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Research Article

# Lie Symmetry Analysis, Exact Traveling Wave Solutions by $\left(\frac{G'}{G}\right)$-Expansion Method and Qualitative Analysis of Hirota-Schrödinger Equation

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**Abstract.** This manuscript is based on the Hirota's equation with Kerr law nonlinearity, specifically contemplated to be a kind of nonlinear Schrödinger equation. A systematic investigation of Lie symmetry method is pertained to derive the symmetry reduction of the given equation. By utilizing these symmetry reductions, the nonlinear partial differential equation is altered into the ordinary differential equations. By utilised the $\left(\frac{G'}{G}\right)$-expansion technique to gain exact solutions of the Hirota-Schrödinger equation. Novel solitary wave solutions were successfully extracted, characterized by hyperbolic, rational, and trigonometric function forms. Furthermore, a qualitative analysis of the reduced system is performed by converting it into an autonomous system, allowing the investigation of the stability and behavior of critical points. Several phase portraits are presented for various parameter values to illustrate the system's dynamics.

**Keywords.** Hirota-Schrödinger equation, Lie symmetry analysis, $\left(\frac{G'}{G}\right)$-Expansion method, Exact solution, Qualitative analysis

**Mathematics Subject Classification (2020).** 70G65, 83C15, 34C23

## 1. Introduction

The behavior of solitons in optical fibers is primarily described by the *Nonlinear Schrödinger Equation* (NLSE), which is fundamental in modeling the propagation of optical pulses in nonlinear and dispersive media. The use of the NLSE in the context of optical fiber communication was initially investigated by Hasegawa and Tappert in the early 1970s [7]. Thus, a large number of NLSE-based models that describe a wide range of physical events have been refined and scrutinised in the literature. The topic of optical solitons has experienced remarkable development in study due to its profound implications in optical fibres, which are crucial in transferring information across continents. With all optical communications currently being utilised for trans-continental and trans-oceanic data transfer via long-haul optical fibres, there is a pressing need for a thorough, systematic research of these dispersive optical solitons ([4]). Over the past 25 years, there have been significant advancements in communication and internet technology. For these technologies, fibre optics is a crucial component and an excellent building element. The idea of soliton transmission becomes significant, particularly in the context of sending massive amounts of data packets across extremely long distances. While many physical processes may be modeled using the Schrödinger equation, one of the key models that takes solitons propagation in optical fibres into account is the Schrödinger Hirota equation, which differs from the standard Schrödinger equations. Consequently, various efficient techniques have been developed by researchers to derive exact solutions. In this work, we employed the $\left(\frac{G'}{G}\right)$-expansion method ([8], [11], [12]) to construct exact traveling wave solutions.

## 2. Hirota's Schrödinger Equation

The Hirota's Schrödinger equation with Kerr law nonlinearity [2] is given by

$$\iota z_t + \bar{a} z_{xx} + \bar{c}|z|^2 z + \iota[\bar{\gamma} z_{xxx} + \bar{\sigma}|z|^2 z_x] = 0, \tag{1}$$

where $z$ is complex valued wave function, $z_t$ is its linear temporal evolution, $\bar{\gamma}$ is the parameter associated with third order dispersion (3OD), $\bar{\sigma}$ is the nonlinear dispersion coefficient and $\iota$ denotes the imaginary unit, $z_{xx}$ represents the linear dispersion term, the nonlinear term $\bar{c}|z|^2 z$ represents the Kerr nonlinearity, where $|z|^2$ is the intensity of the wave or field, $\bar{c}$ is the parameter characterizing Kerr law nonlinear behavior and $\bar{a}$ is a parameter characterizing the strength of the nonlinear effect.

## 3. Classical Lie Symmetry Analysis

The analysis of differential equations through symmetry methods has seen a surge in interest in recent years. In this work, demonstrate how to systematically derive the symmetries and perform symmetry reductions of the Hirota-Schrödinger equation using the classical Lie group approach in an algorithmic manner. To identify the symmetries admitted by equation (1). Now consider

$$z(x,t) = p + \iota q, \tag{2}$$

which splits (1) into its real and imaginary parts as:

$$\begin{cases} -q_t + \bar{a} p_{xx} + \bar{c} p^3 + \bar{c} p q^2 - \bar{\gamma} q_{xxx} + \bar{\sigma} p^2 p_x + \bar{\sigma} p_x q^2 = 0, \\ p_t + \bar{a} q_{xx} + \bar{c} q^3 + \bar{c} q p^2 + \bar{\gamma} p_{xxx} + \bar{\sigma} p^2 q_x + \bar{\sigma} q_x q^2 = 0. \end{cases} \tag{3}$$

