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Research Article

Application of the Deffuant Model in Money Exchange

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Abstract. A money transfer system involves interactions between buyers and sellers, where a buyer purchases goods or services from a seller. The amount of money lost by the buyer is exactly equal to the amount gained by the seller. Over time, these transactions accumulate, influencing the overall distribution of wealth within the system. At each time step, a randomly chosen pair of socially connected agents interacts through a financial transaction, following predefined exchange rules. We extend the Deffuant opinion dynamics model to formulate a money exchange system, where interactions between agents determine the redistribution of wealth. Through this framework, we investigate the conditions under which asymptotic stability emerges, ensuring that the system reaches a steady state. Additionally, we explore whether and when an equal wealth distribution can be achieved under different interaction dynamics. By incorporating elements of social connectivity and iterative financial exchanges, our study provides insights into how simple micro-level interactions can lead to macroscopic patterns of wealth distribution. The findings contribute to the broader understanding of economic dynamics, wealth inequality, and social influence on financial transactions. This study bridges the gap between opinion dynamics and wealth exchange models, offering a novel perspective on financial stability in social systems.

Keywords. Equal wealth, Social mobility, Money transfer, The rich and poor, Conservative system, Social network

Mathematics Subject Classification (2020). 91D30, 91C20, 91D10, 37N40, 93D30

1. Introduction

The Deffuant model is one of the popular models in opinion dynamics (Deffuant *et al*. [3], Shang [13,14]). The original Deffuant model consists of a finite number of agents whose opinion is a number in $[0,1]$. Based on a fixed social relationship among all agents, two socially connected agents are selected at each time step and interact if and only if their opinion distance does not exceed some confidence threshold ϵ. The interacting mechanism goes as follows:

$$\left.\begin{aligned} x_i(t+1) &= x_i(t)+\mu(x_j(t)-x_i(t))\mathbb{1}\{|x_i(t)-x_j(t)|\le\epsilon\}, \\ x_j(t+1) &= x_j(t)+\mu(x_i(t)-x_j(t))\mathbb{1}\{|x_i(t)-x_j(t)|\le\epsilon\}, \end{aligned}\right\} \tag{1.1}$$

for $x_i(t)\in[0,1]$ the opinion of agent i at time t and $\mu\in(0,1/2]$ the convergence parameter. Namely, agents i and j move equally toward each other. The Hegselmann–Krause (HK) model is another widely studied bounded-confidence opinion dynamics model (Blondel *et al*. [1], Han *et al*. [5], Lorenz [10], Proskurnikov and Tempo [12]). In contrast to the Deffuant model, each agent simultaneously considers all agents whose opinions are within a confidence bound and updates its opinion to the average of those neighbors. Li [9] introduces a variant of the HK model that generalizes both the Deffuant and the HK models. Extensions to structured populations, such as leader–follower networks, have also been analyzed (Li [8]).

A money transfer involves a buyer and a seller. The buyer pays for goods or services offered by the seller. Therefore, the buyer loses exactly the amount that the seller gains. Considering a finite set of agents, two agents are *socially connected* if a social relationship (e.g., trust) exists between them. Say m_i is the money of agent i. $m_i<0$ if agent i is in debt. It is realistic to set a lower bound for m_i since no one is allowed to borrow money endlessly. Also, we assume the money transfer system is conservative. Namely, influx equals efflux, therefore the sum of all agents' money constant over time. Before depicting the money transfer system evolved from the Deffuant model, we introduce the following terms. Let $[n]=\{1,2,\dots,n\}$ denote the collection of all agents.

Definition 1.1. A *social graph* at time t, $G(t)=([n],E(t))$, is an undirected graph with vertex set and edge set,

$$[n] \text{ and } E(t)=\{(i,j)\in[n]^2 : i\neq j \text{ and vertices } i \text{ and } j \text{ are socially connected}\}.$$

A *social graph for update* at time t, $\widetilde{G}(t)=([n],\widetilde{E}(t))$, is a subgraph of the social graph at time t for money update.

Definition 1.2. A graph is *δ-trivial* if any two vertices in the graph are at a distance of at most δ apart.

