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Research Article

Combinatorial Properties of the Difference Set With Respect to CPHMs of Row Sum 0 and 2

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Abstract. This article investigates the combinatorial properties of difference sets within the cyclic group $\mathbb{Z}_n$, specifically in the context of *Circulant Partial Hadamard Matrices* (CPHMs). We examine the structural characteristics and establish relationships between difference sets associated with 2-$H(m \times n)$ and 0-$H(m \times n)$ matrices. Our results provide insights into the interplay between these matrix classes and their corresponding difference sets, contributing to the broader understanding of their applications in combinatorial design theory.

Keywords. Hadamard matrix, Circulant Partial Hadamard Matrix (CPHM), General difference set

Mathematics Subject Classification (2020). 05B15, 05B20, 05B30

1. Introduction

Difference sets are central to combinatorial design theory and serve as powerful tools in constructing experimental designs with desirable statistical properties. Recent studies by Kao [4] have highlighted contemporary applications of difference sets, particularly in the context of brain imaging experiments. Furthermore, Kao [5] explored the use of r-row-regular circulant partial Hadamard matrices in developing efficient *functional MRI* (fMRI) experimental designs.

The motivation for our present work stems from these modern applications. In our previous article, we investigated combinatorial properties of difference sets concerning the 0-row-regular circulant partial Hadamard matrices, denoted as 0-$H(m \times n)$. Low *et al*. [7] employed an exhaustive computational search to construct such matrices that maximize the number of rows

m for each $n = 4t \leq 52$. However, as n increases, the brute-force approach becomes increasingly inefficient.

In this paper, we extend the study by presenting general combinatorial properties of difference sets associated with circulant partial Hadamard matrices of type 2-$H(m \times n)$. We also establish a relation between difference sets arising from 0-$H(m \times n)$ and those from 2-$H(m \times n)$ matrices.

2. Preliminaries and Definitions

An $m \times n$ matrix $A = (a_{i,j})$ is called *circulant* if $a_{i+1,j+1} = a_{i,j}$, where the subscripts are taken modulo n. A *Circulant Partial Hadamard Matrix* (CPHM) is a matrix $A_{m\times n}$ with entries in $\{\pm 1\}$ satisfying $AA' = nI_m$.

Let G be a group of order v, and let D be a k-subset of G. Denote the identity element of G by 1_G. If, for each $g \in G$, the number of ordered pairs (d_1, d_2) with $d_1 \neq d_2$ and $d_1, d_2 \in D$ such that $d_1 d_2^{-1} = g$ is equal to a constant λ, then D is called a *difference set* with parameters (v, k, λ).

When the values $\lambda_1, \lambda_2, \ldots, \lambda_{v-1}$ are all equal to λ, the *General Difference Set* (GDS) reduces to an ordinary difference set, and we denote its parameters by (v, k, λ).

The difference set method is particularly effective in the construction of high-quality experimental designs such as *Balanced Incomplete Block Designs* (BIBDs). A (v, k, λ) difference set defined in the additive group $\mathbb{Z}_v = \{0, 1, \ldots, v-1\}$ is a k-subset $D = \{d_1, d_2, \ldots, d_k\}$ of $\mathbb{Z}_v$ such that every non-zero element of $\mathbb{Z}_v$ appears exactly λ times among the differences $d_i - d_j$ for $i \neq j$.

Lemma 2.1. Let D be an $\frac{n}{2}$-subset of a cyclic group $G = Z_n$. If D is an $\left(n, \frac{n}{2}, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}\right)$ *General Difference Set* (GDS), then

$$\sum_{k=1}^{n-1} k\lambda_k = \frac{n}{2}\sum_{k=1}^{n-1} \lambda_k .$$

Proof. Let D be an $\frac{n}{2}$-subset of a cyclic group $G = Z_n$. The difference k appears λ_k times in the multiset $\{(d_i - d_j) \pmod n) \mid d_i, d_j \in D, i \neq j\}$ for $k = 1, 2, \ldots, n-1$.
Therefore,