The Lie group (Adem and Khalique [1], Bluman and Cole [5], Olver [9], and Ovsiannikov [10]) of continuous transformations will now be examined in order to find the classical symmetries in the following manner:

$$\begin{cases} \tilde{p} = p + \epsilon\bar{\eta}(x,t,p,q) + o(\epsilon^2), \\ \tilde{q} = q + \epsilon\bar{\phi}(x,t,p,q) + o(\epsilon^2), \\ \tilde{x} = x + \epsilon\bar{\xi}(x,t,p,q) + o(\epsilon^2), \\ \tilde{t} = t + \epsilon\bar{\tau}(x,t,p,q) + o(\epsilon^2). \end{cases} \tag{4}$$

that preserves the invariance of system (3) under this one-parameter transformation. This yields a linear system with redundant equations for the infinitesimal $\bar{\eta}(x,t,p,q)$, $\bar{\phi}(x,t,p,q)$, $\bar{\xi}(x,t,p,q)$, $\bar{\tau}(x,t,p,q)$. Hence, the invariance condition associated with (1) yields:

$$\begin{cases} -\bar{\phi}^t + \bar{a}\bar{\eta}^{xx} + 3\bar{c}p^2\bar{\eta} + 2\bar{c}pq\bar{\phi} + \bar{c}\bar{\eta}q^2 - \bar{\gamma}\bar{\phi}^{xxx} + \bar{\sigma}p^2\bar{\eta}^x + 2\bar{\sigma}pp_x\bar{\eta} + \bar{\sigma}q^2\bar{\eta}^x + 2\bar{\sigma}p_xq\bar{\phi} = 0, \\ \bar{\eta}^t + \bar{a}\bar{\phi}^{xx} + \bar{c}p^2\bar{\eta} + 2\bar{c}pq\bar{\eta} + 3\bar{c}q^2\bar{\phi} + \bar{\gamma}\bar{\eta}^{xxx} + \bar{\sigma}p^2\bar{\phi}^x + 2\bar{\sigma}pq_x\bar{\eta} + \bar{\sigma}q^2\bar{\phi}^x + 2\bar{\sigma}qq_x\bar{\phi} = 0. \end{cases} \tag{5}$$

Through the substitution of the infinitesimal values $\bar{\eta}^t$, $\bar{\phi}^t$, $\bar{\eta}^x$, $\bar{\phi}^x$, $\bar{\eta}^{xx}$, $\bar{\phi}^{xx}$, $\bar{\eta}^{xxx}$ and $\bar{\phi}^{xxx}$ into (5). By matching the same powers of the various differentials and setting them to zero and arrive at the desired system of over-determined PDEs. The solution to this system is as follows:

$$\bar{\eta} = pe_1 + qe_2, \quad \bar{\phi} = -pe_2 + qe_1, \quad \bar{\xi} = -e_1x + e_3, \quad \bar{\tau} = -2e_1t + e_4, \tag{6}$$

where $e_1$, $e_2$, $e_3$ and $e_4$ are arbitrary constants. The associated infinitesimal generators are presented as follows:

$$D_1 = -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}, \quad D_2 = q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q,}, \quad D_3 = \frac{\partial}{\partial x}, \quad D_4 = \frac{\partial}{\partial t}. \tag{7}$$

## 4. Symmetry Reductions and Exact Solutions of Hirota-Schrödinger's Equation

In the subsequent section, main aim to obtain exact solutions ([3], [8]) of equation (1) by utilizing the reduced forms obtained through similarity transformations. The appropriate similarity variables and their functional forms are determined by solving the associated characteristic equations, as obtained below:

$$\frac{dx}{\bar{\xi}} = \frac{dt}{\bar{\tau}} = \frac{dp}{\bar{\eta}} = \frac{dq}{\bar{\phi}}. \tag{8}$$

To facilitate symmetry reductions and obtain exact solutions, consider the following four distinct cases of vector fields:

### 4.1 Reduction Under $D_3$

Solving equation (8) yields the similarity variables in the following form:

$$z(x,t) = h(\bar{\psi})e^{\iota g(\bar{\psi})}, \tag{9}$$

where $\bar{\psi} = t$. Treating $h(\bar{\psi})$ and $g(\bar{\psi})$ as new dependent variables with a new independent variable $\bar{\psi}$ and using (9) in (1) to obtain a nonlinear ordinary differential equation as:

$$\begin{cases} h'(\bar{\psi}) + \iota\bar{c}h^3(\bar{\psi}) = 0, \\ h(\bar{\psi})g'(\bar{\psi}) = 0. \end{cases} \tag{10}$$

In this case, the analysis yields only a constant solution for the equation under consideration.

### 4.2 Symmetry Reduction Under $D_4$

The similarity reduction corresponding the vector field $D_4$ is given by:

$$z(x,t) = h(\bar{\psi})e^{\iota g(\bar{\psi})}, \tag{11}$$

where $\bar{\psi} = x$. Utilising (11) into (1), the obtained system of ordinary differential equations is

$$\begin{cases} \bar{a}h''(\bar{\psi}) - \bar{a}h(\bar{\psi})(g'(\bar{\psi}))^2 + \bar{c}h^3(\bar{\psi}) - 3\bar{\gamma}h''(\bar{\psi})g'(\bar{\psi}) - 3\bar{\gamma}h'(\bar{\psi})g''(\bar{\psi}) \\ \quad +\bar{\gamma}h(\bar{\psi})(g'(\bar{\psi}))^3 - \bar{\sigma}h(\bar{\psi})g'(\bar{\psi}) = 0, \\ \text{and} \\ 2\bar{a}h'(\bar{\psi})g'(\bar{\psi}) + \bar{a}h(\bar{\psi})g''(\bar{\psi}) + \bar{\gamma}h'''(\bar{\psi}) - 3\bar{\gamma}h'(\bar{\psi})(g'(\bar{\psi}))^2 \\ \quad -3\bar{\gamma}h(\bar{\psi})g'(\bar{\psi})g''(\bar{\psi}) + \bar{\sigma}h^2(\bar{\psi})h'(\bar{\psi}) = 0. \end{cases} \tag{12}$$

Due to the complexity of the above system of equations, a non-trivial solution could not be obtained.