The Deffuant model evolves into a money transfer system with the following setting:

Let $m_i(0)$, $i\in[n]$ and $\mu(t)$, $t\ge 0$ be independent continuous real-valued random variables. Agents i and j are selected at time t and interact if and only if they are socially connected. The interaction mechanism is as follows:

$$\left.\begin{aligned} m_i(t+1) &= m_i(t)+\mu(t)(m_j(t)-m_i(t))\mathbb{1}\{(i,j)\in E(t)\}, \\ m_j(t+1) &= m_j(t)+\mu(t)(m_i(t)-m_j(t))\mathbb{1}\{(i,j)\in E(t)\}, \end{aligned}\right\} \tag{1.2}$$

where

$$m_i(t)\ge -d_i \text{ for } d_i>0 \text{ constant and for all } t\ge 0,$$

$$\sum_{i\in[n]} m_i = C \text{ for } C \text{ constant.}$$

Observe that d_i symbolizes the credibility of agent i and $\mu(t)$ corresponds to the behavior that the money two agents agree to transact given that no one is out of credibility. Say agent i is richer than agent j if $m_i \geq m_j$, and poorer than agent j if $m_i \leq m_j$. Given $m_i(t) \leq m_j(t)$, $m_i(t+1) \leq m_j(t+1)$ if $\mu(t) \leq 1/2$, and $m_i(t+1) \geq m_j(t+1)$ if $\mu(t) > 1/2$. The former indicates that the richer agent remains in its richer status, whereas the latter indicates the richer agent becomes poorer, after the transaction, whether it is a buyer or a seller at time t. The money transfer system evolved from the Deffuant model differs from classical money-exchange models in both the conservation of money and the interaction rules. Early models, such as those by Dragulescu and Yakovenko [4] and Chakraborti and Chakrabarti [2], focused on the statistical mechanics of money and the effects of saving propensity on wealth distribution. Search-theoretic approaches (Kiyotaki and Wright [6], Molico [11]) consider agents meeting randomly and trading in a decentralized market, which also leads to characteristic wealth distributions. More recent work connects kinetic-exchange models with opinion dynamics, showing that the distribution of money or opinion can undergo spontaneous symmetry-breaking transitions (Lallouache *et al.* [7]) and that wealth dynamics can be studied within the broader framework of statistical mechanics (Yakovenko and Rosser Jr. [15]). In contrast, our model evolves a Deffuant-type updating mechanism into a money-transfer system on a time-varying social graph, where interactions occur whenever two agents are socially connected, regardless of their wealth difference. This leads to distinct long-term behaviors compared to these classical models.

Let $U_t = \{(i,j) : \text{agent } i \text{ and } j \text{ are selected at time } t \text{ and } \mu(t) \neq 0\}$, $t \geq 0$ be independent and identically distributed random variables with a support

$$S \subset \{\{(i,j)\} : i,j \in [n] \text{ and } i \neq j\} \text{ and } \widetilde{E}(t) = U_t \cap E(t).$$

In other words, U_t indicates a possible pair of candidates for money transfer. Assume that $(\Omega, \mathscr{F}, P)$ is a probability space for $\mathscr{F} \subset \mathscr{P}(\Omega)$ a σ-algebra and P a probability measure. Denote $E(G)$ as the edge set of graph G. Given a pair of agents in transaction, we consider the following conditions in (1.2):

- $\mu \in (0,1)$,
- $\mu > 1$, and
- $\mu < 0$,

corresponding to the richer agent whose money is

- at least the same as the current money of the poorer agent after the transaction,
- at most the same as the current money of the poorer agent after the transaction, and
- at least the same as its current money after the transaction.

2. Main Results

It turns out that equal wealth can be achieved if all agents are willing and possible to transact and the social graph is connected infinitely many times. Namely, all agents eventually have the average money of the total money in the system.

Theorem 2.1. *Assume that* $\sup_{t\geq 0} |\mu(t) - 1/2| < 1/2$*, the social graph connected infinitely many times and* $\bigcup_{a\in S} a \supset \binom{[n]}{2}$*. Then, equal wealth can be achieved eventually.*

It follows that equal wealth can be achieved for a constant connected social graph if all socially connected agents are willing and possible to transact. Let $m_{(i)}$ be the ith smallest among $m_1, m_2, \dots, m_n$. Theorem 2.2 shows conditions under which $m_{(i)}$ is asymptotically stable.

