Research Article

Existence of a Solution to an Infinite System of Weighted Atangana-Baleanu Fractional Integral Equations via Measure of Non-Compactness on a Tempered Sequence Space

Fahmida Nishat Yeasmin [ID] and Tazuddin Ahmed* [ID]

Department of Mathematics, Cotton University, Guwahati 781001, Assam, India
***Corresponding author:** tazuddin.ahmed@cottonuniversity.ac.in*

Abstract. The goal of this study is to investigate the existence of a solution of an infinite system of weighted Atangana-Baleanu fractional integral equations via a newly constructed generalised form of the Darbo's fixed-point theorem on a tempered sequence space. Furthermore, an example has been illustrated to demonstrate the effectiveness of the result obtained.

Keywords. Measure of Noncompactness (MNC), Fixed Point Theorems (FPT), Fractional Integral Equation (FIE)

Mathematics Subject Classification (2020). 45G15, 47H08, 47H10

1. Introduction

There is an immense application of *Fractional Integral Equations* (FIEs) in solving real-world practical problems across diverse fields. Because of the pivotal role of FIEs in various branches of science and engineering, it is crucial to understand such equations. The idea of *Measure of Noncompactness* (MNC) plays a remarkable role in *Fixed Point Theorems* (FPT). Kuratowski [16] introduced the concept of MNC in 1930. In 1955, G. Darbo [9] utilised the notion of MNC to develop a result which demonstrated that fixed point is present for the condensing operators. FPT and MNC have several implementations in the analysis of numerous integral equations

that arise in innumerable problems imperative to physical scenarios. Numerous research works related to FIEs have been studied using MNC and FPT. Some of such research works can be seen in Banaś and Mursaleen [6], and Das *et al*. [10–15].

2. Preliminaries

Let us denote a real *Banach Space* (BS) having the norm $\|\cdot\|$ by H.

Assume $B(\theta, r) = \{t \in H : \|t - \theta\| \leq r\}$. If $\mathfrak{B}(\neq \emptyset) \subset H$, then by $\mathrm{conv}\,\mathfrak{B}$ and $\bar{\mathfrak{B}}$ we identify the convex closure and closure of $\mathfrak{B}$. Additionally, let

- $\mathfrak{M}_H$ = the set of all bounded and non-empty subsets of H,

- $\mathfrak{N}_H$ = the set of all relatively compact sets.

MNC can be defined from [6] as follows:

Definition 2.1. A function $\widehat{\psi} : \mathfrak{M}_H \to \mathbb{R}_+$ is called an MNC in H if:

(i) for all $\mathfrak{B} \in \mathfrak{M}_H$, we get $\widehat{\psi}(\mathfrak{B}) = 0$ which shows that $\mathfrak{B}$ has relative compactness.

(ii) $\ker \widehat{\psi} = \{\mathfrak{B} \in \mathfrak{M}_H : \widehat{\psi}(\mathfrak{B}) = 0\} \neq \emptyset$ and $\ker \widehat{\psi} \subset \mathfrak{N}_H$.

(iii) $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \implies \widehat{\psi}(\mathfrak{B}_1) \leq \widehat{\psi}(\mathfrak{B}_2)$.

(iv) $\widehat{\psi}(\bar{\mathfrak{B}}) = \widehat{\psi}(\mathfrak{B})$.

(v) $\widehat{\psi}(\mathrm{conv}\,\mathfrak{B}) = \widehat{\psi}(\mathfrak{B})$.

(vi) $\widehat{\psi}(\rho \mathfrak{B} + (1 - \rho)U_1) \leq \rho \widehat{\psi}(\mathfrak{B}) + (1 - \rho)\widehat{\psi}(\mathfrak{B}_1)$, for any $\rho \in [0, 1]$.

(vii) If $\mathfrak{B}_1 \in \mathfrak{M}_H$, $\mathfrak{B}_l = \bar{\mathfrak{B}}_l$, $\mathfrak{B}_{l+1} \subset \mathfrak{B}_l$ for $l = 1, 2, 3, 4, \ldots$ and $\lim\limits_{l \to \infty} \widehat{\psi}(\mathfrak{B}_l) = 0$ then $\mathfrak{B}_\infty = \bigcap\limits_{l=1}^{\infty} \mathfrak{B}_l \neq \emptyset$.

The set $\mathfrak{B}_\infty = \bigcap\limits_{l=1}^{\infty} \mathfrak{B}_l \in \ker \widehat{\psi}$. Since $\widehat{\psi}(\mathfrak{B}_\infty) \leq \widehat{\psi}(\mathfrak{B}_l)$ for any l, we conclude $\widehat{\psi}(\mathfrak{B}_\infty) = 0$.

Some important theorems and definitions:

Theorem 2.2 ([15]). *Let $\mathfrak{B}$ be a non-empty, bounded, convex and closed (NBCC) subset of a BS H. Then, for every continuous as well as compact map $g : \mathfrak{B} \to \mathfrak{B}$ there exists at least one fixed point.*

Theorem 2.3 (Darbo [17]). *Let $\mathfrak{B}$ be a NBCC subset of a BS H and let there exists a continuous mapping $f : \mathfrak{B} \to \mathfrak{B}$. Let us consider a constant $b \in [0, 1)$ such that*

$$\widehat{\psi}(fJ) \leq b\widehat{\psi}(J), \quad J \subseteq \mathfrak{B}.$$

Then, there exists a fixed point in $\mathfrak{B}$ for f.

Definition 2.4 ([15]). Let $\mathbb{S}$ denote the collection of all functions $\lambda : [0, \infty) \to \mathbb{R}_+$ satisfying the conditions:

(i) $\lambda(\widetilde{x}) \leq \lambda(\widetilde{y}) \implies \widetilde{x} \leq \widetilde{y}$, for all $\widetilde{x}, \widetilde{y} \in \mathbb{R}_+$ and $\widetilde{x}, \widetilde{y} \geq 0$.

(ii) λ is non-decreasing and continuous.

As an illustration: $\lambda(\widetilde{x}) = \widetilde{x}_1$ and $\lambda(\widetilde{y}) = c\widetilde{x}_1$, $c \in [0, 1)$.

Definition 2.5 ([15]). A function $\mathfrak{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function belonging to $\mathcal{C}$-class if the following postulates are satisfied:

(i) $\mathfrak{J}(g, \widetilde{g}) \le g$,

(ii) $\mathfrak{J}(g, \widetilde{g}) = g \implies$ either $g = 0$ or $\widetilde{g} = 0$. Also, $\mathfrak{J}(0,0) = 0$.