### 4.3 Symmetry Reduction Under $D_3 + D_4$

The similarity variables for vector field $D_3 + D_4$ is given by:

$$z(x,t) = h(\bar{\psi})e^{\iota g(\bar{\psi})}, \tag{13}$$

where $\bar{\psi} = x - t$. Using the given similarity variable, the reduced system of ordinary differential equation is given by:

$$\begin{cases} h(\bar{\psi})g'(\bar{\psi}) + \bar{a}h''(\bar{\psi}) - \bar{a}h(\bar{\psi})(g'(\bar{\psi}))^2 + ch^3(\bar{\psi}) - 3\bar{\gamma}h''(\bar{\psi})g'(\bar{\psi}) - 3\bar{\gamma}h'(\bar{\psi})g''(\bar{\psi}) \\ \quad +\bar{\gamma}h(\bar{\psi})(g'(\bar{\psi}))^3 - \bar{\gamma}h(\bar{\psi})g'''(\bar{\psi}) - \bar{\sigma}h(\bar{\psi})g'(\bar{\psi}) = 0, \\ \text{and} \\ -h'(\bar{\psi}) + 2\bar{a}h'(\bar{\psi})g'(\bar{\psi}) + \bar{a}h(\bar{\psi})g''(\bar{\psi}) + \bar{\gamma}h'''(\bar{\psi}) - 3\bar{\gamma}h'(\bar{\psi})(g'(\bar{\psi}))^2 \\ \quad -3\bar{\gamma}h(\bar{\psi})g'(\bar{\psi})g''(\bar{\psi}) + \bar{\sigma}(h(\bar{\psi}))^2h'(\bar{\psi}) = 0. \end{cases} \tag{14}$$

The complexity of the system makes it difficult to determine a non-trivial solution.

### 4.4 Symmetry Reduction Under $\bar{\omega}D_3 + D_4$

The similarity variables for vector field $\bar{\omega}D_3 + D_4$ is given by:

$$z(x,t) = h(\bar{\psi})e^{\iota g(\bar{\psi})}, \tag{15}$$

where $\bar{\psi} = x - \bar{\omega}t$. The obtained ordinary differential equations corresponding to the similarity variable (15) is:

$$\begin{cases} \bar{\omega}h(\bar{\psi})g'(\bar{\psi}) + \bar{a}h''(\bar{\psi}) - \bar{a}h(\bar{\psi})g'(\bar{\psi}) + ch^3(\bar{\psi}) - 3\bar{\gamma}h''(\bar{\psi})g'(\bar{\psi}) - 3\bar{\gamma}h'(\bar{\psi})g''(\bar{\psi}) \\ \quad +\bar{\gamma}h(\bar{\psi})(g'(\bar{\psi}))^2 - \bar{\sigma}h^3(\bar{\psi})g'(\bar{\psi}) = 0, \\ -\bar{\omega}h'(\bar{\psi}) + 2\bar{a}h'(\bar{\psi})g'(\bar{\psi}) + \bar{a}h(\bar{\psi})g''(\bar{\psi}) + \bar{\gamma}h'''(\bar{\psi}) - 2\bar{\gamma}h'(\bar{\psi})(g'(\bar{\psi}))^2 - \bar{\gamma}h'(\bar{\psi})g'(\bar{\psi}) \\ \quad -\bar{\gamma}h(\bar{\psi})g''(\bar{\psi}) + \bar{\gamma}h(\bar{\psi})g'''(\bar{\psi}) - \bar{\gamma}h(\bar{\psi})g'(\bar{\psi})g''(\bar{\psi}) + \bar{\sigma}h^2(\bar{\psi})h'(\bar{\psi}) = 0. \end{cases} \tag{16}$$

The complexity of the system makes it difficult to determine a non-trivial solution.

From these reductions we are unable to find the exact traveling wave solutions so we use $\left(\frac{G'}{G}\right)$-expansion method to obtain the traveling wave solution of Hirota-Schrödinger equation describe as follows:

Consider the nonlinear PDE as

$$F(z, z_t, z_x, z_{xt}, z_{xx}, \dots) = 0, \tag{17}$$

where $z = z(x,t)$ is unknown function, along with its partial derivatives.

*Step* 1: Consider the following wave transformation

$$z(x,t) = \psi(\xi), \quad \xi = x + vt \tag{18}$$

with constants $k$ and $v$ will convert PDE to ODE which is of the form

$$F(\psi, \psi', \psi'', \psi''', \dots) = 0. \tag{19}$$

*Step* 2: Suppose solution of ODE (19) has been written of the form

$$\widetilde{F}(\chi) = a_q \left(\frac{G'}{G}\right)^q + a_{q-1}\left(\frac{G'}{G}\right)^{q-1} + \dots, \tag{20}$$

where $a_q$ are constants to be determine, where $q$ ranges from 0 to infinity. As $G = G(\chi)$ persuades the LDE of second order which is of the form written below:

$$G'' + \widehat{\lambda} G' + \widehat{\mu} G = 0. \tag{21}$$

where $a_q$ $(a_q \neq 0)$; where '$q$' is called the balance number, $a_{q-1}, \dots, a_0$, $\widehat{\lambda}$ and $\widehat{\mu}$ are constants to be determine later.

