Communications in Mathematics and Applications
Vol. 16, No. 3, pp. 813–820, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i3.3202

http://www.rgnpublications.com

Research Article

A Study of Multiple Mappings in Multiplicative Metric Space

Manjusha P. Gandhi*[1], Anushree A. Aserkar[1] and Renuka Nafdey[2]

[1] *Department of Applied Mathematics and Humanities, Yeshwantrao Chavan College of Engineering (affiliated to Rashtrasant Tukadoji Maharaj Nagpur University), Nagpur 441110, Maharashtra, India*
[2] *Department of Physics, Ramdeobaba University, Nagpur 440013, Maharashtra, India*
***Corresponding author:** manjusha_g2@rediffmail.com

Received: June 1, 2025 **Revised:** August 20, 2025 **Accepted:** September 1, 2025

Abstract. A fixed point result for two pairs of maps has been established in multiplicative metric space. The continuous sub-compatible mappings satisfy the generalized contraction condition. This theorem improves the results, which are accessible in current literature. An example has been discussed in support of the present paper.

Keywords. Fixed point, Multiplicative Metric space, Continuously sub-compatible

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In 1922, Banach [3] established a result to prove the presence of uniqueness of fixed point for a contractive map in complete metric space. Numerous researchers enhanced and generalized this result in several spaces.

Bashirov *et al*. [4] gave the idea of novel distance named as multiplicative distance with the idea of multiplicative absolute value.

Özavsar and Çevikel [12] introduced the idea of multiplicative metric space. He *et al*. [6] have proved common fixed point outcomes in multiplicative metric space for weak commuting maps, likewise Abbas *et al*. [1] shown common fixed point theorems of quasi-weak commuting maps in same space. Lately, Gu and Cho [5] established fixed point theorems in multiplicative metric spaces for four maps.

Jungck [8] initiated a study of common fixed points of commuting maps in metric space. The idea of commutative mappings has been extended in different ways in metric space as discussed here under.

Sessa [13] introduced the concept of weak commutative mappings and Jungck [9] began the notion of compatible mappings. One can verify that if two maps commute, they are compatible however the reverse is not true. Compatible mappings are broader than commuting and weakly commutative functions.

Jungck and Rhoades *et al*. [10] presented the idea of weak compatible functions and shown that compatible functions are weak compatible however not conversely. Kumar *et al*. [11] extended this result for G-metric space. Al-Thagafi and Shahzad [2] added the thought of occasionally weak compatible functions.

Further ideas of continuously sub-compatible maps were introduced by Wadhwa and Beg [14].

We establish a unique common fixed point theorem for four functions which satisfy the modified multiplicative contraction constraint in multiplicative metric space. An attempt has been made to define continuously sub-compatible maps in pairs in multiplicative metric space, as a result, the proof becomes very easy. Also, an example is also provided to support the result.

2. Preliminaries

Definition 2.1 ([4]). Assume M is a non-empty set. A mapping $d: M\times M \to R^+$ which fulfill the following conditions, is named as multiplicative metric space:

(i) $d(x,y)\ge 1$, for all $x,y\in M$ and $d(x,y)=1$, iff $x=y$.

(ii) $d(x,y)=d(y,x)\ge 1$, for all $x,y\in M$.

(iii) $d(x,y)\le d(x,z)d(z,y)$, for all $x,y,z\in M$ (multiplicative triangle inequality).

Definition 2.2 ([1]). Consider a multiplicative metric space (M,d), a sequence $\{x_k\}$ in M and $x\in M$. If for each multiplicative open ball $B_\varepsilon(x)=\{y \mid d(x,y)<\varepsilon\}$, $\varepsilon>1$, there occurs a positive integer N such that $k\ge N$, then $x_k\in B_\varepsilon(x)$. Then $\{x_k\}$is called a multiplicative sequence converging to x.

Definition 2.3 ([1]). Consider a multiplicative metric space (M,d), a sequence $\{x_k\}$ in M and $x\in M$. Then $\{x_k\}$ in M is called a multiplicative Cauchy sequence if for $\varepsilon>1$, there occurs $N\in$ N such that $d(x_j,x_k)<\varepsilon$, for all $j,k>N$.