Theorem 2.2. *Assume that* $\inf_{t\geq 0}|\mu(t)-1/2| \geq 1/2$, $\bigcup_{a\in S} a \supset \binom{[n]}{2}$ *and the social graph complete after some time. Then,* $m_{(i)}$ *is asymptotically stable for all* $i \in [n]$ *almost surely.*

The $\inf_{t\geq 0}|\mu(t)-1/2| \geq 1/2$ indicates that the richer agent can be either much richer or poorer than the poor agent after a transaction.

3. The Model

The crucial part of proving Theorem 2.1 is to find a nonnegative nonincreasing function and construct an inequality involving the current money and the updated money. Then, we are able to show circumstances under which asymptotic stability holds in the money transfer system.

Lemma 3.1. *Let* $Z(t) = \sum_{i,j\in[n]} (m_i(t)-m_j(t))^2$. *Then,* $Z(t)$ *is monotone with respect to* t. *In particular,*

$$Z(t)-Z(t+1) = 2n\left(\frac{1}{\mu(t)}-1\right)\mathbb{1}\{\mu(t)\neq 0\}\sum_{i\in[n]}(m_i(t)-m_i(t+1))^2.$$

Proof. Let $\widetilde{E}(t) = \{(p,q)\}$, $\mu = \mu(t)$, $m_i = m_i(t)$ and $m_i^\star = m_i(t+1)$, for all $i \in [n]$. Clearly, $Z(t)-Z(t+1) = 0$ for $\mu = 0$. For $\mu \neq 0$, we have

$$\begin{aligned}(m_p-m_q)^2-(m_p^\star-m_q^\star)^2 &= [1-(1-2\mu)^2](m_p-m_q)^2\\ &= (2-2\mu)2\mu/\mu^2(m_p-m_p^\star)^2\\ &= 4\frac{(1-\mu)}{\mu}(m_p-m_p^\star)^2,\end{aligned}$$

$$(m_p-m_i)^2-(m_p^\star-m_i)^2 = (m_p-m_p^\star)^2+2(m_p-m_p^\star)(m_p^\star-m_i),$$

$$(m_q-m_i)^2-(m_q^\star-m_i)^2 = (m_q-m_q^\star)^2+2(m_q-m_q^\star)(m_q^\star-m_i).$$

Since

$$m_p-m_p^\star = -(m_q-m_q^\star),$$

$$\begin{aligned}(m_p-m_i)^2-(m_p^\star-m_i)^2+(m_q-m_i)^2-(m_q^\star-m_i)^2 &= 2(m_p-m_p^\star)^2+2(m_p-m_p^\star)(m_p^\star-m_q^\star)\\ &= [2+2(1-2\mu)/\mu](m_p-m_p^\star)^2\\ &= 2\left(\frac{1}{\mu}-1\right)(m_p-m_p^\star)^2.\end{aligned}$$

In particular,

$$\begin{aligned}Z(t)-Z(t+1) &= \sum_{i,j\in[n]}[(m_i-m_j)^2-(m_i^\star-m_j^\star)^2]\\ &= 2\Big\{(m_p-m_q)^2-(m_p^\star-m_q^\star)^2\\ &\quad + \sum_{i\in[n]-\{p,q\}}[(m_p-m_i)^2-(m_p^\star-m_i)^2+(m_q-m_i)^2-(m_q-m_i)^2]\Big\}\end{aligned}$$

$$= 2\left[4\left(\frac{1}{\mu}-1\right)+2(n-2)\left(\frac{1}{\mu}-1\right)\right](m_p - m_p^\star)^2$$
$$= 4n\left(\frac{1}{\mu}-1\right)(m_p - m_p^\star)^2. \qquad \square$$

Lemma 3.2. *Assume that* $\sup_{t\ge 0}|\mu(t)-1/2| < 1/2$. *Then, all components of* $\widetilde{G}$ *are* δ*-trivial after some finite time for all* $\delta > 0$ *almost surely.*