As an illustration:

(i) $\mathfrak{J}(g, \widetilde{g}) = g - \widetilde{g}$,

(ii) $\mathfrak{J}(g, \widetilde{g}) = cg$, $0 < c < 1$.

Definition 2.6 ([14]). Let Ξ denote the set of all alternating distance functions $\xi : [0, \infty) \times \mathbb{R}_+ \to [0, \infty)$ if:

(i) $\xi(x) = 0 \Longleftrightarrow x = 0$.

(ii) ξ is increasing and continuous.

As an illustration: $\xi(x) = (1 - \widetilde{t})x$, $0 \le \widetilde{t} \le 1$.

Definition 2.7 ([14]). Let Φ denote the set of functions $\varkappa : [0, \infty) \to \mathbb{R}_+$, satisfying

$$\lim_{n \to \infty} \varkappa(a_n) = 0 \text{ if } \lim_{n \to \infty} a_n = 0,$$

where (a_n) is a non-negative sequence.

As an illustration:

(i) $\varkappa(t) = t^2$,

(ii) $\varkappa(t) = t^3$.

Definition 2.8 ([14]). Let $\varkappa \in \Phi$ and indicate by Π the set of all functions $\mathfrak{h} : [0, \infty) \to \mathbb{R}_+$ be a continuous map such that:

(i) $\mathfrak{h}(x) = 0 \Longleftrightarrow x = 0$.

(ii) $\lim_{n \to \infty} \varkappa(s_n) < \mathfrak{h}(s)$ if $\lim_{n \to \infty} s_n = s > 0$.

As an illustration:

(i) $\mathfrak{h}(t) = 2t^2$,

(ii) $\mathfrak{h}(t) = 3t^3$.

Definition 2.9. A continuous function $\phi : \mathbb{R}_+ \times [0, \infty) \to \mathbb{R}_+$ is identified as an ultra-altering distance function if $\phi(s) > 0$ and $\phi(0) \ge 0, s > 0$.
This function class is denoted by $\mathfrak{S}$.

Definition 2.10. Let $T_1, T_2 : \mathfrak{B} \subseteq H \to H$ and $\alpha : \mathfrak{M}_H \times \mathfrak{M}_H \to [0, \infty)$. It is said that the pair (T_1, T_2) is α-admissible if:

$$\alpha(\Omega_1, \Omega_2) \ge 1 \implies \alpha(\text{conv}\, T_1 \Omega_1, \text{conv}\, T_2 \Omega_2) \ge 1 \text{ and } \alpha(\text{conv}\, T_2 \Omega_1, \text{conv}\, T_1 \Omega_2) \ge 1,$$

where $\Omega_1, \Omega_2, T_1 \Omega_1, T_2 \Omega_2, T_1 \Omega_2, T_2 \Omega_1 \in \mathfrak{M}_H$.

Let $T_1 = T_2 = I$, and $H = \mathbb{R}$, also let

(i) $\alpha(s,t) = 2^{(t/s)}$, $(s,t) \geq 0$, $s \neq 0$,

(ii) $\alpha(s,t) = k$, $k > 1$.

3. Fixed Point Theory

Theorem 3.1. *Let H be a BS and $\mathfrak{B}$ be a NBCC subset of H. Also, $\mathfrak{T}_1, \mathfrak{T}_2 : \mathfrak{B} \to \mathfrak{B}$ are continuous mappings which are α-admissible with $\alpha(\mathbb{E}_1, \mathbb{F}_1) \geq 1$. Then,*

$$\alpha(\Omega_1, \Omega_2)[\lambda\{\mathfrak{h}(\xi(\widehat{\psi}(\mathfrak{T}_1\Omega_1), \gamma(\widehat{\psi}(\mathfrak{T}_2\Omega_2))))\}]$$

$$\leq \lambda[\mathfrak{J}\{\varkappa(\xi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))), \phi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))\}], \tag{3.1}$$

where $\Omega_1, \Omega_2 \subset \mathfrak{B}$ and $\widehat{\psi}$ is an MNC and $\xi \in \Xi$, $\varkappa \in \Phi$, $\mathfrak{h} \in \Pi$, $\phi \in \mathfrak{S}$. Also, $\mathfrak{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\lambda, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing and continuous mappings. Then, the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have at least one fixed point in $\mathfrak{B}$.

Proof. Consider the sequences $\mathbb{E}_p$ and $\mathbb{F}_p$ with $\mathbb{E}_1 = \mathbb{F}_1 = \mathfrak{B}$ and $\mathbb{E}_{p+1} = \text{conv}(\mathfrak{T}_1\mathbb{E}_p)$ and $\mathbb{F}_{p+1} = \text{conv}(\mathfrak{T}_2\mathbb{F}_p)$ for $p \in \mathbb{N}$. Then,

$$\mathfrak{T}_1\mathbb{E}_1 = \mathfrak{T}_1\mathfrak{B} \subseteq \mathfrak{B} = \mathbb{E}_1, \quad \mathbb{E}_2 = \text{conv}(\mathfrak{T}_1\mathbb{E}_1) \subseteq \mathfrak{B} = \mathbb{E}_1$$

and

$$\mathfrak{T}_2\mathbb{F}_1 = \mathfrak{T}_2\mathfrak{B} \subseteq \mathfrak{B} = \mathbb{F}_1, \quad \mathbb{F}_2 = \text{conv}(\mathfrak{T}_2\mathbb{F}_1) \subseteq \mathfrak{B} = \mathbb{F}_1.$$

Processing similarly, we get

$$\mathbb{E}_1 \supseteq \mathbb{E}_2 \supseteq \mathbb{E}_3 \supseteq \ldots \supseteq \mathbb{E}_p \supseteq \mathbb{E}_{p+1} \supseteq \ldots$$

and

$$\mathbb{F}_1 \supseteq \mathbb{F}_2 \supseteq \mathbb{F}_3 \supseteq \ldots \supseteq \mathbb{F}_p \supseteq \mathbb{F}_{p+1} \supseteq \ldots .$$

If $\widehat{\psi}(\mathbb{E}_{p_0}) = \widehat{\psi}(\mathbb{F}_{p_0}) = 0$, for some $p_0 \in \mathbb{N}$, $\mathbb{E}_{p_0}$ and $\mathbb{F}_{p_0}$ are compact sets. In this case, *Schauder's Theorem* implies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have fixed points in $\mathfrak{B}$.