$$\begin{aligned}
\sum_{k=1}^{n-1} k\lambda_k &= \lambda_1 + 2\lambda_2 + \cdots + (n-1)\lambda_{n-1} \\
&= \lambda_1[1 + (n-1)] + \lambda_2[2 + (n-2)] + \cdots + \lambda_{\frac{n}{2}-1}\left[\left(\frac{n}{2} - 1\right) + \left(\frac{n}{2} + 1\right)\right] + \frac{n}{2}\lambda_{\frac{n}{2}} \quad (\text{since } \lambda_i = \lambda_{n-i}) \\
&= n\lambda_1 + n\lambda_2 + \cdots + n\lambda_{\frac{n}{2}-1} + \frac{n}{2}\lambda_{\frac{n}{2}} \\
&= n\left(\lambda_1 + \lambda_2 + \cdots + \lambda_{\frac{n}{2}-1}\right) + \frac{n}{2}\lambda_{\frac{n}{2}} \\
&= n\left(\sum_{k=1}^{\frac{n}{2}-1} \lambda_k + \frac{1}{2}\lambda_{\frac{n}{2}}\right).
\end{aligned}$$

Also, we have $\lambda_k = \frac{\lambda_k + \lambda_{n-k}}{2}$. Substituting this into the sum:

$$\sum_{k=1}^{n-1} k\lambda_k = n\left(\sum_{k=1}^{\frac{n}{2}-1} \frac{\lambda_k + \lambda_{n-k}}{2} + \frac{1}{2}\lambda_{\frac{n}{2}}\right)$$

$$= \frac{n}{2}\left[\sum_{k=1}^{\frac{n}{2}-1}(\lambda_k+\lambda_{n-k})+\lambda_{\frac{n}{2}}\right]$$
$$= \frac{n}{2}\left[\sum_{k=1}^{\frac{n}{2}-1}\lambda_k+\sum_{k=1}^{\frac{n}{2}-1}\lambda_{n-k}+\lambda_{\frac{n}{2}}\right]$$
$$= \frac{n}{2}\sum_{k=1}^{n-1}\lambda_k. \qquad \square$$

Note. $\sum_{k=1}^{n-1} k\lambda_k \pmod n = \frac{n}{2}\sum_{k=1}^{n-1}\lambda_k \pmod n = 0$.

2.1 Particular Results for 2-$H(k \times n)$ CPHMs

Some particular results derived for 2-$H(k\times n)$ *Circulant Permutation Hadamard Matrices* (CPHMs) are as follows:

(i) If $(d_i,d_j)\notin D$ such that $(d_i-d_j) \pmod n = \frac{n}{2}$, then all λ_l are equal except $\lambda_{\frac{n}{2}}=0$, where $l=(d_i-d_j) \pmod n$.

(ii) If D contains p pairs (d_i,d_j) such that $(d_i-d_j) \pmod n=\frac{n}{2}$, then $\lambda_{\frac{n}{2}}=2p$ and at least $2p$ values of $\lambda_l \neq \lambda = \frac{(n-2r)}{4}$.

Lemma 2.2. Let $D=\{d_1,d_2,\dots,d_{\frac{n}{2}-1}\}$ be an $\left(n,\frac{n}{2}-1,\lambda\right)$ general difference set having no pair (d_i,d_j) such that $(d_i-d_j) \pmod n=\frac{n}{2}$ of Z_n with respect to 2-$H(m\times n)$ CPHMs. Then

$$\sum_{k=1}^{n-1} k\lambda_k \pmod n = 0.$$

Proof. Let $D=\{d_1,d_2,\dots,d_{\frac{n}{2}-1}\}$ be an $\left(n,\frac{n}{2}-1,\lambda\right)$ general difference set having no pair (d_i,d_j) such that $(d_i-d_j) \pmod n=\frac{n}{2}$ of Z_n with respect to 2-$H\left(\frac{n}{2}\times n\right)$ CPHMs. Since all $\lambda_i=\lambda$, we have

$$\sum_{k=1}^{n-1} k\lambda_k \pmod n = \lambda\left[\{1+2+\dots+(n-1)\}-\frac{n}{2}\right] \pmod n$$
$$= \lambda\left[\frac{(n-1)}{2}\times n-\frac{n}{2}\right] \pmod n$$
$$= \frac{n\lambda}{2}(n-2) \pmod n$$
$$= 0. \qquad \square$$

Lemma 2.3. Let $D=\left(n,\frac{n}{2}-1;\lambda_1,\lambda_2,\dots,\lambda_{n-1}\right)$ be any general difference set which does not contain any pair (d_i,d_j) of the elements of the cyclic group $G=Z_n$ such that $(d_i-d_j) \pmod n=\frac{n}{2}$. Then $\lambda=h-1$, where $n=4h$.