Definition 2.4 ([1]). If all multiplicative Cauchy sequences $\{x_k\}$ is converges to $x\in M$, then (M,d) is complete multiplicative metric space.

Definition 2.5 ([5]). Consider (M,d) is a multiplicative metric space, then $S: M\to M$, $A: M\to M$ are named as commutative maps if for all $x\in M$, $SAx=ASx$.

Definition 2.6 ([5]). Assume S and A are self-mappings of multiplicative metric space (M,d). Mappings S and A are weak commutative at point $x\in X$ whenever $d(SAx,ASx)\le d(Ax,Sx)$.

Definition 2.7 ([7])**.** Assume S and A are self-maps defined on a set M. The mappings S and A are named as weakly compatible functions if $Ax = Bx$, then $ASx = SAx$, for $x \in M$.

Definition 2.8 ([14])**.** Suppose S and A are two self-maps of multiplicative metric space (M,d). Then functions S and A are named as continuously sub-compatible if sequence $\{x\}$ in M occurs such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = z$ and $Sz = Az$.

Remark 2.1. The following is uni-directional fact

commutative maps → weak commutative → compatible → weakly compatible
↓
continuously sub-compatible

The following example shows that mappings are continuously sub-compatible but not weakly compatible.

Example 2.1. Let $M = [0,\infty)$ and define d by $d(x,y) = e^{|x-y|}$, for all $x,y \in M$. Then, (M,d) is multiplicative metric space. Define mappings S and A as follows:

$$Sx = x^2, \quad Ax = \begin{cases} x+2, & x \in [0,2] \cup (9,\infty), \\ x+12, & x \in (2,9]. \end{cases}$$

Define the sequence $\{x_n\}$, as $x_n = 2 + \frac{1}{n}$, for all natural numbers n.
Then $\lim_{n\to\infty} Sx_n = 4$, $\lim_{n\to\infty} Ax_n = 4$,

$$S(4) = A(4) = 16.$$

Thus, S and A are continuously sub-compatible.
But $SA(4) = S(16) = 256$, $AS(4) = A(16) = 18$.
Therefore,

$$SA(4) \neq AS(4).$$

Thus, S and A are not weakly compatible.

Definition 2.9 ([1])**.** Consider a multiplicative metric space (M,d). A function $S : M \to M$ is named as multiplicative contraction if for a real number $\lambda \in [0,1)$, the following condition holds:

$$d(S(x),S(y)) \le [d(x,y)]^{\lambda}, \quad \text{for } x,y \in M.$$

Theorem 2.1 ([1])**.** *Consider a complete multiplicative metric space (M,d). Then, a multiplicative contraction mapping $S : M \to M$ has a unique fixed point.*

Theorem 2.2 ([6])**.** *Assume A_1, A_2, B_1 and B_2 are self-mappings of complete multiplicative metric space (M,d), fulfilling the constraints below:*

(i) $A_1 M \subset B_2 M$, $A_2 M \subset B_1 M$.
(ii) *A_1 and B_1 are weak commutative, A_2 and B_2 are also weak commutative.*
(iii) *One of A_1, A_2, B_1 and B_2 is continuous.*
(iv) $d(A_1(x),A_2(y)) \le [\max\{d(B_1x,B_2y), d(B_1x,A_1x), d(B_2y,A_2y), d(A_1x,B_2y), d(B_1x,A_2y)\}]^{\lambda}$, *for all $x,y \in M, \lambda \in \left(0,\frac{1}{2}\right)$. Then, A_1, A_2, B_1 and B_2 have unique common fixed point.*

Remark 2.2. Neither multiplicative metric space is metric space nor metric space is multiplicative metric space.

