{A}}\times[0,\infty)$ satisfies the following axioms for every $\varpi,w,\xi\in\breve{\mathfrak{A}}$ and $s,t>0$:

(FM_1) $\mathbb{M}(\varpi,w,t)>0$,

(FM_2) $\mathbb{M}(\varpi,w,t)=1$ if and only if $\varpi=w$,

(FM_3) $\mathbb{M}(\varpi,w,t)=\mathbb{M}(w,\varpi,t)$,

(FM_4) $\widehat{\mathfrak{S}}(\mathbb{M}(\varpi,w,t),\mathbb{M}(w,\xi,s))\le\mathbb{M}(\varpi,\xi,t+s)$,

(FM_5) $\mathbb{M}(\varpi,w,\cdot):[0,\infty)\to[0,1]$ is continuous.

Definition 1.4 ([8]). A 3-tuple $(\check{\mathfrak{A}}, \mathbb{M}, \widehat{\mathfrak{S}})$ is said to be an FCM-space if a cone $O \subset \mathbb{B}$, $\check{\mathfrak{A}}$ is an arbitrary set, $\widehat{\mathfrak{S}}$ is a t-conorm, and $\mathbb{M}$ is a fuzzy set on $\check{\mathfrak{A}} \times \check{\mathfrak{A}} \times \operatorname{int}(O)$ satisfying the following properties for every $\varpi, w, \xi \in \check{\mathfrak{A}}$, $s, t \gg \theta$ and $s, t \in \operatorname{int}(O)$:

(FC_{M1}) $\mathbb{M}(\varpi, w, t) > 0$,

(FC_{M2}) $\mathbb{M}(\varpi, w, t) = 1 \Longleftrightarrow \varpi = w$,

(FC_{M3}) $\mathbb{M}(\varpi, w, t) = \mathbb{M}(w, \varpi, t)$,

(FC_{M4}) $\widehat{\mathfrak{S}}(\mathbb{M}(\varpi, w, t), \mathbb{M}(w, \xi, s)) \le \mathbb{M}(\varpi, \xi, t+s)$,

(FC_{M5}) $\mathbb{M}(\varpi, w, \cdot) : \operatorname{int}(O) \to [0,1]$ is continuous.

If we consider $\mathbb{B}$ a set of real numbers, $O = [0, \infty)$ and $\widehat{\mathfrak{S}}(\varpi, w) = \varpi w$, then every FCM space becomes FMS.

Definition 1.5 ([1]). A pair of self-maps $(\widetilde{\wp}, T)$ of an FCM space $(\check{\mathfrak{A}}, \mathbb{M}, \widehat{\mathfrak{S}})$ is said to be compatible if $\lim_{m\to\infty} \mathbb{M}(\widetilde{\wp} T \mathfrak{p}_m, T \widetilde{\wp} \mathfrak{p}_m, t) = 1$ for $t \gg \theta$, whenever $\{\mathfrak{p}_m\}$ be a sequence in $\check{\mathfrak{A}}$ such that $\lim_{m\to\infty} T\mathfrak{p}_m = \lim_{m\to\infty} \widetilde{\wp}\mathfrak{p}_m = \varpi$, for some $\varpi \in \check{\mathfrak{A}}$.

Definition 1.6 ([1]). Let $\widetilde{\wp}$ and T be two self-maps on $\check{\mathfrak{A}}$. If $\varpi = wT = w\widetilde{\wp}$, for some $w \in \check{\mathfrak{A}}$ then w is called a coincidence point of $\widetilde{\wp}$ and T and ϖ is called a point of coincidence of $\widetilde{\wp}$ and T. The self-maps $\widetilde{\wp}$ and T are said to be weakly compatible, if they commute at their coincidence point, i.e., $wT = w\widetilde{\wp}$, for any $w \in \check{\mathfrak{A}}$ then $T\widetilde{\wp}w = T\widetilde{\wp}w$.

Definition 1.7 ([1]). Let $\widetilde{\wp}$ and T be weakly compatible self-maps of a set $\check{\mathfrak{A}}$. If $\widetilde{\wp}$ and T have a unique point of coincidence $\varpi = wT = w\widetilde{\wp}$, then ϖ is the unique CFP of $\widetilde{\wp}$ and T.

Definition 1.8 ([17]). A 3-tuple $(\check{\mathfrak{A}}, \mathcal{M}, \widehat{\mathfrak{S}})$ is said to be generalized-FCM space ($\mathcal{M}$-FCM space) if a cone $O \subset \mathbb{B}$, $\check{\mathfrak{A}} \neq \{\emptyset\}$, $\widehat{\mathfrak{S}}$ is a t-conorm, $\mathcal{M}$ is a fuzzy set on $\check{\mathfrak{A}}^3 \times \operatorname{int}(O)$ satisfying the following axioms for every $\varpi, w, \xi, u \in \check{\mathfrak{A}}$, $s, t \gg \theta$ and $s, t \in \operatorname{int}(O)$:

(FC_{M1}) $\mathcal{M}(\varpi, w, \xi, t) > 0$,

(FC_{M2}) $\mathcal{M}(\varpi, w, \xi, t) = 1$ iff $\varpi = w = \xi$,

(FC_{M3}) $\mathcal{M}(\varpi, w, \xi, t) = \mathcal{M}(p\{w, \varpi, \xi\}, t)$ where p is a permutation,

(FC_{M4}) $\widehat{\mathfrak{S}}(\mathcal{M}(\varpi, w, u, t), \mathcal{M}(u, \xi, \xi, s)) \le \mathcal{M}(\varpi, w, \xi, t+s)$,

(FC_{M5}) $\mathcal{M}(\varpi, w, \xi, \cdot) : \operatorname{int}(O) \to [0,1]$ is continuous.

The function $\mathcal{M}(\varpi, w, \xi, t)$, denotes the degree of nearness between ϖ, w and ξ with respect to t.

Example 1.9 ([17]). Let $\mathbb{B} = \mathbb{R}^2$ and the cone $O = \{(s,t) \in \mathbb{R}^2 : s, t \ge 0\}$. Consider the t-conorm defined as $\widehat{\mathfrak{S}}(\mathfrak{d}_1, \mathfrak{d}_2) = \mathfrak{d}_1, \mathfrak{d}_2$. Defined the function $\mathcal{M} : \mathbb{R}^3 \times \operatorname{int}(O) \to [0,1]$ for every $\varpi, w, \xi \in \mathbb{R}$ and $t \in \operatorname{int}(O)$ as:

$$\mathcal{M}(\varpi, w, \xi, t) = \left\{ \frac{1}{e^{\frac{|\varpi - w| + |w - \xi| + |\xi - \varpi|}{\|t\|}}} \right\}.$$

Then, $(\mathbb{R}, \mathcal{M}, \widehat{\mathfrak{S}})$ it is an $\mathcal{M}$-FCM space.

Definition 1.10 ([17]). A symmetric $\mathcal{M}$-FCM space is an $\mathcal{M}$-FCM space $(\check{\mathfrak{A}}, \mathcal{M}, \widehat{\mathfrak{S}})$ that satisfies:

$$\mathcal{M}(\varpi, w, w, t) = \mathcal{M}(w, \varpi, \varpi, t), \quad \text{for any } \varpi, w \in \check{\mathfrak{A}} \text{ and } t \in \operatorname{int}(O).$$

Definition 1.11 ([17])**.** Let $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ be an $\mathcal{M}$-FCM space. A self-mapping $\widetilde{\wp}_1:\breve{\mathfrak{A}}\to\breve{\mathfrak{A}}$ is said to be $\mathcal{M}$-fuzzy cone contractive ($\mathcal{M}$-FCC) if there exists $0<k<1$ such that

$$\left\{\frac{1}{\mathcal{M}(\widetilde{\wp}_1(\varpi),\widetilde{\wp}_1(w),\widetilde{\wp}_1(\xi),t)}-1\right\}\le k\left\{\frac{1}{\mathcal{M}(\varpi,w,\xi,t)}-1\right\},$$

for every $\varpi,w,\xi\in\breve{\mathfrak{A}}$ and $t\in\operatorname{int}(O)$. Also, constant k excludes the value zero. If $k=0$ then it is possible to have

$$\left\{\frac{1}{\mathcal{M}(\widetilde{\wp}_1(\varpi),\widetilde{\wp}_1(w),\widetilde{\wp}_1(\xi),t)}-1\right\}> k\left\{\frac{1}{\mathcal{M}(\varpi,w,\xi,t)}-1\right\},$$

for all distinct $\varpi,w,\xi\in\breve{\mathfrak{A}}$, $t\in\operatorname{int}(O)$ and k can not have any FP.