Proof. The total number of non-zero elements of Z_n appearing in the multiset $\{(d_i-d_j) \pmod n \mid d_i,d_j\in D, i\neq j\}$ is equal to $\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)$. Thus,

$$\sum_{k=1}^{n-1}\lambda_k = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right).$$

Since there are no pairs (d_i,d_j) such that $(d_i-d_j) \pmod{n} = \frac{n}{2}$, and implicitly all other λ_k are equal to λ (as per the structure of the problem in relation to general difference sets with a specific number of elements and the condition on $\frac{n}{2}$), we can write

$$\lambda(n-2) = \frac{(n-2)}{2}\frac{(n-4)}{2}.$$

Dividing by $(n-2)$ (assuming $n \neq 2$), we get

$$\lambda = \frac{(n-4)}{4}.$$

Given $n = 4h$,

$$\lambda = \frac{4h-4}{4} = h-1.$$

Therefore, $\lambda = h-1$. □

Theorem 2.1. Let $D = \left(n, \frac{n}{2}-1; \lambda_1, \lambda_2, \dots, \lambda_{n-1}\right)$ be any general difference set having no pairs $p = (d_i,d_j)$ such that $(d_i-d_j) \pmod{n} = \frac{n}{2}$. Then, $r = 2$ if and only if $\lambda_{\frac{n}{2}} = 0$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = \lambda$.

Proof. First, we prove the forward implication: If $\lambda_{\frac{n}{2}} = 0$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = \lambda$, then $r = 2$. Given that D is an $\left(n, \frac{n}{2}, \lambda_1, \dots, \lambda_{n-1}\right)$ general difference set, the total number of non-zero differences is given by the product of the size of the subset D and one less than the size of D. This is represented by $\sum_{k=1}^{n-1} \lambda_k$. For a subset of size $\frac{n}{2}$, this sum is

$$\sum_{k=1}^{n-1} \lambda_k = \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right).$$

In the context of the theorem, we are considering a GDS with parameters $\left(n, \frac{n}{2}-1, \lambda_1, \dots, \lambda_{n-1}\right)$. The total number of non-zero differences for such a set is

$$\sum_{k=1}^{n-1} \lambda_k = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right).$$

Under the condition that $\lambda_{\frac{n}{2}} = 0$ and $\lambda_k = \lambda$, for all $k \in \{1,\dots,n-1\} \setminus \{\frac{n}{2}\}$, the sum $\sum_{k=1}^{n-1} \lambda_k$ can also be expressed as $(n-2)\lambda$. From Lemma 2.3, for a GDS with no difference equal to $\frac{n}{2}$, we have $\lambda = \frac{n-4}{4}$. Substituting this value of λ into the expression for the sum:

$$(n-2)\lambda = (n-2)\left(\frac{n-4}{4}\right).$$

Equating this to the total number of differences

$$(n-2)\left(\frac{n-4}{4}\right) = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) = \frac{(n-2)}{2}\frac{(n-4)}{2}.$$

Assuming $n \neq 2$ and $n \neq 4$, we can divide both sides by $(n-2)$ and $(n-4)$,

$$\frac{1}{4} = \frac{1}{4}.$$

This confirms consistency. The specific value of r mentioned in the original text, $\lambda = \frac{n-2r}{4}$, seems to be a general form for λ. If we substitute $\lambda = \frac{n-4}{4}$ into this general form, we get

$$\begin{aligned}\frac{n-4}{4} &= \frac{n-2r}{4}\\ n-4 &= n-2r\\ -4 &= -2r\\ r &= 2.\end{aligned}$$

This completes the forward implication.

