



A Novel Four-Step Iteration Scheme With Numerical Examples

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Received: June 2, 2025

Revised: August 18, 2025

Accepted: September 29, 2025

Abstract. This study presents a four-step iteration method for approximating the fixed points of a contractive-like mapping. The proposed scheme is conceptually related to classical iterations such as Picard, Mann, and Ishikawa, while extending them in a more general framework. The strong convergence, rate of convergence and stability of the proposed method are established, demonstrating theoretical improvements over existing approaches. Moreover, we present some numerical illustrations to validate our method and these numerical illustrations are compared with some similar schemes available in the literature.

Keywords. Banach space, Iterative method, Strong convergence, Order of convergence

Mathematics Subject Classification (2020). 47H09, 54H25, 47H10, 55M20

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1. Introduction

Fixed point theory plays a dominant role in nonlinear functional analysis. It offers solutions to numerous problems in various disciplines, including game theory, chemistry, economics, integral equations, variational inequality problems, split feasibility problems, monotone inclusion problems and so on (Anjali *et al.* [1, 2], Dăneţ and Popescu [7], Li [12]). It is challenging to solve fixed point problems of nonlinear operators, which creates the need for iterative schemes to find fixed points. The technique of fixed point iteration is an elegant way to calculate the fixed points efficiently.

Assume that K is a nonempty, convex and closed subset of real Banach space $(B, \|\cdot\|)$ and U is a self-mapping on K . One of the simplest and oldest fixed point iterative scheme was given by Picard [15] in 1890, which is generally utilized to approximate the fixed point of contraction mappings. It is defined by

$$u_{n+1} = Uu_n. \quad (1.1)$$

In 1953, Mann iteration scheme was defined by Mann [13], which is formulated as

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Uu_n, \quad (1.2)$$

where $\alpha_n \in [0, 1] \subset \mathbb{R}$ is a sequence. Mann iteration scheme reduces to Picard iteration (1.1) if $\alpha_n = 1$.

In 1974, Ishikawa [10] proposed an iteration known as Ishikawa iteration, a generalization of Mann iteration scheme. It is given as

$$\begin{cases} v_n = (1 - \beta_n)u_n + \beta_n Uu_n, \\ u_{n+1} = (1 - \alpha_n)u_n + \alpha_n Uv_n, \end{cases} \quad (1.3)$$

where $\alpha_n, \beta_n \in [0, 1]$ are sequences of real numbers. Ishikawa iteration reduces to Mann iteration (1.2) if $\beta_n = 0$. A robust iteration scheme should have the following attributes:

- It should converge to the fixed point of a mapping;
- Its convergence should be faster than other iteration processes;
- It should be U -stable.

A large number of iterative processes, including one-step, two-step, three-step and four-step have been developed by many researchers over the years to approximate fixed points of different nonlinear mappings, for instances.

In 2023, Anku *et al.* [3] proposed one, two and three-step iterative schemes as under:

One-Step Iterative Scheme:

$$u_{n+1} = \frac{(p+1)u_n U(u_n)}{u_n + pU(u_n)}, \quad p \geq 0. \quad (1.4)$$

Two-Step Iterative Schemes: For $u_0 \neq 0 \in K$, the sequence $u_n \in K$ is given by

$$\begin{cases} u_{n+1} = U\left(\frac{U(u_n) + v_n}{2}\right), \\ v_n = \frac{(p+1)u_n U(u_n)}{u_n + pU(u_n)}, \quad p \geq 0. \end{cases} \quad (1.5)$$

and

$$\begin{cases} u_{n+1} = U\left(\frac{u_n + v_n}{2}\right), \\ v_n = \frac{(p+1)u_n U(u_n)}{u_n + pU(u_n)}, \quad p \geq 0. \end{cases} \quad (1.6)$$

Three-Step Iterative Scheme: For $u_0 \neq 0 \in K$, the sequence $u_n \in K$ is given by

$$\begin{cases} u_{n+1} = Uv_n, \\ v_n = Uw_n, \\ w_n = \frac{(p+1)u_n U(u_n)}{u_n + pU(u_n)}, \quad p \geq 0. \end{cases} \quad (1.7)$$

In 2023, Sharma *et al.* [16] proposed the following three-step iterative scheme:

For $u_0 \in K$, the sequence $\{u_n\} \in K$ is defined by

$$\begin{cases} w_n = \frac{ru_n + Uu_n}{r+1}, \\ v_n = Uw_n, \\ u_{n+1} = Uv_n, \end{cases} \quad (1.8)$$

where $r > 0$ is a real number. Motivated and inspired by the above iterative schemes, we introduce a new four-step iteration scheme, establish its strong convergence and prove the stability for this scheme. Finally, we provide numerical examples to validate our iteration scheme. The proposed four-step iteration scheme generalizes several well-known iterative methods available in the literature, thereby providing a broader theoretical framework for stability and convergence analysis. Furthermore, in contrast to several recent iterative methods available in the literature, the proposed four-step scheme offers improved convergence, enhanced stability and greater efficiency, thereby reinforcing its originality and practical relevance.