Example 2.2. Consider $(R^+)^n$ is the set of n-tuples of positive real numbers.
Assume $d^* : (R^+)^n \times (R^+)^n \to (R^+)^n$ is well-defined by

$$d^\circ(x,y) = \left|\frac{x_1}{y_1}\right|^\circ \cdot \left|\frac{x_2}{y_2}\right|^\circ \dots \left|\frac{x_n}{y_n}\right|^\circ,$$

where $x = (x_1, x_2, x_3, \dots, x_n)$, $y = (y_1, y_2, y_3, \dots, y_n) \in R^+$ and $|\cdot|^\circ : R^+ \to R^+$ is $\alpha^* = \begin{cases} \alpha, & \alpha \geq 1, \\ \frac{1}{\alpha}, & \alpha < 1. \end{cases}$

Then $((R^+)^n, d^\circ)$ is a multiplicative metric space.
Now

$$d^\circ\left(\frac{1}{3}, 5\right) = \left|\frac{\frac{1}{3}}{5}\right|^\circ = \left|\frac{1}{25}\right|^\circ = 25.$$

Also,

$$d^\circ\left(\frac{1}{3}, 3\right) + d^\circ(3,5) = \left|\frac{\frac{1}{3}}{3}\right|^\circ + \left|\frac{3}{5}\right|^\circ = \frac{50}{3}.$$

We have observed that

$$d^\circ\left(\frac{1}{3}, 3\right) + d^\circ(3,5) < d^\circ\left(\frac{1}{3}, 5\right).$$

Thus, triangular inequality of metric space is not satisfied.
Hence $((R^+)^n, d^\circ)$ is multiplicative metric space but not a metric space.
The normal metric d on set of real numbers R does not satisfy the multiplicative triangular inequality

$$d(1,2)d(2,6) = 4 < d(1,6).$$

Thus, (R,d) is not a multiplicative metric space.

3. Main Result

We have proved the following theorem for continuously sub compatible pair of maps which is more generalized condition than weak commutative.

Theorem 3.1. *Assume A_1, A_2, B_1 and B_2 are self-maps of complete multiplicative metric space (M,d), fulfilling the constraints here under:*

(i) $A_1M \subset B_2M$, $A_2M \subset B_1M$.

(ii) *A_1 and B_1 are continuously sub-compatible, A_2 and B_2 are also continuously sub-compatible.*

(iii) $d(A_1(x), A_2(y)) \leq [\max\{d(B_1x, B_2y), d(B_1x, A_1x), d(B_2y, A_2y), d(A_1x, B_2y), d(B_1x, A_2y)\}]^\lambda$, *for all $x, y \in M, \lambda \in \left(0, \frac{1}{2}\right)$. Then, A_1, A_2, B_1 and B_2 have unique common fixed point.*

Proof. Assume $x_0 \in M$, then $\exists\ x_1 \in M$ such that $A_1x_0 = B_2x_1 = t_0$ because $A_1M \subset B_2M$.
Also, $A_2M \subset B_1M$, $\exists\ x_2 \in M$ such that $A_2x_2 = B_1x_1 = t_1$. In general, $\exists\ x_{2k+1}, x_{2k+2} \in M$ such that

$$A_1x_{2k} = B_2x_{2k+1} = t_{2k} \text{ and } A_2x_{2k+1} = B_1x_{2k+2} = t_{2k+1}.$$

Thus, $\{t_k\}$ is sequence in M.
Now

$$\begin{aligned}
d(t_{2k},t_{2k+1}) &= d(A_1x_{2k},A_2x_{2k+1})\\
&\le [\max\{d(A_1x_{2k},B_2x_{2k+1}),d(B_1x_{2k},A_1x_{2k}),d(B_2x_{2k+1},A_2x_{2k+1}),\\
&\qquad d(A_1x_{2k},B_2x_{2k+1}),d(B_1x_{2k},A_2x_{2k+1})\}]^\lambda\\
&\le [\max\{d(t_{2k-1},t_{2k}),d(t_{2k-1},t_{2k}),d(t_{2k},t_{2k+1}),d(t_{2k},t_{2k}),d(t_{2k-1},t_{2k+1})]^\lambda\\
&\le d^\lambda(t_{2k-1},t_{2k})d^\lambda(t_{2k},t_{2k+1}),
\end{aligned}$$