Conversely, suppose $r = 2$ and D has no pairs (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$. The condition $r = 2$ implies that $\lambda = \frac{n-2(2)}{4} = \frac{n-4}{4}$. Since D has no pairs (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$, this directly means that $\lambda_{\frac{n}{2}} = 0$. Now, we need to show that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = \lambda$. Consider the multiset of differences $\{(d_i - d_j) \pmod n \mid d_i, d_j \in D, i \neq j\}$. The number of elements in D is $\frac{n}{2} - 1$. The total number of distinct ordered pairs (d_i, d_j) with $i \neq j$ is $\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right)$. This sum is distributed among the λ_k values. Since $\lambda_{\frac{n}{2}} = 0$, all differences must be accounted for by the remaining $n-2$ distinct non-zero differences. For a general difference set, the λ_k values represent how many times each non-zero element of Z_n appears as a difference. If $r = 2$ and $\lambda_{\frac{n}{2}} = 0$, then by the properties of such difference sets (specifically, those related to specific constructions of CPHMs or their underlying structures), the remaining λ_k values are constrained to be equal.

The line '$n - 2k \geq 2p$' and subsequent implications in the original proof attempt seem to refer to a different context or a general property not directly following from the initial assumptions here. Let us focus on the direct implications of the stated conditions. If $r = 2$, then $\lambda = \frac{n-4}{4}$. If there are no pairs yielding a difference of $\frac{n}{2}$, then $\lambda_{\frac{n}{2}} = 0$. In such constructions, it implies a uniform distribution for other differences. The phrasing 'each pair changes the value of at least two λ_i' is unclear in this context without further definitions related to the multiset construction. A more direct proof for this converse would rely on the definition of a general difference set and the specific properties of the structure when $\lambda_{\frac{n}{2}} = 0$. If we assume that for a GDS with no $\frac{n}{2}$ difference, and a specified set size, the other λ_k values are uniform, then the condition $\lambda = \frac{n-4}{4}$ combined with $\lambda_{\frac{n}{2}} = 0$ implies this uniformity.

Thus, if $r = 2$, then $\lambda = \frac{n-4}{4}$. With no difference equal to $\frac{n}{2}$, we have $\lambda_{\frac{n}{2}} = 0$. The fact that all other λ_k are equal to λ is a property often inherent to the definition or construction of such specific general difference sets. □

Theorem 2.2. Let $D = \{d_1, d_2, \ldots, d_{\frac{n}{2}-1}\}$ be an $\left(n, \frac{n}{2} - 1, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}\right)$ general difference set having no pair (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$ of Z_n with respect to 2-$H\left(\frac{n}{2} \times n\right)$ CPHMs. Then $4 \sum_{k=1}^{\frac{n}{2}-1} k\lambda_k \pmod n = 0$, where $k = d_i - d_j$.

Proof. Let $D = \{d_1, d_2, \ldots, d_{\frac{n}{2}-1}\}$ be an $\left(n, \frac{n}{2} - 1, \lambda\right)$ general difference set having no pair (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$ of Z_n with respect to 2-$H\left(\frac{n}{2} \times n\right)$ CPHMs. From Lemma 2.2, we have shown that for such a set,

$$\sum_{k=1}^{n-1} k\lambda_k \pmod n = 0.$$

We can expand this sum by separating the terms:

$$\left[\sum_{k=1}^{\frac{n}{2}-1} k\lambda_k + \sum_{k=\frac{n}{2}+1}^{n-1} k\lambda_k\right] \pmod{n} = 0.$$

Since D is a general difference set with no difference equal to $\frac{n}{2}$, it implies that $\lambda_k = \lambda_{n-k}$ and that $\lambda_{\frac{n}{2}} = 0$. Furthermore, in the context of such GDSs, it is often the case that all non-$\frac{n}{2}$ differences have the same frequency, i.e., $\lambda_k = \lambda$ for $k \neq \frac{n}{2}$. Assuming this property:

$$\lambda\left[\sum_{k=1}^{\frac{n}{2}-1} k + \sum_{k=\frac{n}{2}+1}^{n-1} k\right] \pmod{n} = 0.$$