The structure of the paper is as under: Section 2 recalls some definitions and lemmas that are useful in establishing our main results. In Section 3, we prove the strong convergence of our novel four-step iterative scheme. We then demonstrate its stability result. In Section 4, we compare our iteration scheme with some similar schemes existing in the literature.

2. Preliminaries

This section provides essential definitions and lemmas that will be helpful in the sequel.

Imoru and Olatinwo [9] enlarged the class of mappings defined by Osilike [14] and Berinde [4]. They used the subsequent contractive condition:

Let $(B, \|\cdot\|)$ be a Banach space and U be a mapping on B which satisfies the following:

$$\|U\zeta - Uv\| \leq \varphi(\|\zeta - U\zeta\|) + \chi\|\zeta - v\|, \quad \text{for all } \zeta, v \in B, \quad (2.1)$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is continuous and monotonically increasing function with $\varphi(0) = 0$ and $\chi \in [0, 1)$.

Definition 2.1 ([4]). Assume that B is a Banach space. A sequence $\{\zeta_n\} \subset B$ is said to be strongly convergent to a point ζ^* iff $\|\zeta_n - \zeta^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.2 ([8]). Suppose that $\{u_n\} \in K$ is an arbitrary sequence. Then an iteration scheme $\zeta_{n+1} = f(U, \zeta_n)$ converging to a fixed point ζ^* is called U -stable, if for $t_n = \|u_{n+1} - f(U, u_n)\|$, $n \in \mathbb{N}$, we have $\lim_{n \rightarrow \infty} t_n = 0$ iff $\lim_{n \rightarrow \infty} u_n = \zeta^*$.

Definition 2.3 ([6]). Assume that $\{\zeta_n\}$ and $\{v_n\}$ are two convergent sequences in \mathbb{R} having limits ζ and v respectively. Then $\{\zeta_n\}$ is said to converge faster than $\{v_n\}$ if $\lim_{n \rightarrow \infty} \left| \frac{\zeta_n - \zeta}{v_n - v} \right| = 0$.

Definition 2.4 ([6]). Suppose there are two fixed point iteration procedures $\{\zeta_n\}$ and $\{v_n\}$ both converging to the same fixed point ζ^* such that $\|\zeta_n - \zeta^*\| \leq a_n$, $\|v_n - \zeta^*\| \leq b_n$, for all $n \geq 1$, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$ then $\{\zeta_n\}$ converges faster than $\{v_n\}$.

Lemma 2.1 ([5]). Let $\{t_n\}$ be a sequence of positive numbers converging to zero, then for any sequence of positive numbers $\{u_n\}$ satisfying

$$u_{n+1} \leq \chi u_n + t_n, \quad n \in \mathbb{N},$$

where $\chi \in [0, 1)$, we have $\lim_{n \rightarrow \infty} u_n = 0$.

Lemma 2.2 ([17]). Assume that $\{a_n\}$ and $\{b_n\}$ are two sequences in \mathbb{R}^+ such that

$$a_{n+1} \leq (1 - \mu_n)a_n + b_n,$$

where $\mu_n \in (0, 1)$ for each $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \frac{b_n}{\mu_n} = 0$ and $\sum_{n=0}^{\infty} \mu_n = \infty$. Then, $\lim_{n \rightarrow \infty} a_n = 0$.

3. Main Results

This section defines a new four-step iterative scheme and proves strong convergence of it. We also demonstrate stability result and compare the rate of convergence for it.

New Four-Step Iterative Scheme: For $\zeta_0 \in K$, let the sequence $\{\zeta_n\} \in K$ be defined by the following iterative scheme:

$$\begin{cases} \kappa_n = \frac{r\zeta_n + U\zeta_n}{r+1}, \\ v_n = \frac{r\kappa_n + U\kappa_n}{r+1}, \\ \psi_n = Uv_n, \\ \zeta_{n+1} = U\psi_n, \end{cases} \quad (3.1)$$

where $r \in \mathbb{R}^+$.

3.1 Convergence Analysis

Theorem 3.1. Assume that U is a self-mapping on K , which satisfies eq. (2.1) having a fixed point ζ^* . Let the sequence $\{\zeta_n\}$ be defined by iterative scheme (3.1). Then, $\{\zeta_n\}$ converges strongly to unique fixed point ζ^* of U .

