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Research Article

On Some Bounds of Fuzzy Secure Domination Number

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Abstract. In a fuzzy graph $G_f(\rho,\eta)$ with vertex set V, a subset $D \subseteq V$ is called a dominating set if each vertex in the graph is either a member of D or has at least one strong neighbor in D. A secure dominating set S is a dominating set with the additional property that every vertex outside S has a strong neighbor in S, that can swap places with it and still maintain domination. The minimum fuzzy cardinality of such secure dominating sets is called the secure domination number, denoted $\gamma_{sc}(G_f)$. This paper explores bounds for the fuzzy secure domination number and examines its relationship with the domination number, independence number, and matching number.

Keywords. Fuzzy graphs, Fuzzy secure domination, Fuzzy secure total domination, Fuzzy secure total domination number

Mathematics Subject Classification (2020). 03E72, 05C72

1. Introduction

Originating from the Konigsberg bridge problem solved by Leonhard Euler [3], graph theory has grown into a rich and versatile field of research with numerous applications. Graph theory studies relationships between objects, where the vertices represent the objects and the edges represent the connections or relationships between them. Graphs can model social networks, communication networks, biological systems and transportation systems.

In real-world situations, uncertainty, vagueness, and inconsistency are significant factors. Traditional bi-valued logic and crisp set theory are often inadequate to represent them.

The seminal paper by L. A. Zadeh [15] marked the evolution of the modern concept of uncertainty, which introduced the theory of fuzzy sets.

A graph represents the relationship between objects, and since the relationships depicted are fuzzy in most real-life situations, they can be dealt with better using fuzzy graph models. Rosenfeld [12], and Yeh and Bang [14] laid the foundations of *Fuzzy Graph Theory*.

Even though we can trace the evolution of the concept of domination in graphs to the mid 1800s [4] about different chessboard problems, it was in 1962 that Ore [11] gave the name domination, and Cokayne and Hedetneimi [1] gave the notation γ for the domination number of a graph. The fuzzified version of domination was proposed by Somasundaram and Somasundaram [13] using effective edges, and Nagoorgani and Chandrasekaran [10] modified it using strong arcs in 2006. The study of graph protection was initiated in the late twentieth century, and Cockayne *et al*. [2] defined secure domination in 2005. It was introduced as the positioning of movable guards at the vertices of a graph to protect its vertices from possible attacks, and the configuration of guards is a dominating set both before and after the attack (Klostermeyer and Mynhardt [7], Merouane and Chellali [9]). The uncertainties in this mode of protection led to the development of fuzzy secure domination by Karunambigai *et al*. [6], which was later modified using strong arcs (Joy and Isaac [5]). This paper determines certain bounds of the fuzzy secure domination number.

2. Preliminaries

A *Fuzzy Graph* (Mathew *et al*. [8]) $G_f(\rho,\eta)$ is a nonempty set V together with a pair of functions $\rho: V \to [0,1]$ and $\eta: V\times V \to [0,1]$ such that $\eta(u,v) \le \rho(u)\wedge\rho(v)$, i.e., the membership value of an edge is less than or equal to the membership value of the vertices it connects, where ρ denotes the fuzzy vertex set of G_f and η denotes its fuzzy edge set.

The order p and the size q of a fuzzy graph $G_f(\rho,\eta)$ are given by $p = \sum_{v\in V}\rho(v)$ and $q = \sum_{uv\in E}\eta(uv)$.

Let $G_f(\rho,\eta)$ be a fuzzy graph on V and $D\subseteq V$. Then, the fuzzy cardinality of D is defined to be $\sum_{v\in D}\rho(v)$ (Mathew *et al*. [8]).

In a fuzzy graph, a path P of length n ([8]), is a sequence of distinct vertices $v_1,v_2,\dots,v_n$ such that $\eta(v_{i-1},v_i)>0$, for $i=1,2,\dots,n$. The strength of P is defined to be $\bigwedge_{i=1}^{n}\eta(v_{i-1},v_i)$, i.e., the weight of the weakest edge of the path.

The strength of connectedness ([8]) between two vertices u and v is denoted by $\eta^\infty(u,v)$ and is the maximum strength of all paths. A fuzzy graph is said to be connected if the value of $\eta^\infty(u,v)$ is greater than zero, for all u, v in V.