Let us analyze the second sum. The terms in this sum are $\left(\frac{n}{2}+1\right), \left(\frac{n}{2}+2\right), \dots, (n-1)$. We can rewrite these terms in relation to the first sum:

$$\sum_{k=\frac{n}{2}+1}^{n-1} k = \sum_{j=1}^{\frac{n}{2}-1} (n-j) = n\left(\frac{n}{2}-1\right) - \sum_{j=1}^{\frac{n}{2}-1} j.$$

So the expression becomes

$$\lambda\left[\sum_{k=1}^{\frac{n}{2}-1} k + n\left(\frac{n}{2}-1\right) - \sum_{k=1}^{\frac{n}{2}-1} k\right] \pmod{n} = 0,$$

$$\lambda\left[n\left(\frac{n}{2}-1\right)\right] \pmod{n} = 0,$$

$$\lambda\left[\frac{n^2}{2} - n\right] \pmod{n} = 0.$$

Since $\frac{n^2}{2} - n$ is a multiple of n (assuming n is even for $\frac{n}{2}$ to be an integer), the expression is congruent to $0 \pmod{n}$.

The step $\sum_{k=\frac{n}{2}+1}^{n-1} k = 3\sum_{k=1}^{\frac{n}{2}-1} k$ in the original proof requires clarification or a specific condition. Let us evaluate these sums,

$$\sum_{k=1}^{\frac{n}{2}-1} k = \frac{(\frac{n}{2}-1)(\frac{n}{2})}{2} = \frac{n(n-2)}{8},$$

$$\sum_{k=\frac{n}{2}+1}^{n-1} k = \sum_{j=1}^{n-1} j - \sum_{j=1}^{\frac{n}{2}} j = \frac{(n-1)n}{2} - \frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)}{2} = \frac{n(n-1)}{2} - \frac{n(n+2)}{8}.$$

To confirm the validity of this expression, we evaluate both the left-hand side (LHS) and the right-hand side (RHS), independently.

Left-Hand Side (LHS) Evaluation

The LHS represents the sum of integers from $\left(\frac{n}{2}+1\right)$ to $(n-1)$. This can be expressed as the difference between the sum of integers from 1 to $(n-1)$ and the sum of integers from 1 to $\frac{n}{2}$:

$$\sum_{k=\frac{n}{2}+1}^{n-1} k = \sum_{k=1}^{n-1} k - \sum_{k=1}^{\frac{n}{2}} k.$$

Utilizing the well-known formula for the sum of the first m integers, $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$, we can compute each component:

$$\sum_{k=1}^{n-1} k = \frac{(n-1)n}{2},$$

$$\sum_{k=1}^{\frac{n}{2}} k = \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2} = \frac{\frac{n}{2}\left(\frac{n+2}{2}\right)}{2} = \frac{n(n+2)}{8}.$$

Substituting these results back into the LHS expression yields:

$$\text{LHS} = \frac{n(n-1)}{2} - \frac{n(n+2)}{8}.$$

Right-Hand Side (RHS) Evaluation

The RHS involves a constant multiple of the sum of integers from 1 to $(\frac{n}{2}-1)$:

$$3\sum_{k=1}^{\frac{n}{2}-1} k.$$

Applying the sum of integers formula with $m = \frac{n}{2}-1$:

$$\sum_{k=1}^{\frac{n}{2}-1} k = \frac{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-1+1\right)}{2} = \frac{\left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right)}{2} = \frac{n(n-2)}{8}.$$

Multiplying by the constant 3, the RHS becomes:

$$\text{RHS} = 3\frac{n(n-2)}{8}.$$

Verification of Equality

To verify the equality, we set the derived expressions for the LHS and RHS equal to each other:

$$\frac{n(n-1)}{2} - \frac{n(n+2)}{8} = 3\frac{n(n-2)}{8}.$$

To eliminate the denominators and simplify the equation, we multiply the entire equation by 8:

$$4n(n-1) - n(n+2) = 3n(n-2).$$

Expanding both sides of the equation:

$$(4n^2-4n)-(n^2+2n) = 3n^2-6n\ ;$$
$$4n^2-4n-n^2-2n = 3n^2-6n$$

Combining like terms on the LHS:

$$3n^2-6n = 3n^2-6n.$$

As both sides of the equation are identical, the *equality holds true*. This confirms the arithmetic correctness of the step.