Proof. Firstly, we show the existence of fixed point ζ^* . By using eq. (3.1), we obtain

$$\begin{aligned} \|\kappa_n - \zeta^*\| &= \left\| \frac{r\zeta_n + U\zeta_n}{r+1} - \zeta^* \right\| \\ &= \left\| \frac{r}{r+1}(\zeta_n - \zeta^*) + \frac{1}{r+1}(U\zeta_n - \zeta^*) \right\| \\ &\leq \frac{r}{r+1} \|\zeta_n - \zeta^*\| + \frac{1}{r+1} \|U\zeta_n - \zeta^*\| \\ &\leq \frac{r}{r+1} \|\zeta_n - \zeta^*\| + \frac{1}{r+1} [\varphi(\|\zeta^* - U\zeta^*\|) + \chi \|\zeta^* - \zeta_n\|] \\ &= \frac{r}{r+1} \|\zeta_n - \zeta^*\| + \frac{\chi}{r+1} \|\zeta_n - \zeta^*\| \\ &= \frac{r+\chi}{r+1} \|\zeta_n - \zeta^*\|, \\ \|v_n - \zeta^*\| &= \left\| \frac{r\kappa_n + U\kappa_n}{r+1} - \zeta^* \right\| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{r}{r+1} \|\kappa_n - \zeta^*\| + \frac{1}{r+1} \|U\kappa_n - \zeta^*\| \\
 &= \frac{r}{r+1} \|\kappa_n - \zeta^*\| + \frac{1}{r+1} \|U\zeta^* - U\kappa_n\| \\
 &\leq \frac{r}{r+1} \|\kappa_n - \zeta^*\| + \frac{1}{r+1} [\varphi(\|\zeta^* - U\zeta^*\|) + \chi\|\zeta^* - \kappa_n\|] \\
 &= \frac{r}{r+1} \|\kappa_n - \zeta^*\| + \frac{\chi}{r+1} \|\kappa_n - \zeta^*\| \\
 &= \frac{r+\chi}{r+1} \|\kappa_n - \zeta^*\|, \\
 \|\psi_n - \zeta^*\| &= \|Uv_n - \zeta^*\| \\
 &\leq \varphi(\|\zeta^* - U\zeta^*\|) + \chi\|v_n - \zeta^*\| = \chi\|v_n - \zeta^*\|
 \end{aligned}$$

and

$$\begin{aligned}
 \|\zeta_{n+1} - \zeta^*\| &= \|U\psi_n - \zeta^*\| \\
 &\leq \varphi(\|\zeta^* - U\zeta^*\|) + \chi\|\psi_n - \zeta^*\| \\
 &= \chi\|\psi_n - \zeta^*\| \\
 &\leq \chi^2\|v_n - \zeta^*\| \\
 &\leq \chi^2 \left(\frac{r+\chi}{r+1}\right) \|\kappa_n - \zeta^*\| \\
 &\leq \chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \|\zeta_n - \zeta^*\|.
 \end{aligned} \tag{3.2}$$

Replacing n by $n - 1$ and $n - 2$ in eq. (3.2), we obtain

$$\begin{aligned}
 \|\zeta_n - \zeta^*\| &\leq \chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \|\zeta_{n-1} - \zeta^*\|, \\
 \|\zeta_{n-1} - \zeta^*\| &\leq \chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \|\zeta_{n-2} - \zeta^*\|.
 \end{aligned}$$

Similarly, on putting $n = 0$ in eq. (3.2), we have

$$\|\zeta_1 - \zeta^*\| \leq \chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \|\zeta_0 - \zeta^*\|.$$

Using all above inequalities in eq. (3.2), one can deduce

$$\|\zeta_{n+1} - \zeta^*\| \leq \left[\chi \left(\frac{r+\chi}{r+1}\right)\right]^{2n+2} \|\zeta_0 - \zeta^*\|. \tag{3.3}$$

Since, $\chi < 1$, one obtains $r + \chi < r + 1$, which implies $0 < \frac{r+\chi}{r+1} < 1$.

Therefore, we have

$$\lim_{n \rightarrow \infty} \chi^{2n+2} \left(\frac{r+\chi}{r+1}\right)^{2n+2} = 0. \tag{3.4}$$

By using eq. (3.4) in eq. (3.3), we conclude

$$\lim_{n \rightarrow \infty} \|\zeta_{n+1} - \zeta^*\| = 0.$$

We now prove that ζ^* is the only fixed point of U . On the contrary, we suppose that ζ^* and ζ' are two fixed points of U with $\zeta^* \neq \zeta'$.

Consider

$$\begin{aligned}\|\zeta^* - \zeta'\| &= \|U\zeta^* - U\zeta'\| \\ &\leq \varphi(\|\zeta^* - U\zeta^*\|) + \chi\|\zeta^* - \zeta'\| \\ &= \chi\|\zeta^* - \zeta'\| \\ &< \|\zeta^* - \zeta'\|, \quad \text{as } \chi < 1.\end{aligned}$$

Thus, we arrive at a contradiction. Hence, $\zeta^* = \zeta'$. \square

3.2 Stability Analysis

Theorem 3.2. Assume that U is a self-mapping on K , which satisfies eq. (2.1) having a fixed point ζ^* . Let the sequence $\{\zeta_n\}$ be defined by eq. (3.1). Then the iterative scheme (3.1) is U -stable.