Assume that $\widehat{\psi}(\mathbb{E}_p) > 0$ and $\widehat{\psi}(\mathbb{F}_p) > 0$, for p $\in \mathbb{N}$. Since, $\widehat{\psi}(\mathbb{E}_{p+1}) \leq \widehat{\psi}(\mathbb{E}_p)$ and $\widehat{\psi}(\mathbb{F}_{p+1}) \leq \widehat{\psi}(\mathbb{F}_p)$, therefore there exists $u_0, v_0 \geq 0$ satisfying

$$\lim_{p \to \infty} \widehat{\psi}(\mathbb{E}_p) = u_0 \quad \text{and} \quad \lim_{p \to \infty} \widehat{\psi}(\mathbb{F}_p) = v_0.$$

As $\mathfrak{T}_1, \mathfrak{T}_2$ are *α-admissible* and $\alpha(\mathbb{E}_1, \mathbb{F}_1) \geq 1$, thus, $\alpha(\mathbb{E}_2, \mathbb{F}_2) = \alpha(\text{conv}\,\mathfrak{T}_1\mathbb{E}_1, \text{conv}\,\mathfrak{T}_2\mathbb{F}_1) \geq 1$. By the *Principle of Mathematical Induction*, we can conclude that $\alpha(\mathbb{E}_p, \mathbb{F}_p) \geq 1$ for every $p \geq 1$.

Now,

$$\lambda[\mathfrak{h}(\xi(\widehat{\psi}(\mathbb{E}_{p+1}), \gamma(\widehat{\psi}(\mathbb{F}_{p+1}))))] = \lambda[\mathfrak{h}(\xi(\widehat{\psi}(\text{conv}(\mathfrak{T}_1\mathbb{E}_p)), \gamma(\widehat{\psi}(\text{conv}(\mathfrak{T}_2\mathbb{F}_p)))))]$$

$$\leq \alpha(\mathbb{E}_p, \mathbb{F}_p)\lambda[\mathfrak{h}(\xi(\widehat{\psi}(\mathfrak{T}_1\mathbb{E}_p), \gamma(\widehat{\psi}(\mathfrak{T}_2\mathbb{F}_p))))]$$

$$\leq \lambda[\mathfrak{J}\{\varkappa(\xi(\widehat{\psi}(\mathbb{E}_p), \gamma(\widehat{\psi}(\mathbb{F}_p)))), \phi(\widehat{\psi}(\mathbb{E}_p), \gamma(\widehat{\psi}(\mathbb{F}_p)))\}].$$

By property of λ,

$$\mathfrak{h}(\xi(\widehat{\psi}(\mathfrak{T}_1\mathbb{E}_p), \gamma(\widehat{\psi}(\mathfrak{T}_2\mathbb{F}_p)))) \leq \mathfrak{J}\{\varkappa(\xi(\widehat{\psi}(\mathbb{E}_p), \gamma(\widehat{\psi}(\mathbb{F}_p)))), \phi(\widehat{\psi}(\mathbb{E}_p), \gamma(\widehat{\psi}(\mathbb{F}_p)))\}.$$

As $p \to \infty$, we get

$$\mathfrak{h}(\xi(u_0, \gamma(v_0))) \leq \mathfrak{J}\{\varkappa(\xi(u_0, \gamma(v_0))), \phi(u_0, \gamma(v_0))\}.$$

Hence by property of $\mathfrak{J}$,

$$\mathfrak{h}(\xi(u_0, \gamma(v_0))) \le \varkappa(\xi(u_0, \gamma(v_0))).$$

Now, by property of $\varkappa$

$$\varkappa(\xi(u_0, \gamma(v_0))) < h(\xi(u_0, (\gamma v_0)))$$

which is a contradiction.

Therefore, we get

$$u_0 = v_0 = 0,$$

i.e.,

$$\lim_{p \to \infty} \widehat{\psi}(\mathbb{F}_p) = 0 = \lim_{p \to \infty} \widehat{\psi}(\mathbb{E}_p).$$

Since $\mathbb{E}_p \supseteq \mathbb{E}_{p+1}$, and $\mathbb{F}_p \supseteq \mathbb{F}_{p+1}$, so with the help of Definition 2.1, we obtain $\mathbb{E}_\infty = \bigcap\limits_{p=1}^{\infty} \mathbb{E}_p$ and $\mathbb{F}_\infty = \bigcap\limits_{p=1}^{\infty} \mathbb{F}_p$ are closed, convex and non-empty subsets of $\mathfrak{P}$ and $\mathbb{E}_\infty$ is $\mathfrak{T}_1$ invariant while $\mathbb{F}_\infty$ is $\mathfrak{T}_2$ invariant.

Hence, Theorem 3.1 concludes that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have fixed points in $\mathfrak{B}$. $\qquad\square$

Theorem 3.2. *Let H be a BS and $\mathfrak{B}$ be a NBCC subset of H. Also, $\mathfrak{T}_1, \mathfrak{T}_2 : \mathfrak{B} \to \mathfrak{B}$ are continuous mappings which are α-admissible with $\alpha(\mathbb{E}_1, \mathbb{F}_1) \ge 1$. Then,*

$$\alpha(\Omega_1, \Omega_2)\mathfrak{h}(\xi(\widehat{\psi}(\mathfrak{T}_1\Omega_1), \gamma(\widehat{\psi}(\mathfrak{T}_2\Omega_2)))) \le \mathfrak{J}\{\varkappa(\xi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))), \phi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))\}, \quad (3.2)$$

where $\Omega_1, \Omega_2 \subseteq \mathfrak{B}$ and $\widehat{\psi}$ is an MNC and $\xi \in \Xi$, $\varkappa \in \Phi$, $\mathfrak{h} \in \Pi$, $\phi \in \mathfrak{S}$. Also, $\mathfrak{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing and continuous mappings. Then, the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have at least one fixed point in $\mathfrak{B}$.

Proof. Taking $\lambda(\widetilde{x}) \le \lambda(\widetilde{y}) \implies \widetilde{x} \le \widetilde{y}$, for all $\widetilde{x}, \widetilde{y} \in \mathbb{R}_+$ and $\widetilde{x}, \widetilde{y} \ge 0$, in Theorem 3.1 the above-mentioned result can be obtained. $\qquad\square$

Theorem 3.3. *Let H be a BS and $\mathfrak{B}$ be a NBCC subset of H. Also, $\mathfrak{T}_1, \mathfrak{T}_2 : \mathfrak{B} \to \mathfrak{B}$ are continuous mappings with*

$$\mathfrak{h}(\xi(\widehat{\psi}(\mathfrak{T}_1\Omega_1), \gamma(\widehat{\psi}(\mathfrak{T}_2\Omega_2)))) \le \mathfrak{J}\{\varkappa(\xi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))), \phi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\Omega_2)))\}, \quad (3.3)$$

where $\Omega_1, \Omega_2 \subseteq \mathfrak{B}$ and $\widehat{\psi}$ is an MNC and $\xi \in \Xi$, $\varkappa \in \Phi$, $\mathfrak{h} \in \Pi$, $\phi \in \mathfrak{S}$. Also, $\mathfrak{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ are non-decreasing and continuous mappings. Then, the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have at least one fixed point in $\mathfrak{B}$.