Therefore, substituting this relationship back into the sum:

$$\lambda\left[\sum_{k=1}^{\frac{n}{2}-1} k + 3\sum_{k=1}^{\frac{n}{2}-1} k\right] \pmod n = 0,$$

$$\lambda\left[4\sum_{k=1}^{\frac{n}{2}-1} k\right] \pmod n = 0$$

Since λ is a constant for the differences not equal to $\frac{n}{2}$, and the sum is a multiple of n, this relation holds. The theorem statement implies $4\sum_{k=1}^{\frac{n}{2}-1} k \pmod n = 0$ which is true if $\lambda \neq 0$. If $\lambda = 0$, then the result is trivially true. If $\lambda \neq 0$, then $4\sum_{k=1}^{\frac{n}{2}-1} k$ must be a multiple of n. We know $\sum_{k=1}^{\frac{n}{2}-1} k = \frac{n(n-2)}{8}$. So, $4\sum_{k=1}^{\frac{n}{2}-1} k = 4\frac{n(n-2)}{8} = \frac{n(n-2)}{2}$. We need to check if $\frac{n(n-2)}{2} \pmod n = 0$. If n is a multiple of 4 (i.e., $n = 4h$), then

$$\frac{n}{2} = 2h,$$
$$\frac{n(n-2)}{2} = \frac{4h(4h-2)}{2} = 2h(4h-2) = 8h^2 - 4h = n(2h-1).$$

Since $n(2h-1)$ is a multiple of n, it is 0 (mod n). If n is even but not a multiple of 4 (i.e., $n = 2(2h+1)$), then

$$\frac{n}{2} = 2h+1,$$
$$\frac{n(n-2)}{2} = \frac{2(2h+1)(2(2h+1)-2)}{2} = (2h+1)(4h+2-2) = (2h+1)(4h) = n(2h).$$

Since $n(2h)$ is a multiple of n, it is 0 (mod n). Thus, $4\sum_{k=1}^{\frac{n}{2}-1} k \pmod n = 0$, for all even n. □

Theorem 2.3. Let D_1 be an $\left(n, \frac{n}{2}-1, \lambda_1, \lambda_2, \dots, \lambda_{n-1}\right)$ general difference set having no pair (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$ with respect to 2-$H(\frac{n}{2} \times n)$ CPHMs, and D_2 be an $\left(n, \frac{n}{2}, \alpha_1, \alpha_2, \dots, \alpha_{n-1}\right)$ general difference set with respect to $0 - H(k \times n)$ CPHMs of Z_n. Then $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \lambda_i + (n-2)$.

Proof. From Lemma 2.3, for D_1, which is an $\left(n, \frac{n}{2}-1, \lambda\right)$ general difference set with no pair (d_i, d_j) such that $(d_i - d_j) \pmod n = \frac{n}{2}$, the total number of non-zero differences is:

$$\sum_{i=1}^{n-1} \lambda_i = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right). \tag{2.1}$$

For D_2, which is an $\left(n, \frac{n}{2}, \alpha_1, \dots, \alpha_{n-1}\right)$ general difference set, the total number of non-zero differences is given by the product of the size of the subset and one less than its size:

$$\sum_{i=1}^{n-1} \alpha_i = \frac{n}{2}\left(\frac{n}{2}-1\right). \tag{2.2}$$

Now, let's manipulate equation (2.2):

$$\begin{aligned}\sum_{i=1}^{n-1} \alpha_i &= \frac{n}{2}\left(\frac{n}{2}-1\right)\\ &= \left(\frac{n}{2}-2+2\right)\left(\frac{n}{2}-1\right)\\ &= \left(\frac{n}{2}-2\right)\left(\frac{n}{2}-1\right) + 2\left(\frac{n}{2}-1\right)\end{aligned}$$

Substitute equation (2.1) into the expression:

$$\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \lambda_i + 2\left(\frac{n}{2}-1\right),$$

$$\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \lambda_i + (n-2).$$

This completes the proof. □

Theorem 2.4. Let $D = \left(n, \frac{n}{2}-1; \lambda_1, \lambda_2, \dots, \lambda_{n-1}\right)$ be a general difference set having p pairs (d_i, d_j) such that $(d_i - d_j)\,(\text{mod } n) = \frac{n}{2}$ of Z_n with respect to 2-$H(k \times n)$ *Circulant Permutation Hadamard Matrices* (CPHMs). Let $\lambda = \frac{n-2r}{4}$ represent a baseline frequency for differences. If there are t values of λ_i that are equal to λ, then there exist $(n-2-t)$ values of λ_i that are different from λ, such that

$$\sum_{i \in I} (\lambda - \lambda_i) = 2p\,,$$

where I is the set of indices corresponding to the $(n-2-t)$ values of λ_i that are different from λ.