Proof. Suppose $\{u_n\}$ is an arbitrary sequence in K and the sequence given by eq. (3.1) is $\zeta_{n+1} = f(U, \zeta_n)$ converging to unique fixed point ζ^* of the mapping U . Assume that $\iota_n = \|u_{n+1} - f(U, u_n)\|$. We want to prove that the iterative method (3.1) is U -stable. For this, we have to prove $\lim_{n \rightarrow \infty} \iota_n = 0$ iff $\lim_{n \rightarrow \infty} u_n = \zeta^*$. First, we assume that $\lim_{n \rightarrow \infty} \iota_n = 0$. Consider

$$\begin{aligned}\|u_{n+1} - \zeta^*\| &= \|u_{n+1} - f(U, u_n) + f(U, u_n) - \zeta^*\| \\ &\leq \|u_{n+1} - f(U, u_n)\| + \|f(U, u_n) - \zeta^*\|.\end{aligned}$$

By making use of eq. (3.2), we obtain

$$\begin{aligned}\|u_{n+1} - \zeta^*\| &\leq \|u_{n+1} - f(U, u_n)\| + \|u_{n+1} - \zeta^*\| \\ &\leq \iota_n + \chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \|u_n - \zeta^*\|.\end{aligned}$$

Since, $\chi^2 \left(\frac{r+\chi}{r+1}\right)^2 \in [0, 1)$ and $\lim_{n \rightarrow \infty} \iota_n = 0$, therefore by using Lemma 2.1, we have $\lim_{n \rightarrow \infty} u_n = \zeta^*$.

Conversely, suppose that $\lim_{n \rightarrow \infty} u_n = \zeta^*$. We have to prove $\lim_{n \rightarrow \infty} \iota_n = 0$. For this, consider

$$\begin{aligned}\iota_n &= \|u_{n+1} - f(U, u_n)\| \\ &\leq \|u_{n+1} - \zeta^*\| + \|\zeta^* - f(U, u_n)\| \\ &\leq \|u_{n+1} - \zeta^*\| + \|u_{n+1} - \zeta^*\|.\end{aligned}$$

Using $\lim_{n \rightarrow \infty} u_n = \zeta^*$, we have $\lim_{n \rightarrow \infty} \iota_n = 0$. Thus, the iteration scheme (3.1) is U -stable. \square

3.3 Rate of Convergence

Theorem 3.3. Assume that U is a self-mapping on K , which satisfies the eq. (2.1) having a fixed point ζ^* . Let $u_0 = \zeta_0 \in K$ be given. Then, the iterative schemes $\{\zeta_n\}$ and $\{u_n\}$ generated by (3.1) and (1.8), respectively. Then $\{\zeta_n\}$ converges to ζ^* faster than $\{u_n\}$.

Proof. From eq. (3.3) of Theorem 3.1, we have

$$\|\zeta_{n+1} - \zeta^*\| \leq \chi^{2n+2} \left(\frac{r+\chi}{r+1}\right)^{2n+2} \|\zeta_0 - \zeta^*\|$$

and the iterative scheme has the following form

$$\|u_{n+1} - \zeta^*\| \leq \chi^{2n+2} \left(\frac{r+\chi}{r+1}\right)^{n+1} \|u_0 - \zeta^*\|.$$

Let us define

$$a_n = \chi^{2n+2} \left(\frac{r+\chi}{r+1}\right)^{2n+2} \|\zeta_0 - \zeta^*\|,$$

$$b_n = \chi^{2n+2} \left(\frac{r+\chi}{r+1}\right)^{n+1} \|u_0 - \zeta^*\|$$

and

$$\tau_n = \frac{a_n}{b_n} = \left(\frac{r+\chi}{r+1}\right)^{n+1} \frac{\|\zeta_0 - \zeta^*\|}{\|u_0 - \zeta^*\|}.$$

Consider

$$\lim_{n \rightarrow \infty} \frac{\tau_{n+1}}{\tau_n} = \lim_{n \rightarrow \infty} \frac{\left(\frac{r+\chi}{r+1}\right)^{n+2}}{\left(\frac{r+\chi}{r+1}\right)^{n+1}} = \left(\frac{r+\chi}{r+1}\right) < 1.$$

Using ratio test, we have $\lim_{n \rightarrow \infty} \tau_n = 0$, which clearly from Definitions 2.3 and 2.4 shows that $\{\zeta_n\}$ converges faster than $\{u_n\}$. □

4. Numerical Illustrations

In this section, we test the effectiveness of the iterative process proposed in the previous section. For this, we consider some linear and nonlinear equations and calculate the number of iterations required and the CPU time (in seconds) for finding the fixed points of such equations. We compare our iterative process with the iterative procedures (1.4), (1.5), (1.6), (1.7) and (1.8). We denote these procedures by Iter1, Iter2, Iter3, Iter4 and Iter5, respectively. We consider the parameters p and r that satisfy $-(2p + 1) < U'(\zeta_n) < 1$ and $r > \left(\frac{1+U'\zeta^*}{2}\right)$, which are the same as in [11] and [16], respectively. In all experiments, we take $p = 1/5$ and $r = 0.1$. We consider three cases by taking three different stopping criteria:

Case 1: $\|\zeta_n - \zeta_{n-1}\| < 10^{-5}$; Case 2: $\|\zeta_n - \zeta_{n-1}\| < 10^{-10}$; Case 3: $\|\zeta_n - \zeta_{n-1}\| < 10^{-15}$

We summarize all equations with fixed point, initial point and equivalent fixed point iterations in Table 1. We list the number of iterations (NOI) and CPU time (in seconds) for all iterative processes for Case 1, Case 2 and Case 3 in Tables 2, 3, 4 respectively and the corresponding graphical comparisons are illustrated in Figures 1, 2 and 3. It is observed that our iterative scheme outperforms the others. It is observed that our iterative scheme outperforms the others.