*Step* 3: Determining the positive integer '$q$' from (19), by equalizing the higher order nonlinear terms to the higher order derivatives.

Replacing (20) into (19) and use ODE (21), then gather all the entire terms of $\left(\frac{G'}{G}\right)$ containing homogeneous power, and equalize each coefficient to zero, resulting in a variety of algebraic equations for analysis $a_q, a_{q-1}, \dots, a_0$, $c, \widehat{\lambda}$ and $\widehat{\mu}$.

*Step* 4: The general solution of (21) is noted, then substituting the values of $a_q$ in (20) we will get the exact solutions of (1) which is of the for

$$\left(\frac{G'}{G}\right) = \begin{cases} \dfrac{\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}}{2}\left(\dfrac{C_1 \sinh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2-4\widehat{\mu}}\right)\chi + C_2\cosh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2-4\widehat{\mu}}\right)\chi}{C_1\cosh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2-4\widehat{\mu}}\right)\chi + C_2\sinh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2-4\widehat{\mu}}\right)\chi}\right) - \dfrac{\widehat{\lambda}}{2}, & \widehat{\lambda}^2 - 4\widehat{\mu} > 0, \\ \dfrac{\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}}{2}\left(\dfrac{-C_1 \sin\left(\frac{1}{2}\sqrt{4\widehat{\mu}-\widehat{\lambda}^2}\right)\chi + C_2\cos\left(\frac{1}{2}\sqrt{4\widehat{\mu}-\widehat{\lambda}^2}\right)\chi}{C_1\cos\left(\frac{1}{2}\sqrt{4\widehat{\mu}-\widehat{\lambda}^2}\right)\chi + C_2\sin\left(\frac{1}{2}\sqrt{4\widehat{\mu}-\widehat{\lambda}^2}\right)\chi}\right) - \dfrac{\widehat{\lambda}}{2}, & \widehat{\lambda}^2 - 4\widehat{\mu} < 0, \\ \dfrac{c_2}{c_1 + c_2\chi} - \dfrac{\widehat{\lambda}}{2}, & \widehat{\lambda}^2 - 4\widehat{\mu} = 0. \end{cases} \tag{22}$$

In this context, $C_1$ and $C_2$ both illustrates the arbitrary constants.

The traveling wave transformation for Hirota-Schrödinger equation (1) is performed as follows:

$$z(x,t) = F(\xi)e^{\iota\theta}, \tag{23}$$

where $\xi = x + vt$, $\theta = \beta x + rt$. Here, '$r$' is the wave number of the soliton and $r$, $v$ and $\beta$ are constants which are determined.

Substituting equation (23) into equation (1), yields the following:

$$(a - 3\gamma\beta)F''(\xi) + (-r - a\beta^2 + \gamma\beta^3)F(\xi) + (c - \sigma\beta)F^3(\xi) = 0. \tag{24}$$

By using the homogeneous balance principal in (24), we have $m = 1$, then solution of ODE (24) is of the form

$$F(\xi) = a_0 + a_1\left(\frac{G'}{G}\right). \tag{25}$$

Substituting (25) in (24), then collect all the coefficients with similar power of $\left(\frac{G'}{G}\right)$ together. Since (25) is the solution of (24), substituting (25) into (24), we obtained the following system of algebraic equation

$$\begin{cases} \left(\frac{G'}{G}\right)^3 : ca_1^3 + 2aa_1 - 6\gamma\beta a_1 - \sigma\beta a_1^3 = 0, \\ \left(\frac{G'}{G}\right)^2 : 3aa_1\lambda - 3\sigma\beta a_0 a_1^2 + 3ca_0a_1^2 - 9\gamma\beta\lambda a_1 = 0, \\ \left(\frac{G'}{G}\right)^1 : \gamma\beta^3 a_1 + 3ca_1a_0^2 - aa_1\beta^2 - 3\gamma\beta a_1\lambda^2 - ra_1 - 6\gamma\beta\mu a_1 - 3\sigma\beta a_1 a_0^2 + aa_1\lambda^2 + 2a\mu a_1 = 0, \\ \left(\frac{G'}{G}\right)^0 : aa_1\lambda\mu + \gamma\beta^3 a_0 - ra_0 + ca_0^3 - aa_0\beta^2 - 3\gamma\beta\lambda\mu a_1 - \sigma\beta a_0^3 = 0. \end{cases} \tag{26}$$

The solutions corresponding to equation (23) are given below:

$$\begin{cases} \gamma = \gamma, \ \ a = a, \ \ \mu = \mu, \ \ c = \dfrac{-2a + g\gamma\beta + \sigma\beta a_1^2}{a_1^2}, \\ r = \dfrac{\gamma\beta^2 a_1^2 + 2aa_0^2 - 6\gamma\beta a_0^2 + a\beta^2 a_1^2 + 6\gamma\beta\mu a_1^2 - 2a\mu a_1^2}{a_1^2}, \ \lambda = \dfrac{2a_0}{a_1}, \ \sigma = \sigma, \ a_0 = a_0, \ a_1 = a_1. \end{cases}$$