For an arc (u,v), if $\eta(u,v)\ge\eta^\infty(u,v)$, then it is said to be strong ([8]). If (u,v) is strong, we say that v is a strong neighbor of u.

The strong neighborhood of u, $N_S(u)=\{u\in V:(u,v)$ is strong$\}$. Also, $N_S(u)$ along with u is the closed strong neighborhood of u denoted by $N_S[u]$.

Let $G_f(\rho,\eta)$ denote a fuzzy graph over the set V. Let $u,v\in V$. We say that u dominates ([10]) v in G_f if (u,v) is a strong arc. A subset D of V is called a dominating set of G_f if every vertex outside D is dominated by some vertex of D.

The domination number, denoted by $\gamma(G_f)$, is the minimum fuzzy cardinality taken over all dominating sets of G_f.

Consider a fuzzy graph $G_f(\rho,\eta)$ on V. Let $D \subseteq V$ be a dominating set and $u \in V - D$. When a strong neighbor v of u exists in D and replacing u for v (a new arrangement of guards after defending an attack) produces a dominating set, we say that v D-defends u. The dominating set D is called a *Secure Dominating Set* (SDS) (Joy and Isaac [5]) of G_f, if every vertex outside D is D-defended by some vertex of D.

The secure domination number $\gamma_{sc}(G_f)$ for graph G_f refers to the least fuzzy cardinality among all SDSs of G_f.

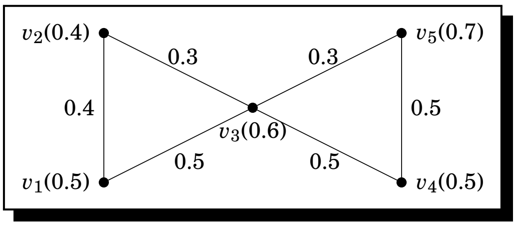

Figure 1. Secure domination in a fuzzy graph

In Figure 1, $\{v_1,v_4\}$, $\{v_2,v_4\}$, $\{v_1,v_5\}$, $\{v_2,v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_4,v_5\}$, $\{v_2,v_4,v_5\}$ are dominating sets with $\gamma(G_f) = 0.9$.

Also, $\{v_2,v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_2,v_5\}$, $\{v_2,v_4,v_5\}$, $\{v_2,v_3,v_4\}$, $\{v_1,v_4,v_5\}$, $\{v_1,v_3,v_4\}$, $\{v_1,v_2,v_4\}$ etc. are secure dominating sets with $\gamma_{sc}(G_f) = 1.4$

Let $G_f(\rho,\eta)$ denote a fuzzy graph over the set V. For $v \in D \subseteq V$, the strong private neighborhood (Joy and Isaac [5]) of v relative to D is defined by $spn(v,D) = N_S[v] - N_S[D - \{v\}]$.

The external strong private neighborhood of v relative to D is given by $espn(v,D) = spn(v,D) - \{v\}$. In other words, for $v \in D \subseteq V$; $w \in V - D$ is a D-external strong private neighbor of v (i.e., D-*espn* of v) if $N_S(w) \cap D = \{v\}$. Then, $ESPN(v,D)$ is the set representing all D-*espn*s of v.

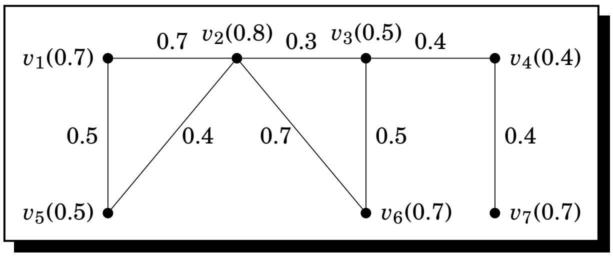

Figure 2. Fuzzy graph

In Figure 2, for $D = \{v_4,v_5,v_6\}$, $spn(v_4,D) = \{v_1,v_4\}$ and $espn(v_4,D) = \{v_1\}$; $spn(v_5,D) = \{v_2,v_5\}$ and $espn(v_5,D) = \{v_2\}$ and $spn(v_6,D) = \{v_6,v_7\}$ and $espn(v_6,D) = \{v_7\}$.