Table 1. Test problems

S. No.	Test problem	Fixed point iteration	Root or fixed point	Initial point ζ_0
1	$f(\zeta) = \zeta^2 - 2\zeta - 3$	$U(\zeta) = (2\zeta + 3)^{1/2}$	3	10
2	$f(\zeta) = \zeta^3 - 3\zeta - 18$	$U(\zeta) = (3\zeta + 18)^{1/3}$	3	10
3	$f(\zeta) = \zeta^2 - 3$	$U(\zeta) = \frac{3}{4} \left(\zeta + \frac{1}{\zeta}\right)$	1.73205080756888	1
4	$f(\zeta) = 2\zeta - 9$	$U(\zeta) = \frac{2}{3}\zeta + \frac{3}{2}$	4.5	10
5	$f(\zeta) = \tan(\zeta) - \zeta$	$U(\zeta) = \pi + \tan^{-1} \zeta$	4.49340945790906	4
6	$f(\zeta) = \sin(\zeta) - 10(\zeta - 1)$	$U(\zeta) = 1 + \frac{\sin \zeta}{10}$	1.0886	1.1

Table 2. Performance comparison (iterations and CPU time) for Case 1

Problem	Number of Iterations (NOI) CPU Time	Iter1	Iter2	Iter3	Iter4	Iter5	Our Iteration Process
1	NOI	17	8	10	6	6	5
	CPU	0.000019	0.00001	0.000009	0.000008	0.000006	0.000006
2	NOI	11	5	6	4	4	3
	CPU	0.000023	0.000012	0.000007	0.000008	0.000007	0.000007
3	NOI	20	9	13	7	7	6
	CPU	0.000022	0.000011	0.00001	0.000008	0.000008	0.000007
4	NOI	38	18	24	13	12	10
	CPU	0.000024	0.000015	0.000013	0.000011	0.00003	0.000008
5	NOI	8	4	5	3	3	3
	CPU	0.001992	0.000568	0.001195	0.000496	0.000389	0.000467
6	NOI	6	3	3	2	2	2
	CPU	0.000013	0.000008	0.000008	0.000007	0.000006	0.000007

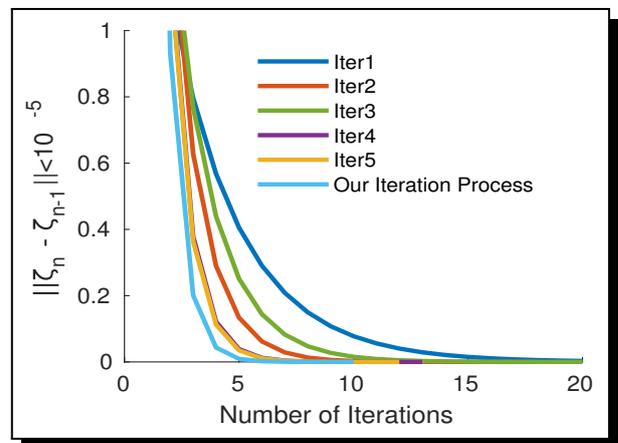
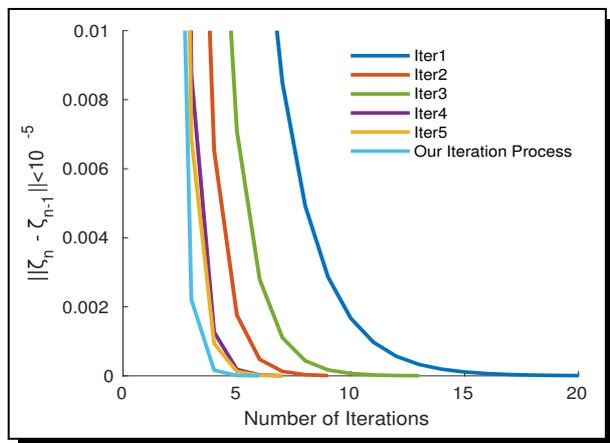
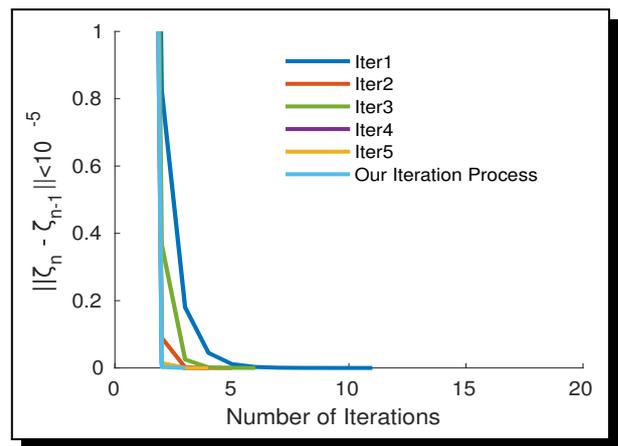
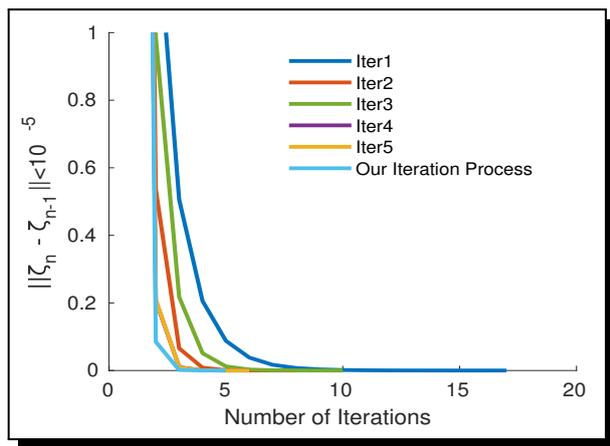