Definition 1.12 ([17])**.** In $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ an $\mathcal{M}$-FCM space. Then, FCM $\mathcal{M}$ is said to be triangular if for each $\varpi,w,\xi,u\in\breve{\mathfrak{A}}$ and $t\in\operatorname{int}(O)$,

$$\left\{\frac{1}{\mathcal{M}(\varpi,w,\xi,t)}-1\right\}<\left\{\frac{1}{\mathcal{M}(\varpi,w,u,t)}-1\right\}+\left\{\frac{1}{\mathcal{M}(u,\xi,\xi,t)}-1\right\}.$$

Definition 1.13 ([17])**.** Let $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ be an $\mathcal{M}$-FCM space, for some $\varpi\in\breve{\mathfrak{A}}$ and $\{\mathfrak{p}_m\}$ be a sequence in $\breve{\mathfrak{A}}$. Then

(i) A sequence $\{\mathfrak{p}_m\}$ is said to converge to ϖ if for every $t\in\operatorname{int}(O)$,

$$\lim_{m\to\infty}\left\{\frac{1}{\mathcal{M}(\mathfrak{p}_m,\varpi,\varpi,t)}-1\right\}=0 \ \text{ i.e.,}$$

$$\lim_{m\to\infty}\mathfrak{p}_m\to\varpi \ \text{ or } \ \mathfrak{p}_m\to\varpi \text{ as } m\to\infty.$$

(ii) A sequence $\{\mathfrak{p}_m\}$ is said to be a Cauchy if for all $t\in\operatorname{int}(O)$ and $n\in\mathbb{N}$, we have

$$\lim_{m\to\infty}\left\{\frac{1}{\mathcal{M}(\mathfrak{p}_{m+n},\mathfrak{p}_m,\mathfrak{p}_m,t)}-1\right\}=0.$$

(iii) $\mathcal{M}$-FCM space $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ in which every Cauchy sequence is convergent is said to be complete.

Definition 1.14 ([17])**.** Let $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ be an $\mathcal{M}$-FCM space. A sequence $\{\mathfrak{p}_m\}$ in $\breve{\mathfrak{A}}$ is $\mathcal{M}$-fuzzy cone contractive if there is $0<k<1$ such that for any $t\in\operatorname{int}(O)$:

$$\left\{\frac{1}{\mathcal{M}(\mathfrak{p}_m,\mathfrak{p}_{m+1},\mathfrak{p}_{m+1},t)}-1\right\}\le k\left\{\frac{1}{\mathcal{M}(\mathfrak{p}_{m-1},\mathfrak{p}_m,\mathfrak{p}_m,t)}-1\right\}.$$

2. Compatibility and Weakly Compatible Maps Results in $\mathcal{M}$-FCM Spaces

In this section, we present CFP theorems for compatible and weakly compatible maps in $\mathcal{M}$-FCM spaces.

Theorem 2.1. *Consider that ζ_1,ζ_2,ζ_3 and ζ_4 are self-maps on $\breve{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies the following conditions for every $\varpi,w\in\breve{\mathfrak{A}}$, $t\gg\theta$ and $t\in\operatorname{int}(O)$:*

$$\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2w,\zeta_2w,t)}-1\right\}\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4w,\zeta_4w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}$$

$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_2 w,\zeta_2 w,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}, \tag{2.1}$$

where $k_i\in[0,\infty)$, for all $i=1,\dots,5$ with $\sum_{i=1}^{5}k_i<1$ and $k_2=k_3$ or $k_4=k_5$. If mappings are satisfying the conditions: $\{\zeta_1,\zeta_3\}$ are compatible, $\{\zeta_2,\zeta_4\}$ are weakly compatible, $\zeta_1(\check{\mathfrak{A}})\subset\zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}})\subset\zeta_3(\check{\mathfrak{A}})$ and map ζ_3 is continuous. Then, maps ζ_1,ζ_2,ζ_3 and ζ_4 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. In complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$, suppose $\mathfrak{p}_0\in\check{\mathfrak{A}}$ any arbitrary point. As from the given hypothesis $\zeta_1(\check{\mathfrak{A}})\subset\zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}})\subset\zeta_3(\check{\mathfrak{A}})$, we generate a sequence $\{\mathfrak{p}_m\}$ from $\check{\mathfrak{A}}$ in such a way that $\mathfrak{q}_{2m+1}=\zeta_4\mathfrak{p}_{2m+1}=\zeta_1\mathfrak{p}_{2m}$ and $\mathfrak{q}_{2m+2}=\zeta_3\mathfrak{p}_{2m+2}=\zeta_2\mathfrak{p}_{2m+1}$, for each non-negative integer m.

Step (i): Meanwhile, from eq. (2.1), we imply that

$$\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$=\left\{\frac{1}{\mathcal{M}(\zeta_1\mathfrak{p}_{2m},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_1\mathfrak{p}_{2m},\zeta_1\mathfrak{p}_{2m},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_1\mathfrak{p}_{2m},\zeta_1\mathfrak{p}_{2m},t)}-1\right\}$$
$$\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1+\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}.$$

After simplifying further, we observe that

$$\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}\le\lambda\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}, \tag{2.2}$$

for $\lambda=\frac{k_1+k_2+k_4}{1-(k_3+k_4)}$.

In the similar manner by using $\mathcal{M}$-FCC, again preceding from eq. (2.1), one can have

$$\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}$$
$$=\left\{\frac{1}{\mathcal{M}(\zeta_1\mathfrak{p}_{2m+2},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}$$

$$\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+2},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_1\mathfrak{p}_{2m+2},\zeta_1\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_1\mathfrak{p}_{2m+2},\zeta_1\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+2},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}$$
$$\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+2},t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}$$
$$+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1+\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}.$$

On solving, we have

$$\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}\leq\mu\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}, \tag{2.3}$$

where $\mu=\left\{\frac{k_1+k_3+k_5}{1-(k_2+k_5)}\right\}$.

Step (ii): By repeated application of $\mathcal{M}$-FCC and using eqs. (2.2) and (2.3), we obtain

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}&\leq\lambda\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}\\&\leq\lambda\mu\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m-1},\zeta_3\mathfrak{p}_{2m},\zeta_3\mathfrak{p}_{2m},t)}-1\right\}\\&\leq\lambda\mu\lambda\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m-2},\zeta_4\mathfrak{p}_{2m-1},\zeta_4\mathfrak{p}_{2m-1},t)}-1\right\}\\&\;\;\vdots\\&\leq\lambda(\mu\lambda)^m\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_0,\zeta_4\mathfrak{p}_1,\zeta_4\mathfrak{p}_1,t)}-1\right\}\end{aligned} \tag{2.4}$$

and

$$\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m+2},\zeta_4\mathfrak{p}_{2m+3},\zeta_4\mathfrak{p}_{2m+3},t)}-1\right\}\leq\mu\left\{\frac{1}{\mathcal{M}(\zeta_4\mathfrak{p}_{2m+1},\zeta_3\mathfrak{p}_{2m+2},\zeta_3\mathfrak{p}_{2m+2},t)}-1\right\}$$
$$\leq\mu\lambda\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_{2m},\zeta_4\mathfrak{p}_{2m+1},\zeta_4\mathfrak{p}_{2m+1},t)}-1\right\}$$
$$\vdots$$
$$\leq(\mu\lambda)^{m+1}\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{p}_0,\zeta_4\mathfrak{p}_1,\zeta_4\mathfrak{p}_1,t)}-1\right\}. \tag{2.5}$$

Now, two cases arise:

Case (i): If $k_2 = k_3$ then

$$\lambda\mu = \left\{\frac{k_1+k_2+k_4}{1-(k_3+k_4)}\right\}\cdot\left\{\frac{k_1+k_3+k_5}{1-(k_2+k_5)}\right\} = \left\{\frac{k_1+k_3+k_5}{1-(k_2+k_5)}\right\}\cdot\left\{\frac{k_1+k_2+k_4}{1-(k_3+k_4)}\right\} < 1\cdot 1 = 1. \tag{2.6}$$