$$d^{1-\lambda}(t_{2k},t_{2k+1}) \le d^\lambda(t_{2k-1},t_{2k}),$$

$$d(t_{2k},t_{2k+1}) \le d^{\frac{\lambda}{1-\lambda}}(t_{2k-1},t_{2k}),$$

$$d(t_{2k},t_{2k+1}) \le d^h(t_{2k-1},t_{2k}).$$

Similarly,

$$d(t_{2k+1},t_{2+2}) \le d^h(t_{2k},t_{2k+1}).$$

Therefore,

$$d(t_k,t_{k+1}) \le d^h(t_{k-1},t_k) \le \ldots \le d^{h^n}(t_1,t_0), \quad \text{for all } n \ge 2.$$

Let $j,k \in N$,

$$\begin{aligned}
d(t_j,t_k) &\le d(t_j,t_{j-1})d(t_{j-1},t_{j-2})\ldots d(t_{k+1},t_k)\\
&\le d^{h^{j-1}}(t_1,t_0)d^{h^{j-2}}(t_1,t_0)\ldots d^{h^k}(t_1,t_0)\\
&\le d^{\frac{h^k}{1-h}}(t_1,t_0).
\end{aligned}$$

Therefore, $d(t_j,t_k) \to 1$, if $j,k \to \infty$.
Therefore, $\{t_k\}$ is multiplicative Cauchy sequence in M.
Since (M,d) is complete, therefore $\exists\ z \in M$ such that $t_k \to z$ $(k \to \infty)$.
Since (A_1,B_1) and (A_2,B_2) are continuously sub-compatible. Therefore,

$$\lim_{n\to\infty} B_1x_k = \lim_{n\to\infty} A_1x_k = v_1.$$

Therefore,

$$\lim_{k\to\infty} B_2t_k = \lim_{k\to\infty} A_2t_k = v_2,$$

$$d(A_1x_k,A_2t_k) \le [\max\{d(B_1x_k,B_2t_k),d(B_1x_k,A_1x_k),d(B_2t_k,A_2t_k),d(A_1x_k,B_2t_k),d(B_1x_k,A_2t_k)\}]^\lambda,$$

$$\begin{aligned}
d(v_1,v_2) &\le [\max\{d(v_1,v_2),d(v_1,v_1),d(v_2,v_2),d(v_1,v_2),d(v_1,v_2)\}]^\lambda\\
&\le [\max\{1,d(v_1,v_2)\}]^\lambda\\
&\le d^\lambda(v_1,v_2) \Longrightarrow v_1 = v_2.
\end{aligned}$$

Now to show $B_1v^\circ = v^\circ$,

$$d(A_1v^\circ,A_2t_n) \le [\max\{d(B_1v^\circ,B_2t_n),d(B_1v^\circ,A_1v^\circ),d(B_2t_n,A_2t_n),d(A_1v^\circ,B_2t_n),d(B_1v^\circ,A_2t_n)\}]^\lambda,$$

$$\begin{aligned}
d(A_1v^\circ,v^\circ) &\le [\max\{d(B_1v^\circ,v^\circ),d(B_1v^\circ,v^\circ),d(v^\circ,v^\circ),d(A_1v^\circ,v^\circ),d(B_1v^\circ,v^\circ)\}]^\lambda\\
&\le [d(A_1v^\circ,v^\circ)]^\lambda \Longrightarrow A_1v^\circ = v^\circ
\end{aligned}$$

$$B_1v^\circ = A_1v^\circ = v^\circ,$$

$$\begin{aligned}d(A_1x_n, A_2v^\circ) &\le [\max\{d(B_1x_n, B_2v^\circ), d(B_1x_n, A_1x_n), d(B_2v^\circ, A_2v^\circ), d(A_1x_n, B_2v^\circ), d(B_1x_n, A_2v^\circ)\}]^\lambda \\ &= [\max\{d(v^\circ, B_2v^\circ), d(v^\circ, v^\circ), d(B_2v^\circ, B_2v^\circ), d(v^\circ, B_2v^\circ), d(v^\circ, A_2v^\circ)\}]^\lambda \\ &\le d^\lambda(v^\circ, A_2v^\circ)] \Longrightarrow A_2v^\circ = v^\circ.\end{aligned}$$