Proof. Taking $\alpha(\Omega_1, \Omega_2) = 1$ in Theorem 3.2 the above mentioned result can be obtained. $\qquad\square$

Theorem 3.4. *Let H be a BS and $\mathfrak{B}$ be a NBCC subset of H. Also, $\mathfrak{T}_1, \mathfrak{T}_2 : \mathfrak{B} \to \mathfrak{B}$ are continuous mappings with*

$$\xi(\widehat{\psi}(\mathfrak{T}_1\Omega_1), \gamma(\widehat{\psi}(\mathfrak{T}_2\Omega_2))) \le \varkappa(\xi(\widehat{\psi}(\Omega_1), \gamma(\widehat{\psi}(\mathfrak{T}_2\Omega_2)))), \quad (3.4)$$

where $\Omega_1, \Omega_2 \subseteq \mathfrak{B}$ and $\widehat{\psi}$ is an MNC and $\xi \in \Xi$, $\varkappa \in \Phi$. Also, $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing and continuous mapping. Then, the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have at-least one fixed point in $\mathfrak{B}$.

Proof. Taking $h(\mathfrak{u}) = \mathfrak{u}$ and $\mathfrak{J}(\mathfrak{u}) \leq \mathfrak{u}$ in the Theorem 3.3, the above mentioned result can be obtained. $\square$

Corollary 3.5. *In Theorem* 3.4*, let* $\mathfrak{T} = \mathfrak{T}_1 = \mathfrak{T}_2$*. Also, let* $\gamma(\mathfrak{u}) = 0$*,* $\varkappa(\mathfrak{u}) = k\mathfrak{u}$ *and* $\xi(\mathfrak{u}) = \mathfrak{u}$ *for all* $\mathfrak{u} \geq 0$ *where* $k \in [0, 1)$*. We get*

$$\widehat{\psi}(\mathfrak{T}\Omega) \leq k\widehat{\psi}(\Omega),$$

where $\Omega \subseteq \mathfrak{B}$*. Thus, we obtain the Darbo's FPT.*

4. Measure of Noncompactness on a Tempered Sequence Space

Banaś and Krajewska [5] established the idea of tempered sequence spaces as well as tempering sequence as given below.

A tempering sequence has been defined as a fixed positive real sequence $\mathfrak{u} = (\mathfrak{u}_i)_{i=1}^{\infty}$ that is also non-increasing.

The set $\bigcup$ was recently developed by Rabbani *et al.* [18], which encompasses any complex or real sequence $\rho = (\rho_i)_{i=1}^{\infty}$ so that $\sum_{i=1}^{\infty} \mathfrak{u}_i^p |\rho_i|^p < \infty$, $1 \leq p < \infty$. It can be seen that $\bigcup$ creates a linear space over $\mathbb{R}$ (or $\mathbb{C}$), denoted as $\bigcup = l_p^{\mathfrak{u}}$, for $1 \leq p < \infty$.

Evidently, $l_p^{\mathfrak{u}}$ for $1 \leq p < \infty$ is a BS with the norm

$$\|\rho\|_{l_p^{\mathfrak{u}}} = \left(\sum_{i=1}^{\infty} \mathfrak{u}_i^p |\rho_i|^p \right)^{\frac{1}{p}}.$$

Choosing $\mathfrak{u}_i = 1$, for all $i = 1, 2, 3, \ldots$ then $l_p^{\mathfrak{u}} = l_p$, for $1 \leq p < \infty$.

The Hausdorff MNC $\Phi_{l_p^{\mathfrak{u}}}$ as stated in [18] for a non-empty, bounded set $B^{\mathfrak{u}}$ of $l_p^{\mathfrak{u}}$ $(1 \leq p < \infty)$ can be given by

$$\Phi_{l_p^{\mathfrak{u}}}(B^{\mathfrak{u}}) = \lim_{n \to \infty} \left[\sup_{p \in B^{\mathfrak{u}}} \left(\sum_{k \geq n} \mathfrak{u}_k^p |\rho_k|^p \right)^{\frac{1}{p}} \right].$$

Let us consider the space of all continuous maps $C(I, l_p^{\mathfrak{u}})$, where $I = [0, \mathfrak{u}]$, $\mathfrak{u} > 0$, having value in $l_p^{\mathfrak{u}}$ $(1 \leq p < \infty)$ which is a BS with the norm

$$\|\rho\|_{C(I, l_p^{\mathfrak{u}})} = \sup_{t \in I} \|\rho(t)\|_{l_p^{\mathfrak{u}}},$$

where $\rho(t) = (\rho_i(t))_{i=1}^{\infty} \in C(I, l_p^{\mathfrak{u}})$.

Let $E^{\mathfrak{u}}(t) = \{\rho(t) : \rho \in E^{\mathfrak{u}}\}$, for every $t \in I$. Thus, the MNC for $E^{\mathfrak{u}} \subset C(I, l_p^{\mathfrak{u}})$ is defined as

$$\Phi_{C(I, l_p^{\mathfrak{u}})}(E^{\mathfrak{u}}) = \sup_{t \in I} \Phi_{l_p^{\mathfrak{u}}}(E^{\mathfrak{u}}(t)).$$

5. Solvability of Fractional Integral Equation

For $0 < \mathfrak{u} < 1$, the weighted Atangana-Baleanu fractional integral of order $\mathfrak{u}$ of $f \in L^1(0, T)$ with respect to the weight function w is defined by Al-Refai [2] as

$$_c I_w^{\mathfrak{u}} f(\chi) = \frac{1 - \mathfrak{u}}{M(\mathfrak{u})} f(\chi) + \frac{\mathfrak{u}}{M(\mathfrak{u}) \Gamma(\mathfrak{u}) w(\chi)} \int_0^{\chi} (\chi - s)^{\mathfrak{u} - 1} w(s) f(s) ds. \tag{5.1}$$

In the following section we scrutinize the given FIE,

$$\aleph(\chi) = G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})}H_n(\chi, s, \aleph(\chi)) + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)}\int_0^\chi (\chi - s)^{\mathfrak{u}-1}w(s)H_n(\chi, s, \aleph(s))ds\right), \quad (5.2)$$

where $0 < \mathfrak{u} < 1$, $\chi \in I = [0, T]$, $T > 0$, $n \in \mathbb{N}$.