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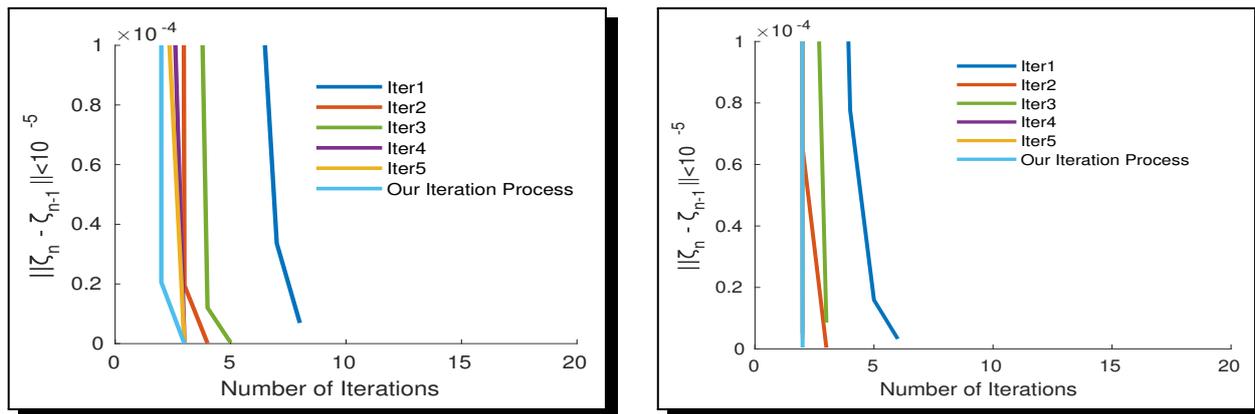


Figure 1. Comparison of our iteration process for Case 1

Table 3. Performance comparison (iterations and CPU time) for Case 2

Problem	Number of Iterations (NOI) CPU Time	Iter1	Iter2	Iter3	Iter4	Iter5	Our Iteration Process
1	NOI	31	13	19	10	9	8
	CPU	0.000022	0.000014	0.000009	0.000007	0.000008	0.000006
2	NOI	19	8	11	6	6	5
	CPU	0.000028	0.000011	0.00001	0.00001	0.000009	0.000009
3	NOI	41	18	25	13	13	10
	CPU	0.000027	0.000011	0.00001	0.00001	0.00003	0.000006
4	NOI	73	33	44	23	22	17
	CPU	0.000046	0.000027	0.00006	0.000023	0.000015	0.000015
5	NOI	16	6	8	4	4	4
	CPU	0.00285	0.002216	0.001864	0.000942	0.000774	0.000899
6	NOI	13	5	7	4	4	3
	CPU	0.00049	0.000008	0.000007	0.00004	0.000009	0.000007