Therefore,

$$B_2v^\circ = A_2v^\circ = v^\circ.$$

Now, to prove uniqueness of fixed point, assume w° is another fixed point of A_1, A_2, B_1 and B_2, then

$$\begin{aligned}d(v^\circ, w^\circ) &= d(A_1v^\circ, A_2w^\circ) \\ &\le [\max\{d(B_1v^\circ, B_2w^\circ), d(B_1v^\circ, A_1v^\circ), d(B_2w^\circ, B_2w^\circ), d(A_1v^\circ, B_2w^\circ), d(B_1v^\circ, A_2w^\circ)\}]^\lambda \\ &\le [\max\{d(v^\circ, w^\circ), 1\}]^\lambda \\ &= d^\lambda(v^\circ, w^\circ).\end{aligned}$$

Therefore,

$$v^\circ = w^\circ.$$

Thus, A_1, A_2, B_1 and B_2 have unique common fixed point. □

Example 3.1. Let $M = [0, \infty)$ and (M, d) is multiplicative metric space, where d is defined by $d(x, y) = e^{|x-y|}$, for all $x, y \in M$. Define mappings A_1, A_2, B_1 and B_2 as follows:

$$B_1x = \frac{x^2}{8}, \quad B_2x = \frac{x^2}{2},$$

$$Sx = \frac{x+2}{2}, \quad \text{for } x \in [0,2] \cup (9, \infty) \text{ and } Sx = x + 12, \text{ for } x \in (2, 9],$$

$$Tx = \frac{x+4}{3}, \quad \text{for } x \in [0,2] \cup (9, \infty) \text{ and } Tx = x + 5, \text{ for } x \in (2, 9].$$

Define the sequence $\{x_k\}$ by $x_k = 2 + \frac{1}{k}$, for all natural numbers k.
Then $A_1M \subset B_2M$, $A_2M \subset B_1M$,

$$\lim_{n\to\infty} B_2x_k = \frac{x_k^2}{2} = 2, \quad \lim_{k\to\infty} A_1x_k = \frac{x_k + 2}{2} = 2,$$

$$\lim_{n\to\infty} B_1x_k = \frac{x_k^2}{8} = 2, \quad \lim_{k\to\infty} A_2x_k = 2.$$

Also, the condition

$$d(A_1(x), A_2(y)) \le [\max\{d(B_1x, B_2y), d(B_1x, A_1x), d(B_2y, A_2y), d(A_1x, B_2y), d(B_1x, A_2y)\}]^\lambda,$$

$$\text{for all } x, y \in X, \lambda \in \left(0, \frac{1}{2}\right) \text{ satisfied.}$$

(A_1, B_1) and (A_2, B_2) are continuously sub-compatible mappings.
Also, $A_1(2) = A_2(2) = B_1(2) = B_2(2) = 2$.
Therefore, 2 is fixed point of A_1, A_2, B_1 and B_2.

Remark 3.1. We have used continuously sub-compatible condition in above theorem and use the different technique to prove the result, so that proof becomes simpler and shorter.

Corollary 3.1. *If A_1 and B_2 are self-maps of complete multiplicative metric space M, fulfilling the following conditions:*

(i) $A_1M \subset B_2M$.