The general outcomes of equation (23) can be provided as:

*Case* I: When $\widehat{\lambda}^2 - 4\widehat{\mu} > 0$,

$$F(\xi) = a_0 + a_1\left(\frac{\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}}{2}\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}\right)\chi + C_2\cosh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}\right)\chi}{C_1\cosh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}\right)\chi + C_2\sinh\left(\frac{1}{2}\sqrt{\widehat{\lambda}^2 - 4\widehat{\mu}}\right)\chi}\right) - \frac{\widehat{\lambda}}{2}\right), \tag{27}$$

then solution is

$$z(x,t) = \left(a_0 + a_1\left(\frac{\sqrt{\left(\frac{2a_0}{a_1}\right)^2 - 4\mu}}{2}\left(\frac{C_1\sinh\left(\frac{1}{2}\sqrt{\left(\frac{2a_0}{a_1}\right)^2 - 4\mu}\right)(x+vt) + C_2\cosh\left(\frac{1}{2}\sqrt{\left(\frac{2a_0}{a_1}\right)^2 - 4\mu}\right)(x+vt)}{C_1\cosh\left(\frac{1}{2}\sqrt{\left(\frac{2a_0}{a_1}\right)^2 - 4\mu}\right)(x+vt) + C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\right)(x+vt)}\right) - \frac{\frac{2a_0}{a_1}}{2}\right)\right)e^{\iota(\beta x + rt)}. \tag{28}$$

*Case* II: When $\widehat{\lambda}^2 - 4\widehat{\mu} < 0$,

$$F(\xi) = a_0 + a_1\left(\frac{\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}}{2}\left(\frac{-C_1\sin\left(\frac{1}{2}\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}\right)\chi + C_2\cos\left(\frac{1}{2}\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}\right)\chi}{C_1\cos\left(\frac{1}{2}\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}\right)\chi + C_2\sin\left(\frac{1}{2}\sqrt{4\widehat{\mu} - \widehat{\lambda}^2}\right)\chi}\right) - \frac{\widehat{\lambda}}{2}\right), \tag{29}$$

then solution is

$$Z(x,t) = \left(a_0 + a_1\left(\frac{\sqrt{4\mu - \left(\frac{2a_0}{a_1}\right)^2}}{2}\left(\frac{-C_1\sin\left(\frac{1}{2}\sqrt{4\mu - \left(\frac{2a_0}{a_1}\right)^2}\right)(x+vt) + C_2\cos\left(\frac{1}{2}\sqrt{4\mu - \left(\frac{2a_0}{a_1}\right)^2}\right)(x+vt)}{C_1\cos\left(\frac{1}{2}\sqrt{4\mu - \left(\frac{2a_0}{a_1}\right)^2}\right)(x+vt) + C_2\sin\left(\frac{1}{2}\sqrt{4\mu - \left(\frac{2a_0}{a_1}\right)^2}\right)(x+vt)}\right) - \frac{\left(\frac{2a_0}{a_1}\right)}{2}\right)\right)e^{\iota(\beta x + rt)}. \tag{30}$$

*Case* III: When $\widehat{\lambda}^2 - 4\widehat{\mu} = 0$,

$$F(\xi) = a_0 + a_1\left(\frac{c_2}{c_1 + c_2\chi} - \frac{\widehat{\lambda}}{2}\right), \tag{31}$$

then solution is

$$z(x,t) = a_0 + a_1\left(\frac{c_2}{c_1 + c_2(x+vt)} - \frac{\left(\frac{2a_0}{a_1}\right)}{2}\right)e^{\iota(\beta x + rt)}. \tag{32}$$

The graphical representations are shown in Figure 1, Figure 2 and Figure 3 this shows 3-D view of 2-D solutions.

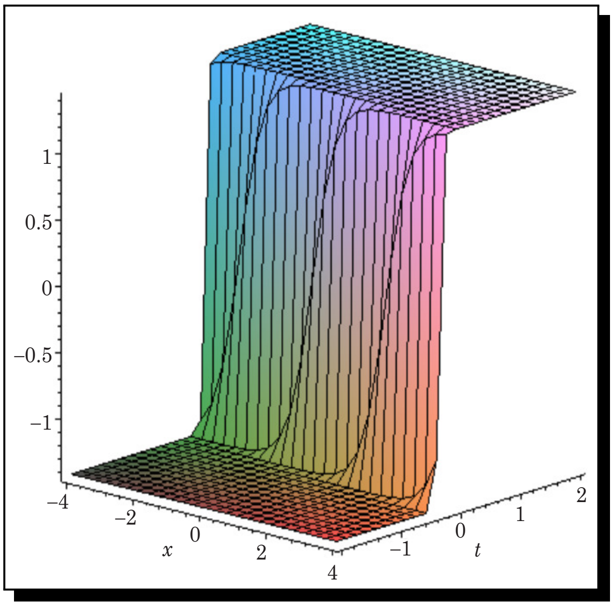


**Figure 1.** Traveling wave solution of $|z(x,t)|$ of equation (28) when $\hat{\lambda}^2 - 4\hat{\mu} > 0$ for $a_0 = 2$, $a_1 = 1$, $\hat{\lambda} = 4$, $\hat{\mu} = 2$, $C_1 = 9$, $C_2 = 4$, $\gamma = 1$, $\beta = 1$, $a = 1$, $c = 5$, $\xi = x + 15t$