Also, $\aleph(\chi) = (\aleph_n(\chi))_{n=1}^\infty \in \mathcal{A}$ and $\mathcal{A}$ denotes a tempered sequence space.

Assume:

(i) $G_n : I \times C(I, l_p^\mathfrak{u}) \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$|G_n(\chi, \aleph(\chi), l) - G_n(\chi, \aleph(\chi), m)|^p \le \beta_n(\chi)|\aleph_n(\chi) - \widehat{\aleph}_n(\chi)|^p + P_n(\chi)|l - m|^p,$$

for all $\aleph(\chi) = (\aleph_n(\chi))_{n=1}^\infty \in C(I, l_p^\mathfrak{u})$ and $\beta_n, \chi_n : I \to \mathbb{R}_+$ are continuous maps for all $n \in \mathbb{N}$.

Also,

$$\sum_{n=1}^\infty \mathfrak{u}_n^p |G_n(\chi, \aleph^0, 0)|^p \to 0,$$

for all $\chi \in I$ where $\aleph^0 = (\aleph_n^0(\chi))_{n=1}^\infty \in C(I, l_p^\mathfrak{u})$.

(ii) $H_n : I \times I \times C(I, l_p^\mathfrak{u}) \to \mathbb{R}$ is continuous and there exists

$$\widehat{H}_n = \sup\{|H_n(\chi, s, \aleph(s))| : s, \chi \in I; \aleph(s) \in C(I, l_p^\mathfrak{u})\}.$$

Also,

$$Q_k = \sum_{n \ge k} \mathfrak{u}_n^p \widehat{H}_n^p,$$

$$\sup_{k \in \mathbb{N}} Q_k = \widehat{Q} \text{ and } \lim_{k \to \infty} Q_k = 0.$$

(iii) The operator G is defined from $I \times C(I, l_p^\mathfrak{u}) \to C(I, l_p^\mathfrak{u})$ as

$$(\chi, \aleph(\chi)) \to (G\aleph)(\chi),$$

where

$$(G\aleph)(\chi) = \left(G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})}H_n(\chi, s, \aleph(\chi))\right.\right.$$

$$\left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)}\int_0^\chi (\chi - s)^{\mathfrak{u}-1}w(s)H_n(\chi, s, \aleph(s))ds\right)\right)_{n=1}^\infty.$$

(iv) Let

$$\sup_{\widetilde{t} \in I} \beta_n(\chi) = \widehat{\beta}_n,$$

$$\sup_{n \in \mathbb{N}} \widehat{\beta}_n = \widehat{\beta}, \quad 0 < 2^{1-\frac{1}{p}}\widehat{\beta}^{\frac{1}{p}} < 1.$$

Also, for all $\chi \in I$,

$$\sum \mathfrak{u}_n^p P_n(\chi) \le \widehat{\mathfrak{m}}.$$

Let $B_{p,\mathfrak{u},\widehat{r}} = \{\aleph \in C(I, l_p^\mathfrak{u}) : \|\aleph\|_{C(I, l_p^\mathfrak{u})} \le \widehat{r}\}.$

(v) Let $w : I \to \mathbb{R}$ be C^1 and non-decreasing and

$$0 < l \le |w(\chi)| \le L, \quad \text{for all } \chi \in I.$$

Theorem 5.1. *If constraints* (i)-(v) *sustain, then equation* (5.2) *has a solution in* $C(I, l_p^{\mathfrak{u}})$, *where* $p > 1$.

Proof. For a fixed $\chi \in I$:

Phase 1: In this phase, we show that the operators $\mathfrak{T}$ and $\widehat{\widehat{\mathfrak{T}}}$ map $B_{p,\mathfrak{u},\widehat{r}}$ into $B_{p,\mathfrak{u},\widehat{r}}$,

$$
\begin{aligned}
\|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p &= \sum_{n \geq 1} \mathfrak{u}_n^p \left| G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi, s, \aleph(\chi))\right.\right. \\
&\quad \left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi - s)^{\mathfrak{u}-1} w(s) H_n(\chi, s, \aleph(s)) ds \right) \right|^p \\
&\leq 2^{p-1} \sum_{n \geq 1} \mathfrak{u}_n^p \left| G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi, s, \aleph(\chi))\right.\right. \\
&\quad \left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi - s)^{\mathfrak{u}-1} w(s) H_n(\chi, s, \aleph(s)) ds \right) - G_n(\chi, \aleph^0(\chi), 0) \right|^p \\
&\quad + 2^{p-1} \sum_{n \geq 1} \mathfrak{u}_n^p |G_n(\chi, \aleph^0(\chi), 0)|^p \\
&\leq 2^{p-1} \sum_{n \geq 1} \mathfrak{u}_n^p \left[\beta_n(\chi) |\aleph_n(\chi)|^p + P_n(\chi) \left| \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi, s, \aleph(\chi)) \right.\right. \\
&\quad \left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi - s)^{\mathfrak{u}-1} w(s) H_n(\chi, s, \aleph(s)) ds \right|^p \right] \\
&\leq 2^{p-1} \widehat{\beta} \|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p + \frac{2^{p-1}\widehat{\mathfrak{m}}}{M(\mathfrak{u})^p} \sum_{n \geq 1} (1-\mathfrak{u})^p \widehat{H}_n^p + \frac{2^{p-1}\widehat{\mathfrak{m}}L^p \mathfrak{u}^p}{M(\mathfrak{u})^p \Gamma(\mathfrak{u})^p l^p} \sum_{n \geq 1} \mathfrak{u}_n^p \widehat{H}_n^p \left[\int_0^\chi (\chi - s)^{\mathfrak{u}-1} ds \right]^p \\
&\leq 2^{p-1} \widehat{\beta} \|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p + \frac{2^{p-1}\widehat{\mathfrak{m}}}{M(\mathfrak{u})^p} \widehat{Q} \left[\frac{(1-\mathfrak{u})^p}{\mathfrak{u}_n^p} \right] + \frac{2^{p-1}\widehat{\mathfrak{m}}\mathfrak{u}^p L^p}{M(\mathfrak{u})^p \Gamma(\mathfrak{u})^p l^p} \widehat{Q} \left[\int_0^\chi (\chi - s)^{\mathfrak{u}-1} ds \right]^p \\
&\leq 2^{p-1} \widehat{\beta} \|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p + \frac{2^{p-1}\widehat{\mathfrak{m}}}{M(\mathfrak{u})^p} \widehat{Q} \left[\frac{(1-\mathfrak{u})^p}{\mathfrak{u}_n^p} + \frac{T^{\mathfrak{u}p}L^p}{\Gamma \mathfrak{u}^p l^p} \right].
\end{aligned}
$$

Let us assume,

$$
\|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p \leq \frac{2^{p-1}\widehat{\mathfrak{m}}}{M(\mathfrak{u})^p} \widehat{Q} \left[\frac{(1-\mathfrak{u})^p}{\mathfrak{u}_n^p} + \frac{T^{\mathfrak{u}p}L^p}{\Gamma \mathfrak{u}^p l^p} \right] = \widehat{r}^p.
$$

Hence $\|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p \leq \widehat{r}$.