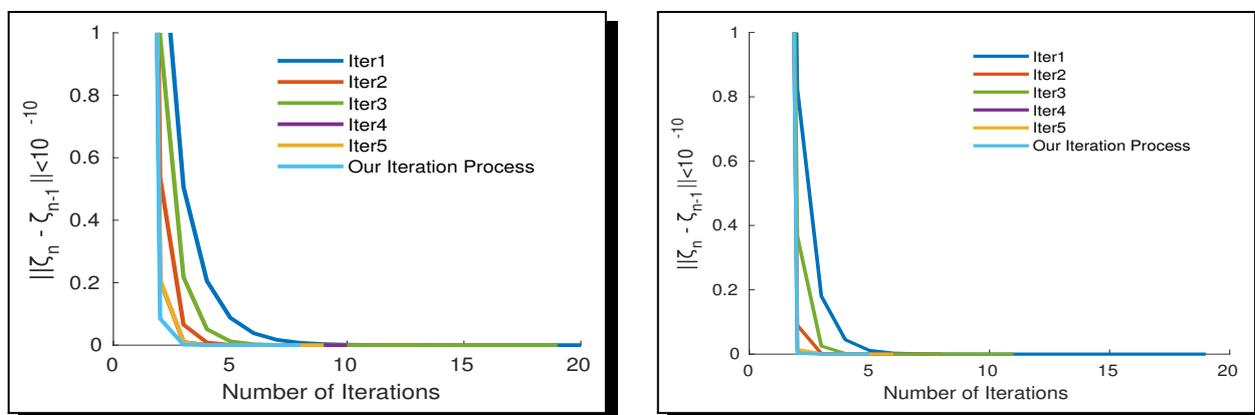


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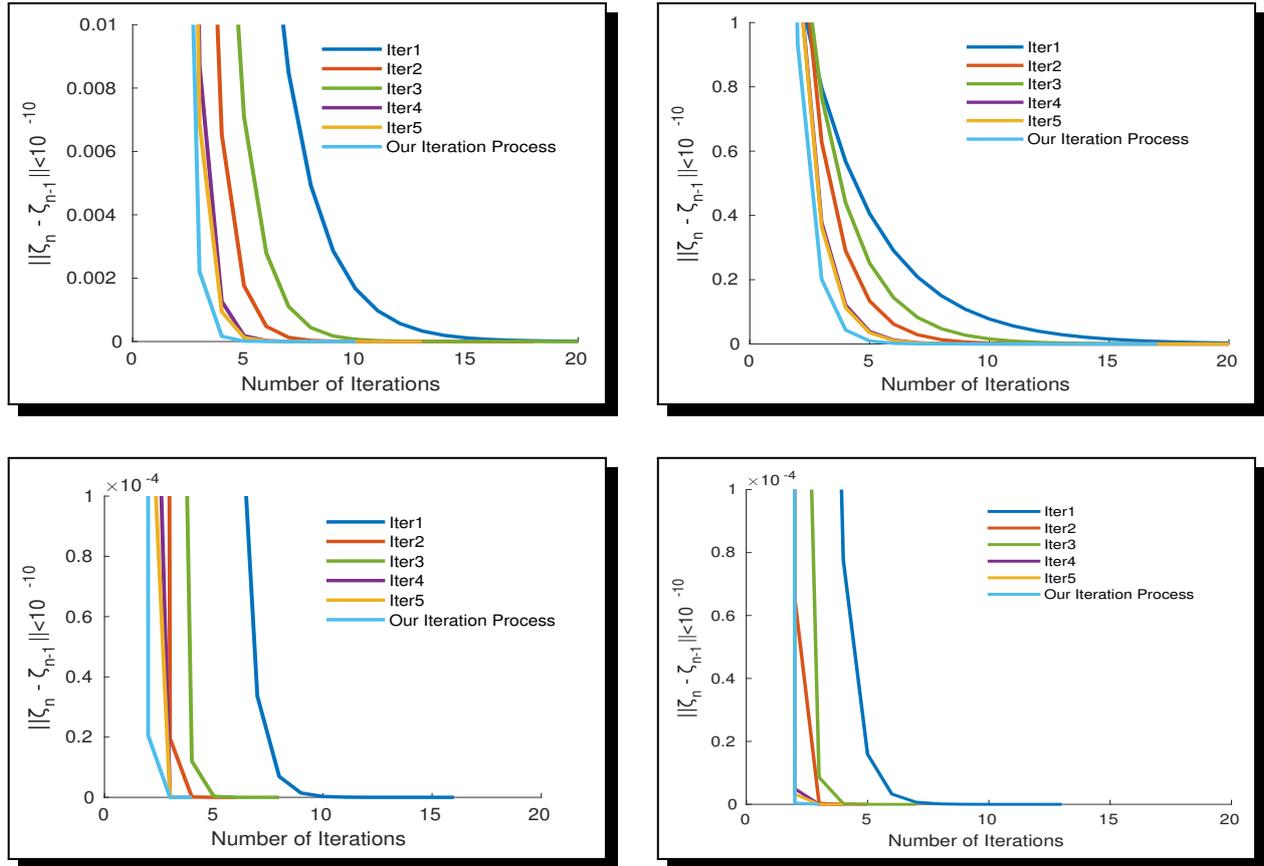