Case (ii): If $k_4 = k_5$ then

$$\lambda\mu = \left\{\frac{k_1+k_2+k_4}{1-(k_3+k_4)}\right\}\cdot\left\{\frac{k_1+k_3+k_5}{1-(k_2+k_5)}\right\} < 1\cdot 1 = 1. \tag{2.7}$$

Since FCM $\mathcal{M}$ is triangular, for $n > m \geq m_0$, one can get

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+1},\mathfrak{q}_{2n+1},\mathfrak{q}_{2n+1},t)}-1\right\} &\leq \left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+1},\mathfrak{q}_{2m+2},\mathfrak{q}_{2m+2},t)}-1\right\}+\dots \\
&\quad + \left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m},\mathfrak{q}_{2m+1},\mathfrak{q}_{2m+1},t)}-1\right\} \\
&\leq \left\{\lambda\sum_{i=m}^{n-1}(\lambda\mu)^i + \sum_{i=m+1}^{n}(\lambda\mu)^i\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\} \\
&\leq \left\{\frac{\lambda(\lambda\mu)^m}{1-\lambda\mu}+\frac{(\lambda\mu)^{m+1}}{1-\lambda\mu}\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\} \\
&= (1+\mu)\left\{\frac{\lambda(\lambda\mu)^m}{1-\lambda\mu}\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\}.
\end{aligned}$$

In the similar way, one can deduce that

$$\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m},\mathfrak{q}_{2n+1},\mathfrak{q}_{2n+1},t)}-1\right\} \leq (1+\lambda)\left\{\frac{(\lambda\mu)^m}{1-\lambda\mu}\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\},$$

$$\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m},\mathfrak{q}_{2n},\mathfrak{q}_{2n},t)}-1\right\} \leq (1+\lambda)\left\{\frac{(\lambda\mu)^m}{1-\lambda\mu}\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\},$$

$$\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+1},\mathfrak{q}_{2n},\mathfrak{q}_{2n},t)}-1\right\} \leq (1+\mu)\left\{\frac{\lambda(\lambda\mu)^m}{1-\lambda\mu}\right\}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\}.$$

Then, for some $m < n$,

$$\begin{aligned}
&\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+1},\mathfrak{q}_{2n+1},\mathfrak{q}_{2n+1},t)}-1\right\} \\
&\quad \leq \max\left[(1+\lambda)\left\{\frac{(\lambda\mu)^m}{1-\lambda\mu}\right\},(1+\mu)\left\{\frac{\lambda(\lambda\mu)^m}{1-\lambda\mu}\right\}\right]\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_0,\mathfrak{q}_1,\mathfrak{q}_1,t)}-1\right\},
\end{aligned}$$

which tends to 0 as $m \to \infty$.

Therefore, it shows that $\{\mathfrak{q}_m\}_{m\geq 0}$ is a Cauchy sequence.

Step (iii): Since by the completeness of $\check{\mathfrak{A}}$, there exists $\xi \in \check{\mathfrak{A}}$ such that $\mathfrak{q}_m \to \xi$ as $m \to \infty$ and for its subsequences, we imply that

$$\zeta_4\mathfrak{p}_{2m+1} \to \xi,\ \ \zeta_3\mathfrak{p}_{2m+2} \to \xi,\ \ \zeta_1\mathfrak{p}_{2m} \to \xi \ \ \text{and} \ \ \zeta_2\mathfrak{p}_{2m-1} \to \xi. \tag{2.8}$$

Since, ζ_3 is a continuous self-map on $\check{\mathfrak{A}}$, therefore we get

$$\zeta_3(\zeta_4\mathfrak{p}_{2m+1}) \to \zeta_3\xi, \zeta_3(\zeta_3\mathfrak{p}_{2m+2}) \to \zeta_3\xi,\ \ \zeta_3(\zeta_1\mathfrak{p}_{2m}) \to \zeta_3\xi \ \ \text{and} \ \ \zeta_3(\zeta_2\mathfrak{p}_{2m-1}) \to \zeta_3\xi.$$

As, $\zeta_3(\zeta_1\mathfrak{p}_{2m}) \to \zeta_3(\xi)$ and (ζ_1,ζ_3) are compatible.

Therefore, one can have

$$\lim_{n\to\infty}\mathcal{M}\{\zeta_1(\zeta_3\mathfrak{p}_{2m}),\zeta_3(\zeta_1\mathfrak{p}_{2m}),\zeta_3(\zeta_1\mathfrak{p}_{2m}),t\} = \lim_{n\to\infty}\mathcal{M}\{\zeta_1(\zeta_3\mathfrak{p}_{2m}),\zeta_3\xi,\zeta_3\xi,t\} = 1,$$

$$\lim_{n\to\infty} \mathcal{M}\{\zeta_1(\zeta_3\mathfrak{p}_{2m}),\zeta_3\xi,\zeta_3\xi,t\} = 1, \quad \text{for } t \gg \theta. \tag{2.9}$$

Now, we claim that $\zeta_3\xi = \xi$, for $t \gg \theta$,

$$\mathcal{M}(\zeta_3\xi,\xi,\xi,2t) \geq \widehat{\mathfrak{S}}[\mathcal{M}\{\zeta_3\xi,\zeta_1(\zeta_3\mathfrak{p}_{2m}),\zeta_1(\zeta_3\mathfrak{p}_{2m}),t\},\mathcal{M}\{\zeta_1(\zeta_3\mathfrak{p}_{2m}),\xi,\xi,t\}]. \tag{2.10}$$

Since maps (ζ_1,ζ_3) are compatible, using $\widehat{\mathfrak{S}}$ operator and eqs. (2.8) to (2.10), we imply that

$$\begin{aligned}\mathcal{M}(\zeta_3\xi,\xi,\xi,2t) &\geq \lim_{n\to\infty} \widehat{\mathfrak{S}}[\mathcal{M}\{\zeta_3\xi,\zeta_1(\zeta_3\mathfrak{p}_{2m}),\zeta_1(\zeta_3\mathfrak{p}_{2m}),t\},\mathcal{M}\{\zeta_1(\zeta_3\mathfrak{p}_{2m}),\xi,\xi,t\}]\\ &= \widehat{\mathfrak{S}}(1,1) = 1, \quad \text{for } t \gg \theta.\end{aligned}$$

Hence, $\mathcal{M}(\zeta_3\xi,\xi,\xi,2t) = 1$ and $\zeta_3\xi = \xi$.

In this way again, we claim that $\zeta_1\xi = \xi$, for $t \gg \theta$,

$$\mathcal{M}(\zeta_1\xi,\xi,\xi,2t) \geq \widehat{\mathfrak{S}}[\mathcal{M}\{\zeta_1\xi,\zeta_3(\zeta_1\mathfrak{p}_{2m}),\zeta_3(\zeta_1\mathfrak{p}_{2m}),t\},\mathcal{M}\{\zeta_3(\zeta_1\mathfrak{p}_{2m}),\xi,\xi,t\}]. \tag{2.11}$$

Again, as maps (ζ_1,ζ_3) are compatible, by $\widehat{\mathfrak{S}}$ operator and eqs. (2.8) to (2.11), we get

$$\begin{aligned}\mathcal{M}(\zeta_1\xi,\xi,\xi,2t) &\geq \lim_{n\to\infty} \widehat{\mathfrak{S}}[\mathcal{M}\{\zeta_1\xi,\zeta_3(\zeta_1\mathfrak{p}_{2m}),\zeta_3(\zeta_1\mathfrak{p}_{2m}),t\},\mathcal{M}\{\zeta_3(\zeta_1\mathfrak{p}_{2m}),\xi,\xi,t\}]\\ &= \widehat{\mathfrak{S}}(1,1) = 1, \quad \text{for } t \gg \theta.\end{aligned}$$

Hence, $\mathcal{M}(\zeta_1\xi,\xi,\xi,2t) = 1$ and $\zeta_1\xi = \xi$.

Thus, $\zeta_1\xi = \zeta_3\xi = \xi$.

Step (iv): Now, we claim that $\zeta_2\xi = \zeta_4\xi$.