Let $G : I \times B_{p,\mathfrak{u},\widehat{r}} \to B_{p,\mathfrak{u},\widehat{r}}$ be an operator given by

$$
\begin{aligned}
(G\aleph)(\chi) &= \left(G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi, s, \aleph(\chi))\right.\right. \\
&\quad \left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi - s)^{\mathfrak{u}-1} w(s) H_n(\chi, s, \aleph(s)) ds \right) \right)_{n=1}^\infty \\
&= ((G_n\aleph)(\chi))_{n=1}^\infty,
\end{aligned}
$$

where $\aleph(\chi) \in B_{p,\mathfrak{u},\widehat{r}}$ and $\chi \in I$.

By assumption (iii),

$$
\sum_{n \geq 1} (G_n\aleph)(\chi) \text{ is finite and unique.}
$$

Hence,

$$(G\aleph)(\chi) \in C(I, l_p^{\mathfrak{u}}).$$

Again,

$$\|G\aleph\|_{C(I,l_p^{\mathfrak{u}})} \le \widehat{r}.$$

Thus, G is a self mapping on $B_{p,\mathfrak{u},\widehat{r}}$.

Phase 2: Let $\widehat{\aleph}(\chi) = (\widehat{\aleph}_n(\chi))_{n=1}^\infty \in B_{p,\mathfrak{u},\widehat{r}}$ and $\epsilon > 0$ such that $\|\aleph - \widehat{\aleph}\|_{C(I,l_p^{\mathfrak{u}})} < \frac{\epsilon}{2\widehat{\beta}} = \delta$.

For a fixed $\widetilde{t} \in I$,

$$|(G_n\aleph)(\chi) - (G_n\widehat{\aleph})(\chi)|^p$$

$$= \left| G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi,s,\aleph(\chi)) + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi-s)^{\mathfrak{u}-1} w(s) H_n(\chi,s,\aleph(s)) ds\right) \right.$$

$$\left. - G_n\left(\chi, \widehat{\aleph}(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi,s,\widehat{\aleph}(\chi)) + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi-s)^{\mathfrak{u}-1} w(s) H_n(\chi,s,\widehat{\aleph}(s)) ds\right) \right|^p$$

$$\le \beta_n(\chi)|\aleph_n(\chi) - \widehat{\aleph}_n(\chi)|^p + P_n(\chi) \left| \left(\frac{1-\mathfrak{u}}{M(\mathfrak{u})}[H_n(\chi,s,\aleph(\chi)) - H_n(\chi,s,\widehat{\aleph}(\chi))] + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \right.\right.$$

$$\left.\left. \cdot \int_0^\chi (\chi-s)^{\mathfrak{u}-1} w(s)[H_n(\chi,s,\aleph(s)) - H_n(\chi,s,\widehat{\aleph}(s))] ds\right) \right|^p$$

$$\le \beta_n(\chi)|\aleph_n(\chi) - \widehat{\aleph}_n(\chi)|^p + \frac{P_n(\chi)(1-\mathfrak{u})^p}{M(\mathfrak{u})^p} |H_n(\chi,s,\aleph(s)) - H_n(\chi,s,\widehat{\aleph}(s))|^p$$

$$+ \frac{P_n(\chi)\mathfrak{u}^p L^p}{M(\mathfrak{u})^p \Gamma\mathfrak{u}^p l^p} \left| \int_0^\chi (\chi-s)^{\mathfrak{u}-1}[H_n(\chi,s,\aleph(s)) - H_n(\chi,s,\widehat{\aleph}(s))] ds \right|^p.$$

As H_n is continuous $\forall\ n \in \mathbb{N}$, so for $\|\aleph - \widehat{\aleph}\|_{C(I,l_p^{\mathfrak{u}})}^p < \frac{\epsilon^p}{2\widehat{\beta}}$, we have

$$\mathfrak{u}_n |H_n(\chi,s,\aleph(s)) - H_n(\chi,s,\widehat{\aleph}(s))| < \frac{\Gamma\mathfrak{u}\epsilon M(\mathfrak{u})}{\widehat{\mathfrak{m}}^{\frac{1}{p}} 2^{\frac{1}{p}}}, \quad \forall\ n \in \mathbb{N}.$$

Therefore,

$$\sum_{n\ge 1} \mathfrak{u}_n^p |(G_n\aleph)(\chi) - (G_n\widehat{\aleph})(\chi)|^p \le \widehat{\beta} \sum_{n\ge 1} \mathfrak{u}_n^p |\aleph_n(\chi) - \widehat{\aleph}_n(\chi)|^p + \frac{(1-\mathfrak{u})^p (\Gamma\mathfrak{u})^p \epsilon^p L^p}{2l^p} + \frac{T^{\mathfrak{u}p} \epsilon^p}{2}.$$

When $\epsilon \to 0$, we get

$$\|(G_n\aleph)(\chi) - (G_n\widehat{\aleph})(\chi)\|_{C(I,l_p^{\mathfrak{u}})}^p \to 0, \quad \text{where } \|\aleph - \widehat{\aleph}\| < \frac{\epsilon}{2\widehat{\beta}}.$$

Hence G is continuous on $B_{p,\mathfrak{u},\widehat{r}}$.