Figure 2. Comparison of our iteration process for Case 2

Table 4. Performance comparison (iterations and CPU time) for Case 3

Problem	Number of Iterations (NOI)	Iter1	Iter2	Iter3	Iter4	Iter5	Our Iteration Process
	CPU Time						
1	NOI	45	19	27	13	13	10
	CPU	0.000591	0.000632	0.000464	0.000463	0.000412	0.000616
2	NOI	28	11	15	8	7	6
	CPU	0.000045	0.000017	0.000019	0.000013	0.000012	0.000012
3	NOI	62	27	37	19	18	14
	CPU	0.00003	0.000016	0.000015	0.000009	0.000008	0.000006
4	NOI	108	47	65	33	32	25
	CPU	0.000054	0.00003	0.000033	0.000018	0.000017	0.000017
5	NOI	22	8	11	6	6	5
	CPU	0.00002	0.00001	0.000011	0.000007	0.000007	0.000008
6	NOI	20	7	10	5	5	4
	CPU	0.001772	0.001444	0.001185	0.001471	0.001136	0.000539

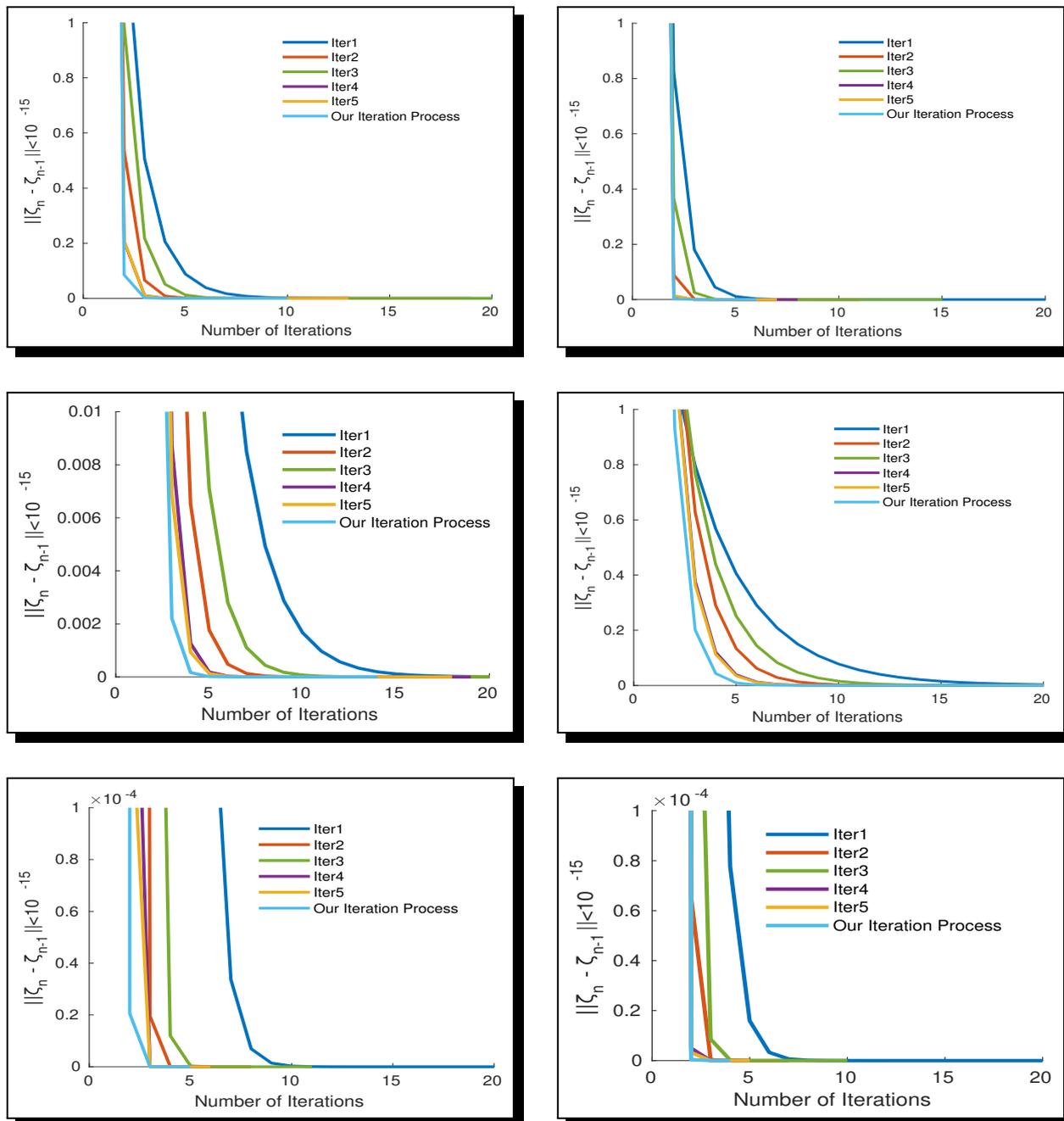


Figure 3. Comparison of our iteration process for Case 3

5. Conclusion

We have defined a fixed-point iteration method and presented its strong convergence in this article. We studied the rate of convergence, stability result and order of convergence for the proposed method. Moreover, we gave some numerical examples to illustrate the efficiency of our method. The proposed method not only demonstrates strong theoretical convergence but also has practical potential for efficiently solving real-world problems involving monotone inclusion and fixed point computations.

Acknowledgement

The first author acknowledges the University Grants Commission (UGC), New Delhi, for providing financial support.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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