Since maps satisfy $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_4(\check{\mathfrak{A}})$, therefore, there exists $u \in \check{\mathfrak{A}}$ such that $\zeta_1\xi = \xi = \zeta_4 u$. Again, from eq. (2.1), we have

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_4u,\zeta_4u,t)} - 1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1\xi,\zeta_2u,\zeta_2u,t)} - 1\right\}\\
&\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_4u,\zeta_4u,t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_1\xi,\zeta_1\xi,t)} - 1\right\}\\
&\quad + k_3\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_2u,\zeta_2u,t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_2u,\zeta_2u,t)} - 1\right\}\\
&\quad + k_5\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_1\xi,\zeta_1\xi,t)} - 1\right\}\\
&= k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\xi,\xi,t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\xi,\zeta_3\xi,\zeta_3\xi,t)} - 1\right\}\\
&\quad + k_3\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_2u,\zeta_2u,t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_2u,\zeta_2u,t)} - 1\right\}\\
&\quad + k_5\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_4u,\zeta_4u,t)} - 1\right\}\\
&= (k_3+k_4)\left\{\frac{1}{\mathcal{M}(\zeta_4u,\zeta_2u,\zeta_2u,t)} - 1\right\}.\end{aligned}$$

Since $(k_3+k_4) < 1$, as from the given hypothesis $\sum_{i=1}^{5} k_i < 1$. Therefore, $\mathcal{M}(\zeta_4u,\zeta_2u,\zeta_2u,t) = 1$, i.e., $\zeta_2\xi = \zeta_4\xi = \xi$ and by using weakly compatibility of maps ζ_2 and ζ_4, we get

$$\zeta_4\xi = \zeta_4(\zeta_2u) = \zeta_2(\zeta_4u) = \zeta_2\xi.$$

Now, we claim that $\zeta_2\xi = \xi$, by eq. (2.1), we deduce that

$$\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\xi,\xi,t)} - 1\right\} = \left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)} - 1\right\}$$

$$\begin{aligned}
&\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_4\xi,\zeta_4\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\
&\quad+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&=k_1\left\{\frac{1}{\mathcal{M}(\xi,\zeta_3\xi,\zeta_3\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\xi,\zeta_3\xi,\zeta_3\xi,t)}-1\right\}\\
&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\xi,\zeta_4\xi,\zeta_4\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\
&\quad+k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\xi,\xi,t)}-1\right\}\\
&=(k_1+k_4+k_5)\left\{\frac{1}{\mathcal{M}(\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}.
\end{aligned}$$

As $(k_1+k_4+k_5)<1$, from given $\sum_{i=1}^{5}k_i<1$.

Then, $\mathcal{M}(\xi,\zeta_2\xi,\zeta_2\xi,t)=1$, i.e., $\zeta_2\xi=\xi$, which implies $\xi=\zeta_4\xi$.

Thus, $\zeta_1\xi=\zeta_2\xi=\zeta_3\xi=\zeta_4\xi=\xi$.

Hence, ξ is the CFP of ζ_1,ζ_2,ζ_3 and ζ_4 self-maps in $\breve{\mathfrak{A}}$.

Step (v) Uniqueness: To prove uniqueness of FP, let $\mathfrak{u}_0$ be another FP of the self-maps ζ_1,ζ_2,ζ_3 and ζ_4, i.e., $\zeta_1\mathfrak{u}_0=\zeta_2\mathfrak{u}_0=\zeta_3\mathfrak{u}_0=\zeta_4\mathfrak{u}_0=\mathfrak{u}_0$. Then from eq. (2.1), for $t\gg\theta$, we have

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)}-1\right\}&=\left\{\frac{1}{\mathcal{M}(\zeta_1\mathfrak{u}_0,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\
&\leq k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{u}_0,\zeta_4\xi,\zeta_4\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{u}_0,\zeta_1\mathfrak{u}_0,\zeta_1\mathfrak{u}_0,t)}-1\right\}\\
&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\mathfrak{u}_0,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\
&\quad+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4\xi,\zeta_1\mathfrak{u}_0,\zeta_1\mathfrak{u}_0,t)}-1\right\}\\
&=(k_1+k_4+k_5)\left\{\frac{1}{\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)}-1\right\}.
\end{aligned}$$

Since $(k_1+k_4+k_5)<1$ as $\sum_{i=1}^{5}k_i<1$, then $\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)=1$, i.e., $\mathfrak{u}_0=\xi$.

Thus, we established that CFP is unique. □

Example 2.2. Let $\breve{\mathfrak{A}}=[-1,1]$ and t-conorm $\widehat{\mathfrak{S}}$ be defined by $\widehat{\mathfrak{S}}(\eth_1,\eth_2)=\eth_1,\eth_2$. Define the function $\mathcal{M}:\breve{\mathfrak{A}}^3\times[0,\infty)\to[0,1]$ for every $\varpi,w\in\breve{\mathfrak{A}}$ and $t>0$ as:

$$\mathcal{M}(\varpi,w,w,t)=\frac{t}{(t+|\varpi-w|)},$$

then $(\breve{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ is an $\mathcal{M}$-FCM space with FCM $\mathcal{M}$ is triangular.

The self-maps ζ_1,ζ_2,ζ_3 and ζ_4 on $\breve{\mathfrak{A}}$ defined as

$$\zeta_1\varpi=\zeta_2\varpi=\begin{cases}\frac{1}{2}\left(\frac{2\varpi}{3}+\frac{1}{4}\right) & \text{if } \varpi\neq 0\\ 0 & \text{if } \varpi=0\end{cases} \quad\text{and}\quad \zeta_3\varpi=\zeta_4\varpi=\begin{cases}\left(\frac{2\varpi}{3}+\frac{1}{4}\right) & \text{if } \varpi\neq 0\\ 0 & \text{if } \varpi=0\end{cases}.$$

Since $\zeta_1(\check{\mathfrak{A}})=\zeta_2(\check{\mathfrak{A}})$, $\zeta_3(\check{\mathfrak{A}})=\zeta_4(\check{\mathfrak{A}})$, we have $\zeta_1(\check{\mathfrak{A}})\subset\zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}})\subset\zeta_3(\check{\mathfrak{A}})$.
Then from eq. (2.1), for every $\varpi, w\in\check{\mathfrak{A}}$ and $t>0$, we have

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}&=\left\{\frac{(t+|\zeta_1\varpi-\zeta_2 w|)}{t}-1\right\}=\frac{|\zeta_1\varpi-\zeta_2 w|}{t}=\frac{|\varpi-w|}{3t}\\&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4 w,\zeta_4 w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}\\&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_2 w,\zeta_2 w,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}\\&\quad+k_5\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}.\end{aligned}$$

Thus, from Theorem 2.1 using $k_1=\frac{1}{2}$, $k_2=k_3=\frac{1}{6}$ and $k_4=k_5=0$ we can imply that 0 is the unique CFP of maps.

Corollary 2.3. *Consider that ζ_1,ζ_2,ζ_3 and ζ_4 are self-maps on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi,w\in\check{\mathfrak{A}}$, $t\gg\theta$ and $t\in\operatorname{int}(O)$:*

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4 w,\zeta_4 w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}\\&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_2 w,\zeta_2 w,t)}-1\right\},\end{aligned}\tag{2.12}$$

where $k_i\in[0,\infty)$, for all $i=1,2,3$ with $\sum_{i=1}^{3}k_i<1$. If maps are satisfying the conditions: $\{\zeta_1,\zeta_3\}$ are compatible, $\{\zeta_2,\zeta_4\}$ are weakly compatible, $\zeta_1(\check{\mathfrak{A}})\subset\zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}})\subset\zeta_3(\check{\mathfrak{A}})$ and map ζ_3 is continuous. Then, maps ζ_1,ζ_2,ζ_3 and ζ_4 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. One can prove the result using Theorem 2.1 by considering $k_2=k_3$ and $k_4=k_5=0$. □

Corollary 2.4. *Consider that ζ_1,ζ_2,ζ_3 and ζ_4 are self-maps on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi,w\in\check{\mathfrak{A}}$, $t\gg\theta$ and $t\in\operatorname{int}(O)$:*

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4 w,\zeta_4 w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}\\&\quad+k_3\left\{\frac{1}{\mathcal{M}(\zeta_4 w,\zeta_2 w,\zeta_2 w,t)}-1\right\},\end{aligned}\tag{2.13}$$

where $k_i\in[0,\infty)$, for all $i=1,2,3$ with $k_1+2(k_2+k_3)<1$. If maps are satisfying the conditions: $\{\zeta_1,\zeta_3\}$ are compatible, $\{\zeta_2,\zeta_4\}$ are weakly compatible, $\zeta_1(\check{\mathfrak{A}})\subset\zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}})\subset\zeta_3(\check{\mathfrak{A}})$ and map ζ_3 is continuous. Then, maps ζ_1,ζ_2,ζ_3 and ζ_4 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. Proof of corollary can be deduced by considering conditions $k_2=k_3$ and $k_4=k_5=0$ in Theorem 2.1. □