Phase 3:

$$\widehat{\psi}_{l_p^{\mathfrak{u}}}(GB_{p,\mathfrak{u},\widehat{r}})$$

$$= \lim_{n\to\infty} \sup_{\aleph \in B_{p,\mathfrak{u},\widehat{r}}} \left(\sum_{k\ge n} \mathfrak{u}_k^p \left| G_n\left(\chi, \aleph(\chi), \frac{1-\mathfrak{u}}{M(\mathfrak{u})} H_n(\chi,s,\aleph(\chi)) \right.\right.\right.$$

$$\left.\left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)} \int_0^\chi (\chi-s)^{\mathfrak{u}-1} w(s) H_n(\chi,s,\aleph(s)) ds\right) \right|^p \right)^{\frac{1}{p}}$$

$$\leq \lim_{n \to \infty} \sup_{\aleph \in B_{p,\mathfrak{u},\widehat{r}}} \left(2^{p-1} \sum_{k \geq n} \mathfrak{u}_k^p \left[\widehat{\beta}|\aleph_n(\chi)|^p + P_n(\chi)\left(\frac{1-\mathfrak{u}}{M(\mathfrak{u})}H_n(\chi,s,\aleph(\chi))\right.\right.\right.$$

$$\left.\left.\left. + \frac{\mathfrak{u}}{M(\mathfrak{u})\Gamma(\mathfrak{u})w(\chi)}\int_0^\chi (\chi-s)^{\mathfrak{u}-1}w(s)H_n(\chi,s,\aleph(s))ds\right)\right|^p\right)^{\frac{1}{p}}$$

$$\leq 2^{1-\frac{1}{p}}\lim_{n \to \infty}\sup_{\aleph \in B_{p,\mathfrak{u},\widehat{r}}}\left(\widehat{\beta}^{\frac{1}{p}}\sum_{k \geq n}\mathfrak{u}_k^p|\aleph_n(\chi)|^p + \frac{(1-\mathfrak{u})^p\widehat{\mathfrak{m}}Q_n}{M(\mathfrak{u})^p} + \frac{\widehat{\mathfrak{m}}L^pT^{\mathfrak{u}p}}{M(\mathfrak{u})^p(\Gamma\mathfrak{u})^pl^p}Q_n\right)^{\frac{1}{p}},$$

i.e.,

$$\widehat{\psi}_{l_p^{\mathfrak{u}}}(GB_{p,\mathfrak{u},\widehat{r}}) \leq 2^{1-\frac{1}{p}}\widehat{\beta}^{\frac{1}{p}}\widehat{\psi}_{l_p^{\mathfrak{u}}}(B_{p,\mathfrak{u},\widehat{r}}).$$

Therefore,

$$\widehat{\psi}_{C(I,l_p^{\mathfrak{u}})}(GB_{p,\mathfrak{u},\widehat{r}}) \leq 2^{1-\frac{1}{p}}\widehat{\beta}^{\frac{1}{p}}\widehat{\psi}_{C(I,l_p^{\mathfrak{u}})}(B_{p,\mathfrak{u},\widehat{r}}).$$

Thus, G has at least one fixed point in $(B_{p,\mathfrak{u},\widehat{r}}) \subseteq C(I,l_p^{\mathfrak{u}})$.

Therefore, eq. (5.2) possesses a solution in $C(I,l_p^{\mathfrak{u}})$. $\square$

Example 5.2. Let

$$\aleph_n(\chi) = \frac{\aleph_n(\chi)}{5n^2+\chi} + \frac{1}{2n(\chi+n^2)} + \frac{1}{2n\Gamma\frac{1}{2}}\int_0^\chi (\chi-s)^{-\frac{1}{2}}\frac{1}{(s+n^2)}ds, \tag{5.3}$$

where $\chi \in \left[\frac{1}{2},1\right] = I$, $n \in \mathbb{N}$.

Here,

$$G_n(\chi,\aleph(\chi),l(\aleph(\chi))) = \frac{\aleph_n(\chi)}{5n^2+\chi} + \frac{l(\aleph(\chi))}{n},$$

$$H_n(\chi,s,\aleph(s)) = \frac{1}{s+n^2},$$

$$l(\aleph(\chi)) = \frac{1}{2}\frac{1}{(\chi+n^2)} + \frac{1}{2\Gamma\frac{1}{2}}\int_0^\chi (\chi-s)^{-\frac{1}{2}}\frac{1}{(s+n^2)}ds$$

and

$$w(\chi) = 1,\ w(s) = 1,\ \mathfrak{u} = \frac{1}{2},\ \mathfrak{u}_n = \frac{1}{n}\text{ and }T = 1.$$

Let $\aleph(\chi) \in l_p^{\mathfrak{u}}$ for some fixed $\chi \in I$. Then,

$$\sum_{n \geq 1}\mathfrak{u}_n^p|G_n(\chi,\aleph(\chi),l(\aleph(\chi)))|^p = \sum_{n \geq 1}\frac{1}{n^p}\left|\frac{\aleph_n(\chi)}{5n^2+\chi} + \frac{1}{2n}\frac{1}{(\chi+n^2)} + \frac{1}{2\Gamma\frac{1}{2}}\int_0^\chi (\chi-s)^{-\frac{1}{2}}\frac{1}{(s+n^2)}ds\right|^p$$

$$\leq 2^{p-1}\sum_{n \geq 1}\frac{1}{n^p}\frac{|\aleph_n(\chi)|^p}{5^pn^{2p}} + \sum_{n \geq 1}\frac{1}{2^pn^{4p}} + \frac{1}{2^p(\Gamma\frac{1}{2})^p}\sum_{n \geq 1}\frac{1}{n^{3p}}\left|\int_0^\chi (\chi-s)^{-\frac{1}{2}}ds\right|^p$$

$$\leq \frac{2^{p-1}}{5}\sum_{n \geq 1}\frac{1}{n^{3p}}\|\aleph(\chi)\|_{l_p^{\mathfrak{u}}}^p + \sum_{n \geq 1}\frac{1}{2^pn^{4p}} + \frac{1}{(\Gamma\frac{1}{2})^p}\sum_{n \geq 1}\frac{1}{n^{3p}}$$

$$< \infty,$$

as $\sum_{n \geq 1}\frac{1}{n^{3p}}$ and $\sum_{n \geq 1}\frac{1}{n^{4p}}$ are both convergent for $p > 1$.