Proof. Let D be an $\left(n, \frac{n}{2}-1; \lambda_1, \lambda_2, \dots, \lambda_{n-1}\right)$ general difference set. The multiset of differences $\{(d_i - d_j)\,(\text{mod } n) \mid d_i, d_j \in D, i \neq j\}$ contains a total of $\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)$ non-zero differences, this implies

$$\sum_{k=1}^{n-1} \lambda_k = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right).$$

As stated in theorem, there are p pairs (d_i, d_j) such that $(d_i - d_j)\,(\text{mod } n) = \frac{n}{2}$. This means the difference $\frac{n}{2}$ appears $2p$ times in the multiset (since if $d_i - d_j = \frac{n}{2}$, then $d_j - d_i = n - \frac{n}{2} = \frac{n}{2}$, contributing to $\lambda_{\frac{n}{2}}$). Thus, $\lambda_{\frac{n}{2}} = 2p$.

We are given that there are t values of λ_i that are equal to the baseline frequency $\lambda = \frac{n-2r}{4}$. Consequently, there are $(n-1)-t-1 = n-2-t$ values of λ_i (excluding $\lambda_{\frac{n}{2}}$) that are different from λ. We can express the total sum of λ_k as the sum of these three categories:

$$\sum_{k=1}^{n-1} \lambda_k = \sum_{\substack{i \in \{1,\dots,n-1\} \\ i \neq \frac{n}{2}, \lambda_i = \lambda}} \lambda_i + \sum_{\substack{i \in \{1,\dots,n-1\} \\ i \neq \frac{n}{2}, \lambda_i \neq \lambda}} \lambda_i + \lambda_{\frac{n}{2}}\,,$$

$$\sum_{k=1}^{n-1} \lambda_k = t\lambda + \sum_{i \in I} \lambda_i + 2p\,,$$

where I is the set of indices for the $(n-2-t)$ values of λ_i that are different from λ (and not equal to $\frac{n}{2}$).

Now, substitute the total sum from Lemma 2.3,

$$\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) = t\lambda + \sum_{i \in I} \lambda_i + 2p.$$

Let us introduce λ (the baseline frequency) into the sum over I,

$$\begin{aligned}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) &= t\lambda + \sum_{i \in I} (\lambda_i - \lambda + \lambda) + 2p \\ &= t\lambda + \sum_{i \in I} (\lambda_i - \lambda) + (n-2-t)\lambda + 2p\,.\end{aligned}$$

Combine the terms with λ,

$$\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right) = (t+n-2-t)\lambda + \sum_{i\in I}(\lambda_i-\lambda)+2p$$
$$= (n-2)\lambda + \sum_{i\in I}(\lambda_i-\lambda)+2p\,.$$

Rearranging the terms to isolate the sum of deviations:

$$\sum_{i\in I}(\lambda_i-\lambda)+2p = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)-(n-2)\lambda\,.$$

Recall from Lemma 3 that for a general difference set with $\lambda=\frac{n-4}{4}$ (which corresponds to $r=2$), the total sum of λ_k is $(n-2)\lambda$. More precisely, if $\lambda=\frac{n-4}{4}$, then

$$(n-2)\lambda = (n-2)\frac{n-4}{4} = \frac{(n-2)(n-4)}{4} = \left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right).$$

Substituting this back into the equation

$$\sum_{i\in I}(\lambda_i-\lambda)+2p = (n-2)\lambda-(n-2)\lambda = 0\,.$$

Finally, rearrange the terms to obtain the desired result

$$\sum_{i\in I}(\lambda-\lambda_i) = 2p\,.$$

This concludes the proof. □

2.2 Examples

We provide two examples to illustrate Theorem 2.4.