Corollary 2.5. *Consider that ζ_1,ζ_2,ζ_3 and ζ_4 are self-maps on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi,w\in\check{\mathfrak{A}}$, $t\gg\theta$ and $t\in\operatorname{int}(O)$:*

$$\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4 w,\zeta_4 w,t)}-1\right\},\tag{2.14}$$

where $k_1 \in [0,1)$. If maps are satisfying the conditions: $\{\zeta_1,\zeta_3\}$ are compatible, $\{\zeta_2,\zeta_4\}$ are weakly compatible, $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}}) \subset \zeta_3(\check{\mathfrak{A}})$ and map ζ_3 is continuous. Then, maps ζ_1,ζ_2,ζ_3 and ζ_4 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. One can prove the result by Theorem 2.1 as considering $k_2 = k_3 = k_4 = k_5 = 0$. □

Example 2.6. Let $\check{\mathfrak{A}} = [0,1]$ and the t-conorm $\widehat{\mathfrak{S}}$ is defined by $\widehat{\mathfrak{S}}(\mathfrak{d}_1,\mathfrak{d}_2) = \mathfrak{d}_1,\mathfrak{d}_2$. Define the function $\mathcal{M} : \check{\mathfrak{A}}^3 \times [0,\infty) \to [0,1]$ for every $\varpi, w \in \check{\mathfrak{A}}$ and $t > 0$ as: $\mathcal{M}(\varpi,w,w,t) = \frac{t}{(t+|\varpi-w|)}$, then $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ is an $\mathcal{M}$-FCM space with FCM $\mathcal{M}$ is triangular.

The self-maps ζ_1,ζ_2,ζ_3 and ζ_4 on $\check{\mathfrak{A}}$ are defined by

$$\zeta_1\varpi = \frac{\varpi}{(\varpi+6)}, \quad \zeta_2 w = \frac{w}{(w+10)}, \quad \zeta_3\varpi = \frac{\varpi}{3} \quad \text{and} \quad \zeta_4 w = \frac{w}{5}.$$

Since $\zeta_1(\check{\mathfrak{A}}) = \zeta_2(\check{\mathfrak{A}})$, $\zeta_3(\check{\mathfrak{A}}) = \zeta_4(\check{\mathfrak{A}})$, we have $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_4(\check{\mathfrak{A}})$, $\zeta_2(\check{\mathfrak{A}}) \subset \zeta_3(\check{\mathfrak{A}})$.

Then from eq. (2.14), for every $\varpi, w \in \check{\mathfrak{A}}$ and $t > 0$, we get

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_2 w,\zeta_2 w,t)} - 1\right\} &= \left\{\frac{(t+|\zeta_1\varpi-\zeta_2 w|)}{t} - 1\right\} \\
&= \frac{|\zeta_1\varpi-\zeta_2 w|}{t} = \frac{1}{t}\left|\frac{\varpi}{(\varpi+6)} - \frac{w}{(w+10)}\right| \\
&= \frac{1}{t}\left|\frac{10\varpi-6w}{(\varpi+6)(w+10)}\right| \le \frac{1}{t}\left|\frac{10\varpi-6w}{60}\right| = \frac{1}{t}\left|\frac{\varpi}{3} - \frac{w}{5}\right| \\
&\le \frac{1}{2}\left\{\frac{1}{\mathcal{M}(\zeta_3\varpi,\zeta_4 w,\zeta_4 w,t)} - 1\right\}.
\end{aligned}$$

Thus, from Corollary 2.5 with $k_1 = \frac{1}{2}$ we can imply that 0 is the unique CFP of maps.

Corollary 2.7. *Consider that ζ_1 and ζ_2 are two self-maps on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi, w \in \check{\mathfrak{A}}$, $t \gg \theta$ and $t \in \operatorname{int}(O)$:*

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_1 w,\zeta_1 w,t)} - 1\right\} \le{}& k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_2 w,\zeta_2 w,t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_1\varpi,\zeta_1\varpi,t)} - 1\right\} \\
&+ k_3\left\{\frac{1}{\mathcal{M}(\zeta_2 w,\zeta_1 w,\zeta_1 w,t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_1 w,\zeta_1 w,t)} - 1\right\} \\
&+ k_5\left\{\frac{1}{\mathcal{M}(\zeta_2 w,\zeta_1\varpi,\zeta_1\varpi,t)} - 1\right\}, \qquad (2.15)
\end{aligned}$$

where $k_i \in [0,\infty)$, for all $i = 1,\dots,5$ with $\sum_{i=1}^{5} k_i < 1$ and $k_2 = k_3$ or $k_4 = k_5$. If maps are satisfying the conditions: $\{\zeta_1,\zeta_2\}$ are weakly compatible, $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_2(\check{\mathfrak{A}})$ and map ζ_2 is continuous. Then, maps ζ_1 and ζ_2 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. One can easily prove the result by considering $\zeta_3 = \zeta_1$, $\zeta_4 = \zeta_2$ and compatible condition for two ζ_1, ζ_2 maps in Theorem 2.1. □

3. Generalized Contraction and Fixed Point Results In $\mathcal{M}$-FCM Spaces

In this section, we establish some CFP theorems for weakly compatible maps in $\mathcal{M}$-FCM spaces. Here, we demonstrate CFP results without using map continuity, in addition some more conditions on hypothesis are also inserted. Furthermore, we prove a new-type of FCC theorem in $\mathcal{M}$-FCM spaces.

Theorem 3.1. *Consider that ζ_1 and ζ_2 are two self-mappings on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies the following condition for every $\varpi,w \in \check{\mathfrak{A}}$, $t \gg \theta$ and $t \in \operatorname{int}(O)$:*

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_1 w,\zeta_1 w,t)}-1\right\} \le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\}\\ +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2 w,\zeta_1 w,\zeta_1 w,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_1 w,\zeta_1 w,t)}-1\right\}\\ +k_5\left\{\frac{1}{\mathcal{M}(\zeta_2 w,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\},\end{aligned} \tag{3.1}$$

where $k_i \in [0,\infty)$, for all $i=1,\dots,5$ with $\sum_{i=1}^{5} k_i < 1$ and $k_2=k_3$ or $k_4=k_5$. If mappings satisfy the conditions: $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_2(\check{\mathfrak{A}})$, $\zeta_1(\check{\mathfrak{A}})$ or $\zeta_2(\check{\mathfrak{A}})$ is complete and $\{\zeta_1,\zeta_2\}$ are weakly compatible. Then, maps ζ_1 and ζ_2 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. In complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$, suppose $\mathfrak{p}_0 \in \check{\mathfrak{A}}$ any arbitrary point. As from the given hypothesis $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_2(\check{\mathfrak{A}})$, we construct a sequence $\{\mathfrak{q}_m\}$ from $\zeta_2(\check{\mathfrak{A}})$ in such a way that $\mathfrak{q}_{2m+1}=\zeta_2\mathfrak{p}_{2m+1}=\zeta_1\mathfrak{p}_{2m}$ and $\mathfrak{q}_{2m+2}=\zeta_2\mathfrak{p}_{2m+2}=\zeta_1\mathfrak{p}_{2m+1}$, for each non-negative integer m. Since $\zeta_2(\check{\mathfrak{A}})$ is complete, therefore there exists $\xi,u \in \check{\mathfrak{A}}$ such that $\mathfrak{q}_{2m+1} \to u=\zeta_2\xi$ as $m\to\infty$,

$$\lim_{m\to\infty}\mathcal{M}(\mathfrak{q}_{2m+1},u,u,t)=\lim_{m\to\infty}\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},u,u,t)=1, \quad \text{for } t\gg\theta. \tag{3.2}$$

Since FCM $\mathcal{M}$ is triangular, for $t\gg\theta$, we have

$$\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\} < \left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\mathfrak{q}_{2m+2},\mathfrak{q}_{2m+2},t)}-1\right\}+\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+2},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}. \tag{3.3}$$

From eqs. (3.1) and (3.2), we have

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\mathfrak{q}_{2m+2},\zeta_1\xi,\zeta_1\xi,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1\mathfrak{p}_{2m+1},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\ &\quad +k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_1\mathfrak{p}_{2m+1},\zeta_1\mathfrak{p}_{2m+1},t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\quad +k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\mathfrak{p}_{2m+1},\zeta_1\mathfrak{p}_{2m+1},t)}-1\right\}\\ &= k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\ &\quad +k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_2\mathfrak{p}_{2m+2},\zeta_2\mathfrak{p}_{2m+2},t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{2m+1},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\quad +k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_2\mathfrak{p}_{2m+2},\zeta_2\mathfrak{p}_{2m+2},t)}-1\right\}\end{aligned}$$

$$\to (k_3+k_4)\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\} \text{ as } m\to\infty.$$

Therefore,

$$\lim_{m\to\infty}\sup\left\{\frac{1}{\mathcal{M}(q_{2m+2},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\le (k_3+k_4)\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\} \text{ as } m\to\infty.$$

From eqs. (3.2) and (3.3), one can have

$$\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\le (k_3+k_4)\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\} \tag{3.4}$$

as, $(k_3+k_4)<1$, implies $\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)=1$.