If vertex v D-defends vertex u, where u is outside the set D, it's apparent that $spn(v,D)$ is u's strong neighbor. Consequently, $espn(v,D)$ also serves as u's strong neighbor. This concept leads to a prominent characteristic of SDS.

Proposition 2.1 ([5])**.** *Let D be a dominating set. Vertex $v \in D$ defends $u \in V - D$ if and only if $G_f[ESPN(v,D) \cup \{u,v\}]$ is complete.*

Corollary 2.2 ([5])**.** *Let D be a dominating set. D is an SDS if and only if for each $u \in V - D$, there exists, $v \in D$ such that $G_f[ESPN(v,D) \cup \{u,v\}]$ is complete.*

3. Secure Domination and Independence in Fuzzy Graphs

Graph theory defines an independent subset of V as a collection devoid of adjacent vertices. This idea was fuzzified by Somasundaram and Somasundaram [13] and modified by Nagoorgani and Chandrasekaran [10] using the idea of strong arcs.

Definition 3.1 ([10])**.** Consider a fuzzy graph $G_f(\rho,\eta)$ with vertex set V. If a strong arc does not connect two vertices, they are considered fuzzy independent.

A subset of V is identified as a fuzzy independent set of G_f if every pair of vertices within it is fuzzy independent.

The fuzzy independence number, denoted as $\beta(G_f)$, refers to the largest possible fuzzy cardinality of a fuzzy independent set of G_f.

Figure 3 gives an example, where $\{v_1,v_5\}$, $\{v_2,v_3\}$, $\{v_1,v_2\}$, $\{v_1,v_3\}$, $\{v_3,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_2,v_3\}$, are fuzzy independent sets with $\beta = 2.0$.

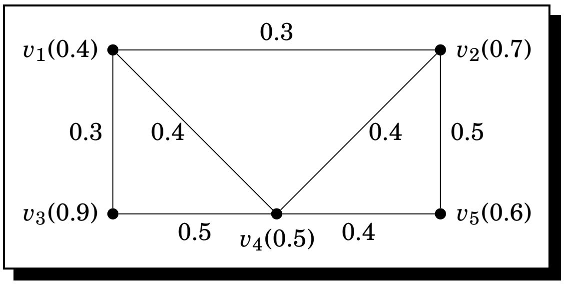

Figure 3. Independent sets in a fuzzy graph

Theorem 3.2. *For any fuzzy graph G_f, $\gamma_{sc}(G_f) \le 2\beta(G_f)$.*

Proof. Let X be a maximum independent set of G_f. Obviously, it will be a dominating set.
For each $x \in X$, $G_f[spn(x,X)]$ is complete. Otherwise, let $y,z \in spn(x,X)$ and yz is not a strong arc. Then $(X-\{x\}) \cup \{y,z\}$ results in an independent set larger than X.
If $espn(x,X) \neq \phi$, choose arbitrary $a_x \in espn(x,X)$.
Define $D = X \cup \{a_x : x \in X \text{ and } espn(x,X) \neq \phi\}$. since X is a dominating set and $espn(v,D) = \phi$, for every $v \in D$ we have for every $u \in V-D$, there exists $v \in D$ such that $G_f[espn(v,D) \cup \{u,v\}]$ is complete.
Hence D is an SDS with cardinality at most 2β. □

Theorem 3.3. *For every fuzzy graph G_f, $\gamma_{sc}(G_f) \le \gamma(G_f) + \beta(G_f)$.*

Proof. Let P be a $\gamma(G_f)$-set and Q be a maximal independent set of $G_f[V(G_f)-P]$ and let $R = V - (P \cup Q)$. Then, every vertex in R has at least two strong neighbors in $P \cup Q$, one in P and the other in Q. Then, $P \cup Q$ is an SDS of G_f and hence $\gamma_{sc} \le \gamma + \beta$. □

Remark 3.4. The previous theorem can be obtained as a corollary to this theorem since $\gamma(G_f) \leq \beta(G_f)$ for every fuzzy graph.

4. Secure Domination and Matching in Fuzzy Graphs

Somasundaram and Somasundaram [13] introduced the concept of matching in fuzzy graphs using effective edges. In this section, we will define the concept using strong arcs and study its relation with γ_{sc}.