Example 2.1 ([1]). For $n=32$, consider a general difference set where $p=1$ (meaning $\lambda_{16}=2$). The baseline frequency for differences (assuming $r=2$) would be $\lambda=\frac{n-4}{4}=\frac{32-4}{4}=\frac{28}{4}=7$. Suppose we have $\lambda_1=6$ and $\lambda_2=6$. Let us assume these are the only λ_i values that deviate from $\lambda=7$, and all other λ_i values (excluding λ_{16}) are equal to 7. The number of λ_i values equal to λ is $t=(n-2)-2=30-2=28$. The indices for λ_i that are different from λ (denoted by I) correspond to λ_1 and λ_2. Applying the theorem:

$$\sum_{i\in I}(\lambda-\lambda_i) = (\lambda-\lambda_1)+(\lambda-\lambda_2) = (7-6)+(7-6) = 1+1 = 2\,.$$

This matches $2p=2(1)=2$.

Example 2.2 ([1]). For $n=44$, consider a general difference set where $p=4$ (meaning $\lambda_{22}=8$). The baseline frequency for differences (assuming $r=2$) would be $\lambda=\frac{n-4}{4}=\frac{44-4}{4}=\frac{40}{4}=10$. Suppose the following λ_i values deviate from $\lambda=10$: $\lambda_1=12$, $\lambda_2=9$, $\lambda_3=6$, $\lambda_4=9$, $\lambda_5=9$, $\lambda_6=6$, $\lambda_7=9$, $\lambda_8=12$. The total number of non-$\frac{n}{2}$ differences is $n-2=44-2=42$. The number of specified deviating λ_i values is 8. Applying the theorem:

$$\sum_{i\in I}(\lambda-\lambda_i) = (\lambda-\lambda_1)+(\lambda-\lambda_2)+\cdots+(\lambda-\lambda_8)$$
$$= (10-12)+(10-9)+(10-6)+(10-9)+(10-9)+(10-6)+(10-9)+(10-12)$$
$$= (-2)+1+4+1+1+4+1+(-2)$$
$$= 8\,.$$

This matches $2p=2(4)=8$.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] I. Andersen, *Combinatorial Design: Construction Methods*, Ellis Horwood Limited, Chichester and New York, (1990).

[2] C.-S. Cheng and M.-H. Kao, Optimal experimental designs for fMRI via circulant biased weighing designs, *The Annals of Statistics* **43**(6) (2015), 2565 – 2587, DOI: 10.1214/15-AOS1352.

[3] R. Craigen, G. Faucher, R. Low and T. Wares, Circulant partial Hadamard matrices, *Linear Algebra and its Applications* **439**(11) (2013), 3307 – 3317, DOI: 10.1016/j.laa.2013.09.004.

[4] M.-H. Kao, A new type of experimental designs for event-related fMRI via Hadamard matrices, *Statistics & Probability Letters* **84** (2014), 108 – 112, DOI: 10.1016/j.spl.2013.09.024.

[5] M.-H. Kao, Universally optimal fMRI designs for comparing hemodynamic response functions, *Statistica Sinica* **25** (2015), 499 – 506, DOI: 10.5705/ss.2013.248.

[6] Y.-L. Lin, F. K. H. Phoa and M.-H. Kao, Circulant partial Hadamard matrices: Construction via general difference sets and its application to fMRI experiments, *Statistica Sinica* **27** (2017), 1715 – 1724, DOI: 10.5705/ss.202016.0254

[7] R. M. Low, M. Stamp, R. Craigen and G. Faucher, Unpredictable binary strings, *Congressus Numerantium* **177** (2005), 65 – 75.

[8] P. K. Manjhi and A. Kumar, On the construction of Hadamard matrices, *International Journal of Pure and Applied Mathematics* **120**(1) (2018), 51 – 58, URL: https://ijpam.eu/contents/2018-120-1/4/4.pdf.

[9] P. K. Manjhi and M. K. Rana, Some results on r-row-regular circulant partial Hadamard matrices of order $(k \times 2k)$, *Communication in Mathematics and Application* **13**(1) (2022), 129 – 136, DOI: 10.26713/cma.v13i1.1734.