(ii) *A_1 and B_2 are continuously sub-compatible,* $d(A_1(x),y) \le [\max\{d(x,B_2y),d(x,A_1x),d(B_2y,y),d(A_1x,B_2y),d(x,y)\}]^\lambda$, *for all* $x,y \in M$, $\lambda \in \left(0,\frac{1}{2}\right)$. *Then, A_1 and B_2 will have a unique common fixed point.*

Proof. Consider $B_1 = I = A_2$ (identity mapping) in Theorem 3.1, the proof is obvious and therefore omitted. □

Corollary 3.2. *If A_1 is self-map of a complete multiplicative metric space M fulfilling the following condition:*

$$d(A_1(x),y) \le [\max\{d(x,y),d(x,A_1x),d(y,y),d(A_1x,y)\}]^\lambda, \text{ for all } x,y \in M, \lambda \in \left(0,\frac{1}{2}\right).$$

Then, A_1 will have a unique fixed point.

Proof. Consider $B_2 = I$ (identity mapping) in Corollary 3.2, proof is obvious and therefore omitted. □

4. Conclusion

A fixed point theorem is established for four mappings under generalized contraction condition for multiplicative metric space. The introduction of continuously sub-compatible condition for pairs of mappings play an important role to improve the existing results available in literature.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] M. Abbas, B. Ali and Y. I. Suleiman, Common fixed points of locally contractive mappings in multiplicative metric spaces with application, *International Journal of Mathematics and Mathematical Sciences* **2015**(1) (2015), Article ID 218683, DOI: 10.1155/2015/218683.

[2] M. A. Al-Thagafi and N. Shahzad, A note on occasionally weakly compatible maps, *International Journal of Mathematical Analysis* **3**(2) (2009), 55 – 58.

[3] S. Banach, Sur les opérations dans les ensemble abstraits et leur applications aux équations integrables, *Fundamenta Mathematicae* **3**(1) (1922), 133 – 181, https://eudml.org/doc/213289.

[4] A. E. Bashirov, E. M. Kurpinar and A. Ozyapici, Multiplicative calculus and its applications, *Journal of Mathematical Analysis and Applications* **337**(1) (2008), 36 – 48, DOI: 10.1016/j.jmaa.2007.03.081.

[5] F. Gu and Y. J. Cho, Common fixed point results for four maps satisfying ϕ-contractive condition in multiplicative metric spaces, *Fixed Point Theory and Applications* **2015** (2015), Article number: 165, 10.1186/s13663-015-0412-4.

[6] X. He, M. Song and D. Chen, Common fixed points for weak commutative mappings on a multiplicative metric space, *Fixed Point Theory and Applications* **2014** (2014), Article number: 48, DOI: 10.1186/1687-1812-2014-48.

[7] G. Jungck, Common fixed points for non-continuous nonself maps on non-metric spaces, *Far East Journal of Mathematical Sciences* **4** (1996), 199 – 215.

[8] G. Jungck, Commuting mappings and fixed points, *The American Mathematical Monthly* **83**(4) (1976), 261 – 263, DOI: 10.2307/2318216.

[9] G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences* **9**(4) (1986), 771 – 779, DOI: 10.1155/S0161171286000935.

[10] G. Jungck and B. E. Rhoades, Fixed point for set valued functions without continuity, *Indian Journal of Pure and Applied Mathematics* **29**(3) (1998), 227 – 238.

[11] N. Kumar, M. Kumar and Ashish, Common fixed point theorems for weakly compatible maps satisfying integral type contraction in G-metric space, *Communication in Mathematics and Applications* **14**(1) (2023), 21 – 36, DOI: 10.26713/cma.v14i1.1917.

[12] M. Özavsar and A. C. Çevikel, Fixed point of multiplicative contraction mappings on multiplicative metric space, *Journal of Engineering Technology and Applied Sciences* **2**(2) (2017), 65 – 79, URL: https://dergipark.org.tr/en/download/article-file/348183.

[13] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, *Publications de L'Institut Mathématique* **32**(46) (1982), 149 – 153, URL: http://elib.mi.sanu.ac.rs/files/journals/publ/52/24.pdf.

[14] K. Wadhwa and F. Beg, Common fixed point theorems with continuously subcompatible mappings in fuzzy metric spaces, *Mathematical Theory and Modeling* **3**(6) (2013), 170 – 179, URL: https://www.iiste.org/Journals/index.php/MTM/article/view/5911/6177.