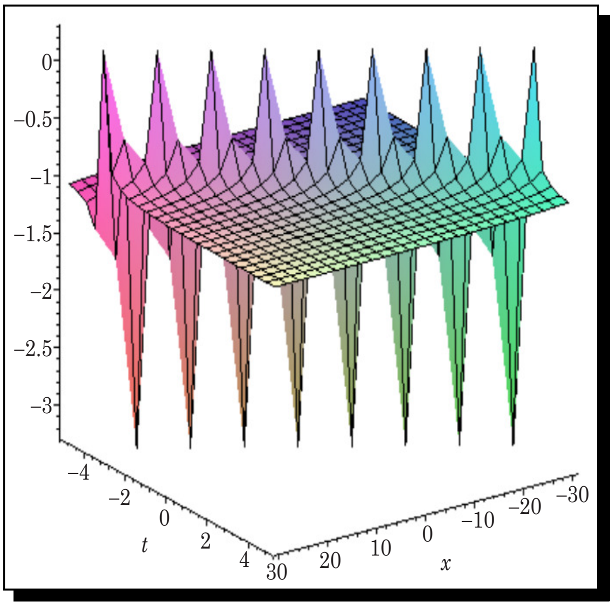


**Figure 2.** Traveling wave solution of $|z(x,t)|$ of equation (30) when $\hat{\lambda}^2 - 4\hat{\mu} < 0$ for $a_0 = 1$, $a_1 = 1$, $a = 1$, $C_1 = 4$, $C_2 = 7$, $\hat{\mu} = 4$, $\hat{\lambda} = 2$, $\gamma = 1$, $\beta = 1$, $\sigma = 1$, $\xi = x - 9t$

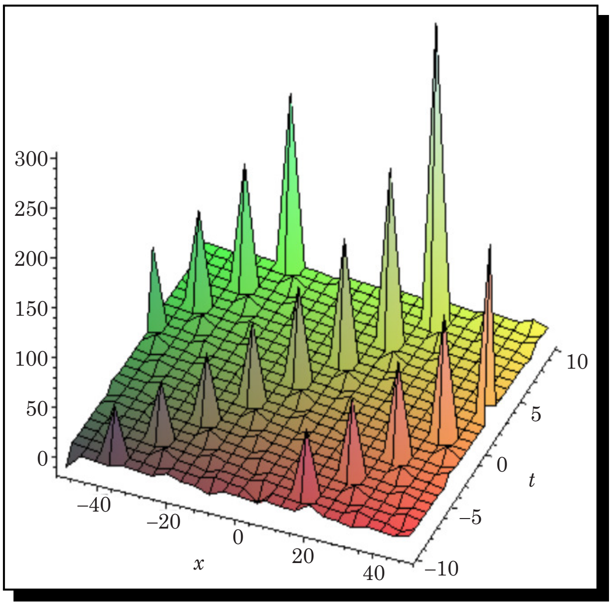


**Figure 3.** Traveling wave solution of $|z(x,t)|$ of equation (32) when $\hat{\lambda}^2 - 4\hat{\mu} = 0$ for $a_0 = 1$, $a_1 = 1$, $\beta = 1$, $\sigma = 1$, $C_1 = 4$, $C_2 = 5$, $\xi = x + 9t$

## 5. Qualitative Analysis

In this section, a qualitative approach (Du *et al.* [6]) is adopted to study the Hirota-Schrödinger equation based on the bifurcation theory of planar dynamical systems. After converting the equation into an autonomous system, bifurcation analysis is applied to the resulting ordinary differential equation. The autonomous system is then analyzed to identify its equilibrium points. These critical points are classified into four types: nodes, saddles, centers, and spirals. Phase portraits corresponding to various points are constructed to visually illustrate the nature and behavior of the system.

Let us examine the following complex transformation:

$$z(x,t) = e^{\iota\bar{\omega}}V(\bar{\delta}) \tag{33}$$

where $\bar{\omega} = jx - kt + \bar{\omega}_0$ and $\bar{\delta} = t - \bar{\xi}x + \bar{\delta}_0$. A nonlinear ODE is obtained by substituting (33) into (1). After decomposing the equation into its real and imaginary components, the following result emerges:

$$(k - \bar{a}j^2 + \bar{\gamma}j^3)V(\bar{\delta}) + (\bar{a}\bar{\xi}^2 + \bar{\gamma}j\bar{\xi}^2)V''(\bar{\delta}) + (\bar{c} - \bar{\sigma}j)V^3(\bar{\delta}) = 0, \tag{34}$$

$$(1 - 2\bar{a}\bar{\xi}j + 3\bar{\gamma}\bar{\xi}j^2)V'(\bar{\delta}) - \bar{\sigma}\bar{\xi}V^2(\bar{\delta})V'(\bar{\delta}) - \bar{\gamma}\bar{\xi}^3V'''(\bar{\delta}) = 0. \tag{35}$$

Integrating (35) with respect to $\bar{\delta}$ once and assuming a zero integration constant yields the following:

$$(1 - 2\bar{a}\bar{\xi}j + 3\bar{\gamma}\bar{\xi}j^2)V(\bar{\delta}) - \bar{\sigma}\bar{\xi}V^3(\bar{\delta}) - \bar{\gamma}\bar{\xi}^3V''(\bar{\delta}) = 0. \tag{36}$$