Therefore, $\zeta_2\xi=\zeta_1\xi=u$.

By weakly compatibility of maps ζ_1 and ζ_2, we get

$$\zeta_1 u=\zeta_1(\zeta_2\xi)=\zeta_2(\zeta_1\xi)=\zeta_2 u.$$

Thus, from eq. (3.1),

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1u,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_1u,\zeta_1u,t)}-1\right\}\\
&\quad +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&\quad +k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1u,\zeta_1u,t)}-1\right\}\\
&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_1u,\zeta_1u,t)}-1\right\}\\
&\quad +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\}\\
&\quad +k_5\left\{\frac{1}{\mathcal{M}(u,\zeta_1u,\zeta_1u,t)}-1\right\}\\
&=(k_1+k_4+k_5)\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\}.
\end{aligned}$$

Since $k_1+k_4+k_5<1$, $\mathcal{M}(\zeta_1u,u,u,t)=1$, for $t\gg\theta$.

Thus, $\zeta_2u=\zeta_1u=u$.

Hence, u is the CFP of maps ζ_1 and ζ_2.

Uniqueness: To prove uniqueness of FP, let u_0 be another FP of the self-maps ζ_1 and ζ_2, i.e., $\zeta_1u_0=\zeta_2u_0=u_0$. Then from eq. (3.1), for $t\gg\theta$, we have

$$\begin{aligned}
\left\{\frac{1}{\mathcal{M}(u_0,\xi,\xi,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1u_0,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2u_0,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2u_0,\zeta_1u_0,\zeta_1u_0,t)}-1\right\}\\
&\quad +k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2u_0,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\
&\quad +k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1u_0,\zeta_1u_0,t)}-1\right\}
\end{aligned}$$

$$= (k_1 + k_4 + k_5)\left\{\frac{1}{\mathcal{M}(\mathrm{u}_0, \xi, \xi, t)} - 1\right\}.$$

Since $(k_1 + k_4 + k_5) < 1$ as $\sum_{i=1}^{5} k_i < 1$, then $\mathcal{M}(\mathrm{u}_0, \xi, \xi, t) = 1$, i.e., $\mathrm{u}_0 = \xi$.

Thus, we established the uniqueness of CFP. □

Corollary 3.2. *Consider that ζ_1 and ζ_2 are two self-maps on $\check{\mathfrak{A}}$ such that $\zeta_1(\check{\mathfrak{A}}) \subset \zeta_2(\check{\mathfrak{A}})$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}}, \mathcal{M}, \widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi, w \in \check{\mathfrak{A}}$, $t \gg \theta$ and $t \in \operatorname{int}(O)$:*

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1^m\varpi, \zeta_1^m w, \zeta_1^m w, t)} - 1\right\} \le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi, \zeta_2 w, \zeta_2 w, t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi, \zeta_1^m\varpi, \zeta_1^m\varpi, t)} - 1\right\} \\ + k_3\left\{\frac{1}{\mathcal{M}(\zeta_2 w, \zeta_1^m w, \zeta_1^m w, t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi, \zeta_1^m w, \zeta_1^m w, t)} - 1\right\} \\ + k_5\left\{\frac{1}{\mathcal{M}(\zeta_2 w, \zeta_1^m\varpi, \zeta_1^m\varpi, t)} - 1\right\},\end{aligned} \tag{3.5}$$

where $k_i \in [0, \infty)$, for all $i = 1, \dots, 5$ with $\sum_{i=1}^{5} k_i < 1$ and $k_2 = k_3$ or $k_4 = k_5$.

If mappings satisfy the conditions with $\zeta_1(\zeta_2) = \zeta_2(\zeta_1)$:

$(\mathrm{A}^{3.2.1})$ *$\check{\mathfrak{A}}$ is complete and ζ_2 is continuous,*

$(\mathrm{A}^{3.2.2})$ *$\zeta_1(\check{\mathfrak{A}})$ and $\zeta_2(\check{\mathfrak{A}})$ are complete.*

Then, maps ζ_1 and ζ_2 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. By Theorem 3.1 and Corollary 2.7, we get a point $u \in \check{\mathfrak{A}}$ such that

$$\zeta_1 u = \zeta_1^m u = u. \tag{3.6}$$

Therefore, from eq. (3.5), we have

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1 u, u, u, t)} - 1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1(\zeta_1^m u), \zeta_1^m u, \zeta_1^m u, t)} - 1\right\} \\ &= \left\{\frac{1}{\mathcal{M}(\zeta_1^m(\zeta_1 u), \zeta_1^m u, \zeta_1^m u, t)} - 1\right\} \\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2(\zeta_1 u), \zeta_2 u, \zeta_2 u, t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\zeta_2(\zeta_1 u), \zeta_1^m(\zeta_1 u), \zeta_1^m(\zeta_1 u), t)} - 1\right\} \\ &\quad + k_3\left\{\frac{1}{\mathcal{M}(\zeta_2(\zeta_1 u), \zeta_1^m u, \zeta_1^m u, t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_2(\zeta_1 u), \zeta_1^m u, \zeta_1^m u, t)} - 1\right\} \\ &\quad + k_5\left\{\frac{1}{\mathcal{M}(\zeta_2 u, \zeta_1^m(\zeta_1 u), \zeta_1^m(\zeta_1 u), t)} - 1\right\} \\ &= k_1\left\{\frac{1}{\mathcal{M}(\zeta_1(\zeta_2 u), \zeta_2 u, \zeta_2 u, t)} - 1\right\} + k_2\left\{\frac{1}{\mathcal{M}(\zeta_1(\zeta_2 u), \zeta_1(\zeta_1^m u), \zeta_1(\zeta_1^m u), t)} - 1\right\} \\ &\quad + k_3\left\{\frac{1}{\mathcal{M}(\zeta_2 u, u, u, t)} - 1\right\} + k_4\left\{\frac{1}{\mathcal{M}(\zeta_1(\zeta_2 u), u, u, t)} - 1\right\} \\ &\quad + k_5\left\{\frac{1}{\mathcal{M}(\zeta_2 u, \zeta_1(\zeta_1^m u), \zeta_1(\zeta_1^m u), t)} - 1\right\}\end{aligned}$$

$$= k_1\left\{\frac{1}{\mathcal{M}(\zeta_1 u,u,u,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_1 u,\zeta_1 u,\zeta_1 u,t)}-1\right\} + k_3\left\{\frac{1}{\mathcal{M}(\zeta_2 u,u,u,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_1 u,u,u,t)}-1\right\} + k_5\left\{\frac{1}{\mathcal{M}(u,\zeta_1 u,\zeta_1 u,t)}-1\right\}$$

$$= (k_1+k_4+k_5)\left\{\frac{1}{\mathcal{M}(\zeta_1 u,u,u,t)}-1\right\}.$$

Since $k_1+k_4+k_5<1$, therefore $\mathcal{M}(\zeta_1 u,u,u,t)=1$, for $t\gg\theta$.

Thus, $\zeta_2 u=\zeta_1 u=u$.

Hence, u is the CFP of maps ζ_1 and ζ_2.