Definition 4.1. Let $G_f(\rho,\eta)$ represent a fuzzy graph. An independent set of edges, or matching in G_f, denoted as M, is characterized by a collection of strong arcs where no two arcs in M share common vertices.

The edge independence number of G_f, also known as the strong fuzzy matching number of G_f, denoted by $\beta'(G_f)$, is determined by the maximum fuzzy cardinality attainable for an independent set of edges.

The strong matching number in the crisp sense is denoted by $\beta'_c(G_f)$, which is the number of arcs in the corresponding set.

In Figure 3, $\{v_1v_4, v_2v_5\}$ and $\{v_3v_4, v_2v_5\}$ are matching with edge independence number or strong fuzzy matching number, $\beta'(G_f) = 1.0$ and $\beta'_c = 2$.

Theorem 4.2. *For every fuzzy graph $G_f(\rho,\eta)$ of order p, $\gamma_{sc} \leq p - \beta'$.*

Proof. Let u_iv_i, $i = 1,2,\dots,\beta'_c$ be a maximum strong fuzzy matching of G_f. Then, $X = V - \{u_1,u_2,\dots,u_{\beta'_c}\}$ dominates G_f. For each $i \in \{1,2,\dots,\beta'_c\}$, there are two possibilities:

(i) u_i is the unique X-*espn* of v_i.
Then, u_i is a strong neighbor of each of $\{v_i\} \cup (espn(v_i,X) - \{u_i\})$. This implies that $(X - \{v_i\}) \bigcup \{u_i\}$ is a dominating set and $u_i \in N_S(v_i)$, i.e., v_i X-defends u_i.

(ii) v_i has no X-*espn*.
Then, u_i is a strong neighbor of each of $\{v_i\} \cup (espn(v_i,X) - \{u_i\})$. v_i X-defends u_i.

Thus, X is a dominating set and $v_i \in X$ defends $u_i \in V - X$. By Proposition 2.1, $G_f[ESPN(v_i,X) \cup \{u_i,v_i\}]$ is complete and X is an SDS with cardinality p-β'. □

Definition 4.3. In a fuzzy graph $G_f(\rho,\eta)$, a strong matching M that matches all vertices of G_f is a perfect matching.

The following corollary is obvious from Theorem 4.2.

Corollary 4.4. *For a fuzzy graph $G_f(\rho,\eta)$, with perfect matching, $\gamma_{sc} \leq \frac{p}{2}$.*

In Figure 4, $\{v_1v_6, v_2v_5, v_3v_4\}$ forms a perfect matching, whereas $\{v_1v_6, v_4v_5\}$ and $\{v_3v_6, v_2v_5\}$ are maximal matchings.

Here $\gamma(G_f) = 1.5$, $\gamma_{sc}(G_f) = 1.7 \leq 1.9 = \frac{p}{2}$.

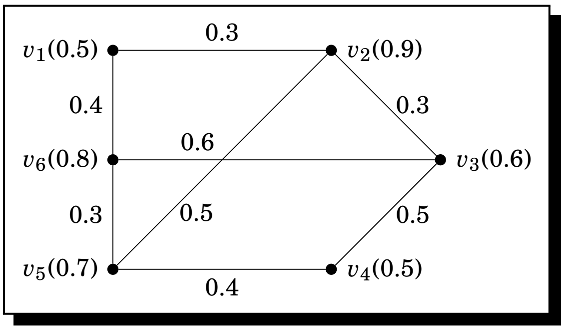

Figure 4. Perfect matching in a fuzzy graph

5. Conclusion

We have studied the fuzzy secure domination number and its association with domination, independence, and matching numbers. Fuzzy secure domination holds immense theoretical and practical significance as it has various applications in practical scenarios, including network security, resource allocation, telecommunication networks, disaster management, infrastructure protection, etc. If we specifically consider network security, fuzzy secure domination provides a framework for designing attack-resilient networks. It helps ensure that monitoring, communication, and control functions in a network remain secure even when some vertices are compromised. In practical networks, minimizing the number of secure dominating vertices is desirable to reduce costs, and the bounds give guarantees on resource requirements. Future work will focus on establishing more accurate bounds.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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