Equations (36) and (35) gives us

$$\frac{(k - \bar{a}j^2 + \bar{\gamma}j^3)}{(1 - 2\bar{a}\bar{\xi}j + 3\bar{\gamma}\bar{\xi}j^2)} = -\frac{(\bar{a}\bar{\xi}^2 + \bar{\gamma}j\bar{\xi}^2)}{\bar{\gamma}\bar{\xi}^3} = -\frac{(\bar{c} - \bar{\sigma}j)}{\bar{\sigma}\bar{\xi}}. \tag{37}$$

The dynamical system that results from applying the Galilean transformation to (18) is as follows:

$$\begin{cases} V'(\bar{\delta}) = U(\bar{\delta}), \\ U'(\bar{\delta}) = \bar{W}_1V^3(\bar{\delta}) + \bar{W}_2V(\bar{\delta}), \end{cases} \tag{38}$$

where

$$\bar{W}_1 = \frac{(\bar{\sigma}j - \bar{c})}{(\bar{a}\bar{\xi}^2 + \bar{\gamma}j\bar{\xi}^2)}, \quad \bar{W}_2 = \frac{(-k + \bar{a}j^2 - \bar{\gamma}j^3)}{(\bar{a}\bar{\xi}^2 + \bar{\gamma}j\bar{\xi}^2)}. \tag{39}$$

The Hamiltonian function for (38) is given by

$$H(V,U) = \frac{U^2}{2} - \bar{W}_1\frac{V^4}{4} - \bar{W}_2\frac{V^2}{2} = h,$$

where $h$ is hamiltonian constant. By solving the system the derived equilibrium points are

$$\epsilon_1 = (0,0), \quad \epsilon_2 = \left(-\iota\frac{\bar{W}_2}{\bar{W}_1}, 0\right), \quad \epsilon_2 = \left(\iota\frac{\bar{W}_2}{\bar{W}_1}, 0\right).$$

The determinant of the Jacobian matrix for the system in equation (38) is

$$D(V,U) = -\bar{W}_1V^2(\delta) - \bar{W}_2. \tag{40}$$

Depending upon determinant we have different cases:

(i) $(V,U)$ acts as a saddle point if $D(V,U) < 0$,

(ii) $(V,U)$ acts as a center point if $D(V,U) > 0$,

(iii) $(V,U)$ acts as a cuspid point if $D(V,U) = 0$.

The following outcomes can be achieved by modifying the relevant parameter:

*Case* 1: $\widehat{W}_1 > 0$ and $\widehat{W}_2 > 0$
Through choosing particular parameter values, it is observed that the only real equilibrium point is $(0,0)$, as shown in Figure 4. Clearly, $(0,0)$ corresponds to a saddle point.

*Case* 2: $\widehat{W}_1 < 0$ and $\widehat{W}_2 > 0$
The findings demonstrate three equilibrium points where $(0,0)$ works as a saddle point, as depicted in Figure 5. In addition, $(-2.4494,0)$ and $(2.4494,0)$ act as center points.

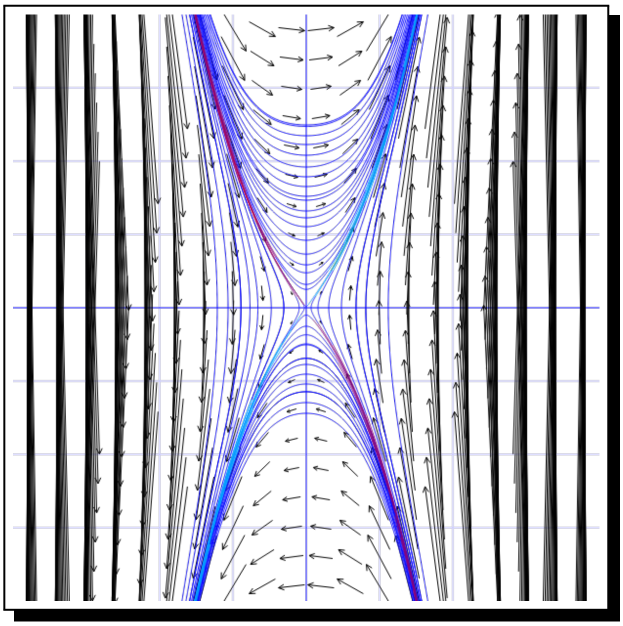

**Figure 4.** Phase portraits illustrating the bifurcations of the proposed system under the conditions $\widehat{W}_1 > 0$ and $\widehat{W}_2 > 0$ based on varying parameter values