Uniqueness: To prove uniqueness of FP, let $\mathfrak{u}_0$ be another FP such that $\zeta_1\mathfrak{u}_0=\zeta_2\mathfrak{u}_0=\mathfrak{u}_0$ and $\zeta_1^m\mathfrak{u}_0=\zeta_2\mathfrak{u}_0=\mathfrak{u}_0$ as in eq. (3.6). Then from eq. (3.5), for $t\gg\theta$, we have

$$\left\{\frac{1}{\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)}-1\right\}=\left\{\frac{1}{\mathcal{M}(\zeta_1^m\mathfrak{u}_0,\zeta_1^m\xi,\zeta_1^m\xi,t)}-1\right\}$$

$$\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{u}_0,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{u}_0,\zeta_1^m\mathfrak{u}_0,\zeta_1^m\mathfrak{u}_0,t)}-1\right\} + k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1^m\xi,\zeta_1^m\xi,t)}-1\right\}+k_4\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{u}_0,\zeta_1^m\xi,\zeta_1^m\xi,t)}-1\right\} + k_5\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1^m\mathfrak{u}_0,\zeta_1^m\mathfrak{u}_0,t)}-1\right\}$$

$$=(k_1+k_4+k_5)\left\{\frac{1}{\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)}-1\right\}.$$

Since $(k_1+k_4+k_5)<1$ as $\sum_{i=1}^{5}k_i<1$, then $\mathcal{M}(\mathfrak{u}_0,\xi,\xi,t)=1$, i.e., $\mathfrak{u}_0=\xi$.

Thus, we proved that CFP is unique for both maps. □

Theorem 3.3. *Consider that ζ_1 and ζ_2 are two self-mappings on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies following condition for every $\varpi,w\in\check{\mathfrak{A}}$, $t\gg\theta$ and $t\in\operatorname{int}(O)$:*

$$\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_1 w,\zeta_1 w,t)}-1\right\}\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_1\varpi,\zeta_1\varpi,t)}-1\right\} + k_3\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2 w,\zeta_1 w,\zeta_1 w,t),\mathcal{M}(\zeta_2\varpi,\zeta_1\varpi,\zeta_1\varpi,t))}-1\right\}, \tag{3.7}$$

where $k_i\in[0,\infty)$, for all $i=1,2,3$ with $\sum_{i=1}^{3}k_i<1$. If maps satisfy: $\zeta_1(\check{\mathfrak{A}})\subset\zeta_2(\check{\mathfrak{A}})$ and pairs $\{\zeta_1,\zeta_2\}$ are weakly compatible. Then, maps ζ_1 and ζ_2 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. In complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$, suppose $\mathfrak{p}_0\in\check{\mathfrak{A}}$ any arbitrary point. As from the given hypothesis $\zeta_1(\check{\mathfrak{A}})\subset\zeta_2(\check{\mathfrak{A}})$, we construct a sequence $\{\mathfrak{p}_m\}$ from $\zeta_2(\check{\mathfrak{A}})$ such that $\mathfrak{q}_{m+1}=\zeta_2\mathfrak{p}_{m+1}=\zeta_1\mathfrak{p}_m$, for each non-negative integer m. Now, from eq. (3.7),

$$\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m,\zeta_2\mathfrak{p}_{m+1},\zeta_2\mathfrak{p}_{m+1},t)}-1\right\}$$

$$= \left\{ \frac{1}{\mathcal{M}(\zeta_1\mathfrak{p}_{m-1}, \zeta_1\mathfrak{p}_m, \zeta_1\mathfrak{p}_m, t)} - 1 \right\}$$
$$\le k_1 \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_m, t)} - 1 \right\} + k_2 \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_1\mathfrak{p}_{m-1}, \zeta_1\mathfrak{p}_{m-1}, t)} - 1 \right\}$$
$$+ k_3 \left\{ \frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_1\mathfrak{p}_m, \zeta_1\mathfrak{p}_m, t), \mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_1\mathfrak{p}_{m-1}, \zeta_1\mathfrak{p}_{m-1}, t))} - 1 \right\}$$
$$\le k_1 \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_m, t)} - 1 \right\} + k_2 \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_m, t)} - 1 \right\}$$
$$+ k_3 \left\{ \frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_{m+1}, \zeta_2\mathfrak{p}_{m+1}, t), 1)} - 1 \right\}.$$

On solving, we imply that

$$\left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_{m+1}, \zeta_2\mathfrak{p}_{m+1}, t)} - 1 \right\} \le \mu \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_m, t)} - 1 \right\},$$

where $\mu = \left\{ \frac{k_1+k_1}{1-k_3} \right\}$.

By continues process, for $t \gg \theta$, we deduce that

$$\left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_{m+1}, \zeta_2\mathfrak{p}_{m+1}, t)} - 1 \right\} \le \mu \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_m, t)} - 1 \right\}$$
$$\vdots$$
$$\le \mu^m \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_0, \zeta_2\mathfrak{p}_1, \zeta_2\mathfrak{p}_1, t)} - 1 \right\},$$

which shows that sequence $\{\zeta_2\mathfrak{p}_m\}$ is an $\mathcal{M}$-FCC condition.

Hence, for $t \gg \theta$,

$$\lim_{m\to\infty} \mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_{m+1}, \zeta_2\mathfrak{p}_{m+1}, t) = 1. \tag{3.8}$$

Since FCM $\mathcal{M}$ is triangular, for $n > m \ge m_0$, we can bring that

$$\left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_n, \zeta_2\mathfrak{p}_n, t)} - 1 \right\} \le \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m, \zeta_2\mathfrak{p}_{m-1}, \zeta_2\mathfrak{p}_{m-1}, t)} - 1 \right\} + \dots$$
$$+ \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{n-1}, \zeta_2\mathfrak{p}_n, \zeta_2\mathfrak{p}_n, t)} - 1 \right\}$$
$$\le (\mu^m + \mu^{m+1} + \dots + \mu^{n-1}) \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_0, \zeta_2\mathfrak{p}_1, \zeta_2\mathfrak{p}_1, t)} - 1 \right\}$$
$$\le \frac{\mu^m}{1-\mu} \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_0, \zeta_2\mathfrak{p}_1, \zeta_2\mathfrak{p}_1, t)} - 1 \right\} \to 0, \quad \text{as } m \to \infty.$$

Thus, $\{\zeta_2\mathfrak{p}_m\}$ it is a Cauchy sequence.

By using completeness of $\check{\mathfrak{A}}$, there exists $\xi, u \in \check{\mathfrak{A}}$ such that as $m \to \infty$, we can get

$$\mathfrak{q}_m = \zeta_2\mathfrak{p}_m \to u = \zeta_2\xi,$$
$$\lim_{m\to\infty} \mathcal{M}(\zeta_2\mathfrak{p}_m, u, u, t) = 1, \quad \text{for } t \gg \theta. \tag{3.9}$$

Since FCM $\mathcal{M}$ is triangular, we have

$$\left\{ \frac{1}{\mathcal{M}(\zeta_2\xi, \zeta_1\xi, \zeta_1\xi, t)} - 1 \right\} \le \left\{ \frac{1}{\mathcal{M}(\zeta_2\xi, \zeta_2\mathfrak{p}_{m+1}, \zeta_2\mathfrak{p}_{m+1}, t)} - 1 \right\} + \left\{ \frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m+1}, \zeta_1\xi, \zeta_1\xi, t)} - 1 \right\}. \tag{3.10}$$

From eqs. (3.7) to (3.9), we can deduce that

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m+1},\zeta_1\xi,\zeta_1\xi,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1\mathfrak{p}_m,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\ &\quad +k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m,\zeta_1\mathfrak{p}_m,\zeta_1\mathfrak{p}_m,t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\xi,\zeta_2\mathfrak{p}_m,\zeta_2\mathfrak{p}_m,t),\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t))}-1\right\}\\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}\\ &\quad +k_2\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_m,\zeta_2\mathfrak{p}_{m+1},\zeta_2\mathfrak{p}_{m+1},t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\xi,\zeta_2\mathfrak{p}_m,\zeta_2\mathfrak{p}_m,t),\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t))}-1\right\}\\ &\to k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\},\quad \text{as } m\to\infty\end{aligned}$$

this implies that

$$\lim_{m\to\infty}\sup\left\{\frac{1}{\mathcal{M}(\zeta_2\mathfrak{p}_{m+1},\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\le k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\quad \text{as } m\to\infty.$$

From eqs. (3.9) and (3.10), we imply that

$$\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\le k_3\left\{\frac{1}{\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}.$$

As, $k_3<1$ with $\sum_{i=1}^{3}k_i<1$, implies $\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t)=1$.

Therefore, $\zeta_2\xi=\zeta_1\xi=u$.