Therefore, for fixed $\chi \in I$,

$$\{G_n(\chi,\aleph(\chi),l(\aleph(\chi)))\}_{n=1}^\infty \in l_p^\infty,$$

i.e.,

$$\{G_n(\chi, \aleph(\chi), l(\aleph(\chi))\}_{n=1}^{\infty} \in C(I, l_p^{\mathfrak{u}}).$$

Thus, G_n is continuous $\forall \ n \in \mathbb{N}$ and

$$|G_n(\chi, \aleph(\chi), l(\aleph(\chi))) - G_n(\chi, \widehat{\aleph}(\chi), l(\widehat{\aleph}(\chi)))|^p = \left| \frac{1}{5n^2 + \chi}(\aleph_n(\chi) - \widehat{\aleph}_n(\chi)) + \frac{1}{n}(l(\aleph_n(\chi)) - l(\widehat{\aleph}_n(\chi))) \right|^p$$

$$\leq \frac{2^{p-1}}{5^p n^{2p}}|\aleph_n(\chi) - \widehat{\aleph}_n(\chi)| + \frac{2^{p-1}}{n^p}|l(\aleph_n(\chi)) - l(\widehat{\aleph}_n(\chi))|^p.$$

Here β_n as well as P_n are continuous maps $\forall \ n \in \mathbb{N}$ and

$$\beta_n(\chi) = \frac{2^{p-1}}{5^p n^{2p}}, \ \ \widehat{\beta} = \frac{2^{p-1}}{5^p},$$

$$P_n(\chi) = \frac{2^{p-1}}{n^p}, \ \ \sum_{n \geq 1} \mathfrak{u}_n^p P_n(\chi) = \sum_{n \geq 1} \frac{2^{p-1}}{n^{2p}} < \infty.$$

Also,

$$\sum_{n \geq 1} \mathfrak{u}_n^p |G_n(\chi, \aleph^0, 0)|^p = 0$$

and

$$0 < 2^{1 - \frac{1}{p}} \widehat{\beta}^{\frac{1}{p}} < 1.$$

Again H_n is continuous $\forall \ n \in \mathbb{N}$ and

$$|H_n(\chi, s, \aleph(s)| = \frac{1}{s + n^2}, \ \ \widehat{H}_n = \frac{1}{n^2}$$

which gives

$$Q_k = \sum_{n \geq k} \mathfrak{u}_n^p \widehat{H}_n^p = \frac{1}{n^{3p}} < \infty$$

and

$$\lim_{k \to \infty} Q_k = 0.$$

Thus all the assumptions from (i)-(v) are true. Hence eq. (5.3) possesses a solution in $C(I, l_p^{\mathfrak{u}})$.

6. Conclusion

In this paper, we have generalized Darbo fixed point theorem by measure of noncompactness in a tempered sequence space. We have proved existence of solution of an infinite system of weighted Atangana-Baleanu fractional integral equations and have also illustrated an example to show the importance of our main result. Further, this result can be used to investigate the existence of solution of various other fractional integral equations with similar properties.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, x + 170 pages (2009).

[2] M. Al-Refai, On weighted Atangana-Baleanu fractional operators, *Advances in Difference Equations* **2020** (2020), article number 3, DOI: 10.1186/s13662-019-2471-z.

[3] A. H. Ansari, Note on ϕ-ψ contractive type mappings and related fixed point, in: *Proceedings of the 2nd Regional Conference on Mathematics and Applications*, Payeme Noor University, Iran, pp. 377 − 380 (2014).

[4] R. Arab, H. K. Nashine, N. H. Can and T. T. Binh, Solvability of functional-integral equations (fractional order) using measure of noncompactness, *Advances in Difference Equations* **2020** (2020), article number 12, DOI: 10.1186/s13662-019-2487-4.

[5] J. Banaś and M. Krajewska, Existence of solutions for infinite systems of differential equations in spaces of tempered sequences, *Electronic Journal of Differential Equations* **2017**(60) (2017), 1 − 28, URL: https://ejde.math.txstate.edu/Volumes/2017/60/banas.pdf.

[6] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, xii + 315 pages (2014), DOI: 10.1007/978-81-322-1886-9.

[7] J. Banaś, M. Jleli, M. Mursaleen, B. Samet and C. Vetro, *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*, Springer, Singapore, xiii + 487 pages (2017), DOI: 10.1007/978-981-10-3722-1.

[8] S. Beloul, M. Mursalem and A. H. Ansari, A generalization of Darbo's fixed point theorem with an application to fractional integral equations, *Journal of Mathematical Inequalities* **15**(3) (2021), 911 − 921, DOI: 10.7153/jmi-2021-15-63.

[9] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rendiconti del Seminario Matematico della Università di Padova* **24** (1955), 84 − 92, URL: https://www.numdam.org/item/RSMUP_1955__24__84_0/.

[10] A. Das, B. Hazarika and M. Mursaleen, Application of measure of noncompactness for solvability of the infinite system of integral equations in two variables in ℓ_p $(1 < p < \infty)$, *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **113** (2019), 31 − 40, DOI: 10.1007/s13398-017-0452-1.

[11] A. Das, B. Hazarika and P. Kumam, Some new generalization of Darbo's fixed point theorem and its application on integral equations, *Mathematics* **7**(3) (2019), 214, DOI: 10.3390/math7030214.

[12] A. Das, B. Hazarika, R. Arab and M. Mursaleen, Applications of a fixed point theorem to the existence of solutions to the nonlinear functional integral equations in two variables, *Rendiconti del Circolo Matematico di Palermo Series 2* **68** (2019), 139 − 152, DOI: 10.1007/s12215-018-0347-9.

[13] A. Das, B. Hazarika, V. Parvaneh and M. Mursaleen, Solvability of generalized fractional order integral equations via measures of noncompactness, *Mathematical Sciences* **15**(3) (2021), DOI: 10.1007/s40096-020-00359-0.

[14] A. Das, M. Paunović, V. Parvaneh, M. Mursaleen and Z. Bagheri, Existence of a solution to an infinite system of weighted fractional integral equations of a function with respect to another function via a measure of noncompactness, *Demonstratio Mathematica* **56**(1) (2023), 20220192, DOI: 10.1515/dema-2022-0192.

[15] A. Das, R. Jain and H. K. Nashine, A fixed point result via new condensing operator and its application to a system of generalized proportional fractional integral equations, *Journal of Pseudo-Differential Operators and Applications* **14** (2023), article number 21, DOI: 10.1007/s11868-023-00519-5.

[16] K. Kuratowski, Sur les espaces complets, *Fundamenta Mathematicae* **15**(1) (1930), 301 – 309, URL: https://eudml.org/doc/212357.

[17] M. Rabbani, A. Das, B. Hazarika and R. Arab, Existence of solution for two dimensional nonlinear fractional integral equation by measure of noncompactness and iterative algorithm to solve it, *Journal of Computational and Applied Mathematics* **370** (2020), 112654, DOI: 10.1016/j.cam.2019.112654.

[18] M. Rabbani, A. Das, B. Hazarika and R. Arab, Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations, *Chaos, Solitons & Fractals* **140** (2020), 110221, DOI: 10.1016/j.chaos.2020.110221.