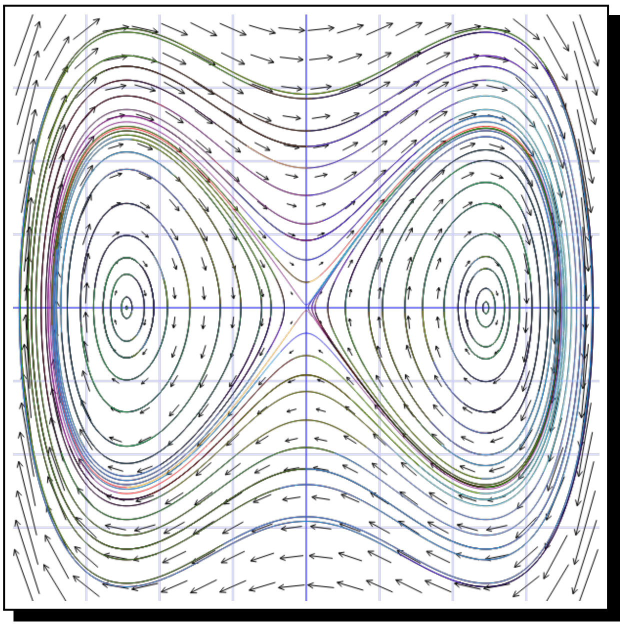

**Figure 5.** Phase portraits showing bifurcation behavior of the proposed system under the conditions $\widehat{W}_1 < 0$ and $\widehat{W}_2 > 0$ under different parameter values

*Case* 3: $\widehat{W}_1 < 0$ and $\widehat{W}_2 < 0$
Upon selecting suitable parameter values, it is observed that the only real (non-complex) Equilibrium point is $(0,0)$, as illustrated in Figure 6. This point clearly corresponds to a center.

*Case* 4: $\widehat{W}_1 > 0$ and $\widehat{W}_2 < 0$
By selecting appropriate parameter values, it is found that three equilibrium points $(0,0)$, $(-0.5773,0)$, and $(0.5773,0)$, as depicted in Figure 7. It is clear that, $(0,0)$ corresponds to a center point, while $(-0.5773,0)$ and $(0.5773,0)$ are saddle points.

## 6. Conclusion

In this paper, Lie symmetry method is implemented to the Hirota-Schrödinger equation to obtain similarity reductions. However, these reductions did not yield exact solutions. Consequently, the $\left(\frac{G'}{G}\right)$-expansion method was employed to derive exact solutions of the Hirota-Schrödinger equation. The obtained solutions are represented by rational, trigonometric, and hyperbolic functions and graphical representation shown in 3-D view. Then, investigate the bifurcation

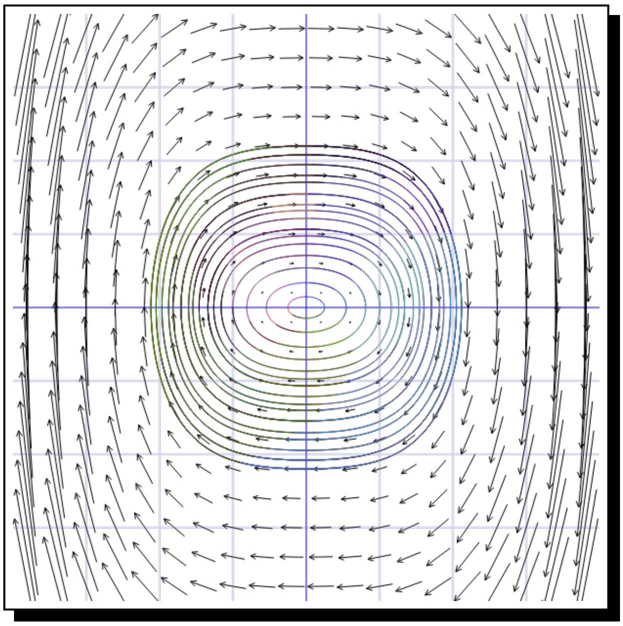

**Figure 6.** Phase portraits of the proposed systems bifurcations with condition $\widehat{W_1} < 0$ and $\widehat{W_2} < 0$ based on different parameter values

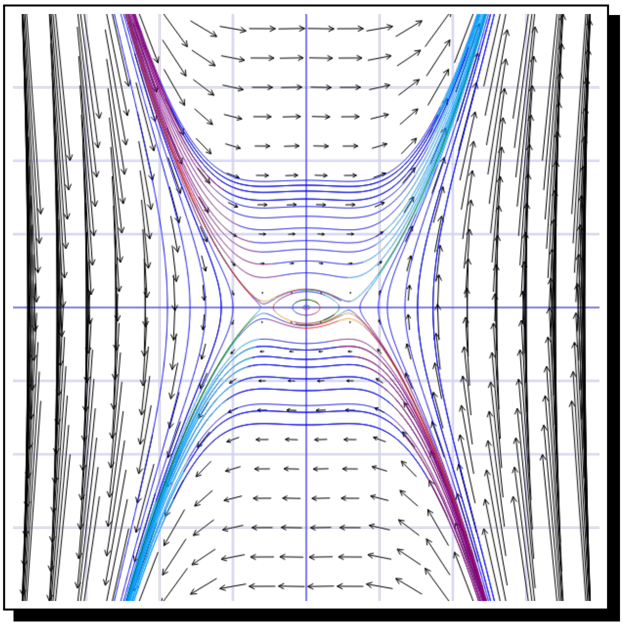

**Figure 7.** Phase portraits of the proposed systems bifurcations with condition $\widehat{W_1} > 0$ and $\widehat{W_2} <$ 0based on different parameter values

theory of the associated dynamical system corresponding to the nonlinear Hirota-Schrödinger equation. Through bifurcation analysis, examined the qualitative behavior of the system. By perturbing the governing dynamical system, various phase portraits were generated to explore the dynamic characteristics of the model.

## Competing Interests

The authors declare that they have no competing interests.

## Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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