Although, using weakly compatibility of maps ζ_1 and ζ_2, one can have

$$\zeta_1u=\zeta_1(\zeta_2\xi)=\zeta_2(\zeta_1\xi)=\zeta_2u.$$

Thus, from eq. (3.7),

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1u,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2u,\zeta_1u,\zeta_1u,t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t),\mathcal{M}(\zeta_2\xi,\zeta_1u,\zeta_1u,t))}-1\right\}\\ &= (k_1+k_3)\left\{\frac{1}{\mathcal{M}(\zeta_1u,u,u,t)}-1\right\}.\end{aligned}$$

Since $k_1+k_3<1$, therefore $\mathcal{M}(\zeta_1u,u,u,t)=1$, for $t\gg\theta$ implies that $\zeta_2u=\zeta_1u=u$.

Hence, u is the CFP of maps ζ_1 and ζ_2.

Uniqueness: To prove uniqueness of FP, let $\mathfrak{u}_0$ be another FP of the self-maps ζ_1 and ζ_2, i.e., $\zeta_1\mathfrak{u}_0=\zeta_2\mathfrak{u}_0=\mathfrak{u}_0$.

Then from eq. (3.7), for $t \gg \theta$, we have

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(u_0,\xi,\xi,t)}-1\right\} &= \left\{\frac{1}{\mathcal{M}(\zeta_1 u_0,\zeta_1\xi,\zeta_1\xi,t)}-1\right\}\\ &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2 u_0,\zeta_2\xi,\zeta_2\xi,t)}-1\right\}+k_2\left\{\frac{1}{\mathcal{M}(\zeta_2 u_0,\zeta_1 u_0,\zeta_1 u_0,t)}-1\right\}\\ &\quad +k_3\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2\xi,\zeta_1\xi,\zeta_1\xi,t),\mathcal{M}(\zeta_2\xi,\zeta_1 u_0,\zeta_1 u_0,t))}-1\right\}\\ &= (k_1+k_3)\left\{\frac{1}{\mathcal{M}(u_0,\xi,\xi,t)}-1\right\}.\end{aligned}$$

Since $k_1+k_3<1$ as $\sum_{i=1}^{3} k_i<1$,

Thus, $\mathcal{M}(u_0,\xi,\xi,t)=1$, i.e., $u_0=\xi$.

Hence, we established that CFP is unique. □

Corollary 3.4. *Consider that ζ_1 and ζ_2 are two self-maps on $\check{\mathfrak{A}}$. FCM $\mathcal{M}$ is triangular in a complete $\mathcal{M}$-FCM space $(\check{\mathfrak{A}},\mathcal{M},\widehat{\mathfrak{S}})$ satisfies the following condition for any $\varpi,w\in\check{\mathfrak{A}}$, $t\gg\theta$ and $t\in \mathrm{int}(O)$:*

$$\begin{aligned}\left\{\frac{1}{\mathcal{M}(\zeta_1\varpi,\zeta_1 w,\zeta_1 w,t)}-1\right\} &\le k_1\left\{\frac{1}{\mathcal{M}(\zeta_2\varpi,\zeta_2 w,\zeta_2 w,t)}-1\right\}\\ &\quad +k_2\left\{\frac{1}{\widehat{\mathfrak{S}}(\mathcal{M}(\zeta_2 w,\zeta_1 w,\zeta_1 w,t),\mathcal{M}(\zeta_2 w,\zeta_1\varpi,\zeta_1\varpi,t))}-1\right\},\end{aligned} \tag{3.11}$$

where $k_i\in[0,\infty)$, for all $i=1,2$ with $\sum_{i=1}^{2} k_i<1$. If maps satisfy: $\zeta_1(\check{\mathfrak{A}})\subset\zeta_2(\check{\mathfrak{A}})$ and pairs $\{\zeta_1,\zeta_2\}$ are weakly compatible. Then, maps ζ_1 and ζ_2 have a unique CFP in $\check{\mathfrak{A}}$.

Proof. Proof of corollary can be illustrated by taking $k_2=0$ in Theorem 3.3. □

4. Discussion of Results

Some related outcomes to these CFP results were already existing for various mappings in FCM spaces, we have extended the research work of Rehman *et al*. [12], in generalized FCM spaces with some suitable examples. These results may be extended to intuitionistic fuzzy cone metric (Intuitionistic FCM) spaces. The outcomes of maps can also be influenced or modified to accommodate multivalued maps.

5. Conclusions

In the paper, we extended and generalized CFP theorems in generalized FCM spaces using compatible and weakly compatible self-maps under continuity and without using continuity of map. Also, show related examples to these results. Some FCC theorems are also established to improve and extend existing results in the form of generalized setting.

Nomenclature

FCM	: Fuzzy Cone Metric	FCC	: Fuzzy Cone Contraction
FMS	: Fuzzy Metric Space	FP	: Fixed Point
CFP	: Common Fixed Point		

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *Journal of Mathematical Analysis and Applications* **341**(1) (2008), 416 – 420, DOI: 10.1016/j.jmaa.2007.09.070.

[2] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* **64**(3) (1994), 395 – 399, DOI: 10.1016/0165-0114(94)90162-7.

[3] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *Journal of Mathematical Analysis and Applications* **332**(2) (2007), 1468 – 1476, DOI: 10.1016/j.jmaa.2005.03.087.

[4] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, *Indian Journal of Pure and Applied Mathematics* **29**(3) (1998), 227 – 238.

[5] F. Kiany and A. Amini-Harandi, Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces, *Fixed Point Theory and Applications* **2011** (2011), article number 94, DOI: 10.1186/1687-1812-2011-94.

[6] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika* **11**(5) (1975), 336 – 344, URL: https://www.kybernetika.cz/content/1975/5/336/paper.pdf.

[7] T. Öner, Some topological properties of fuzzy cone metric spaces, *Journal of Nonlinear Science and Applications* **9**(3) (2016), 799 – 805, DOI: 10.22436/jnsa.009.03.08.

[8] T. Öner, M. B. Kandemir and B. Tanay, Fuzzy cone metric spaces, *Journal of Nonlinear Sciences and Applications* **8**(5) (2015), 610 – 616, DOI: 10.22436/jnsa.008.05.13.

[9] R. P. Pant, Common fixed points of noncommuting mappings, *Journal of Mathematical Analysis and Applications* **188**(2) (1994), 436 – 440, DOI: 10.1006/jmaa.1994.1437.

[10] B. D. Pant and S. Chauhan, Common fixed point theorems for two pairs of weakly compatible mappings in Menger spaces and fuzzy metric spaces, *Scientific Studies and Research. Series Mathematics and Informatics* **21**(2) (2011), 81 – 96, URL: https://pubs.ub.ro/article/3464.

[11] A. Razani and M. Shirdaryazdi, Some results on fixed points in the fuzzy metric space, *Journal of Applied Mathematics and Computing* **20** (2006), 401 – 408, DOI: 10.1007/BF02831947.

[12] S. U. Rehman, S. Jabeen, F. Abbas, H. Ullah and I. Khan, Common fixed point theorems for compatible and weakly compatible maps in fuzzy cone metric spaces, *Annals of Fuzzy Mathematics and Informatics* **19**(1) (2020), 1 – 19, DOI: 10.30948/afmi.2020.19.1.1.

[13] S. U. Rehman, Y. Li, S. Jabeen and T. Mahmood, Common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces, *Abstract and Applied Analysis* **2019**(1) (2019), 2841606, DOI: 10.1155/2019/2841606.

[14] Z. Sadeghi, S. M. Vaezpour, C. Park, R. Saadati and C. Vetro, Set-valued mappings in partially ordered fuzzy metric spaces, *Journal of Inequalities and Applications* **2014** (2014), article number 157, DOI: 10.1186/1029-242X-2014-157.

[15] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific Journal of Mathematics* **10**(1) (1960), 385 – 389, DOI: 10.2140/pjm.1960.10.313.

[16] T. Som, Some results on common fixed point in fuzzy metric spaces, *Soochow Journal of Mathematics* **33**(4) (2007), 553 – 561.

[17] M. Suganthi, M. Jayaraman and A. Ramachandran, Generalized contraction theorems in M-fuzzy cone metric spaces, *Journal of Innovative Applied Mathematics and Computational Sciences* **2**(3) (2022), 29 – 40, DOI: 10.58205/jiamcs.v2i3.48.

[18] L. A. Zadeh, Fuzzy sets, *Information and Control* **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.