Proof. Assume by contradiction that this is not the case. Then, there is $(t_k)_{k\ge 0}$ increasing with $\widetilde{G}(t_k)$ δ-nontrivial for all $k \ge 0$. $R = \sup_{t\ge 0}|\mu(t)-1/2|$ implies $1/2 - R \le \mu(t) \le 1/2 + R$, therefore $1/\mu(t) - 1 \ge \frac{1/2-R}{1/2+R}$ for all $t \ge 0$. It follows from Lemma 3.1 that

$$Z(0) \ge Z(0) - Z(s) = \sum_{t=0}^{s-1}[Z(t) - Z(t+1)], \quad \text{for all } s \ge 1.$$

Letting $s \to \infty$,

$$\infty > Z(0) \ge \sum_{t\ge 0}[Z(t) - Z(t+1)] > \sum_{k\ge 0} 4n\frac{(1/2-R)^3}{1/2+R}\delta^2 = \infty, \text{ a contradiction.} \qquad \square$$

Proof of Theorem 2.1. Due to finiteness of the social graph, it connected infinitely many times implies there is a graph H connected infinitely many times, saying $(t_k)_{k\ge 0}$ the timepoints. For $(i,j) \in E(H)$, let B_t be the event that $(i,j) \in U_t$. Because of $\bigcup_{a\in S} a \supset \binom{[n]}{2}$ and support S finite, via second Borel–Cantelli lemma,

$$\sum_{k\ge 0} P(B_{t_k}) \ge \sum_{k\ge 0} \min_{a\in S} P(U_{t_k} = a) \ge \sum_{k\ge 0} \min_{a\in S} P(U_0 = a) = \infty$$

implies B_{s_ℓ}, $\ell \ge 0$ hold for $(s_\ell)_{\ell\ge 0} \subset (t_k)_{k\ge 0}$; therefore $(i,j) \in \widetilde{E}(s_\ell)$. Through Lemma 3.2, vertices i and j have the same money eventually. Graph H connected implies all vertices achieve the same money. $\square$

Proof of Theorem 2.2. We claim that $\lim_{t\to\infty}|\mu(t)-1/2| = 1/2$. Let

$$A_t = \sup\{|m_i(t) - m_j(t)| : i,j \in [n],\ -d_i < m_i(t) \text{ and } -d_j < m_j(t)\}.$$

For $c = 0$ or $-\infty$, $A_t = c$ implies $A_s = c$ for some t and for all $s \ge t$. It is clear that $P(A_0 = c) = 0$. Since

$$\inf_{t\ge 0}|\mu(t) - 1/2| \ge 1/2,$$

the sequence $(A_t)_{t\ge 0}$ is nondecreasing.

For $A_0 > 0$, assume that $\limsup_{t\to\infty}|\mu(t)-1/2| > 1/2$. Then there exist $\delta > 0$ and an increasing sequence $(t_k)_{k\ge 0}$ such that

$$|\mu(t_k) - 1/2| \ge 1/2 + \delta, \quad \text{for all } k \ge 0.$$

Let

$$i_t = \arg\max\{m_i(t) : i \in [n] \text{ and } -d_i < m_i(t)\},$$
$$j_t = \arg\min\{m_i(t) : i \in [n] \text{ and } -d_i < m_i(t)\},$$

and let E_t denote the event that agents i_t and j_t transact at time t.

Because the support S is finite, $\bigcup_{a\in S} a \supset \binom{[n]}{2}$, and the social graph becomes complete after some finite time, the second Borel–Cantelli lemma implies

$$\sum_{k\geq 0} P(E_{t_k}) = \infty,$$

which in turn implies that the events E_{ℓ_i} occur for infinitely many indices $(\ell_i)_{i\geq 0} \subset (t_k)_{k\geq 0}$.

For such ℓ_i we have

$$A_{\ell_i+1} \geq |1-2\mu(\ell_i)| A_{\ell_i} \geq (1+2\delta)A_{\ell_i}, \quad \text{for all } i \geq 0.$$

Therefore,

$$A_{\ell_k+1} \geq (1+2\delta)^{k+1} A_{\ell_0} \to \infty, \quad \text{as } k \to \infty,$$

a contradiction. This completes the proof. □

4. Conclusion

This paper develops a money-exchange model by extending the Deffuant-type updating rule to a time-varying social graph. The long-term behavior of the agents' wealth depends crucially on two elements: the structure of the social network over time and the interaction parameter $\mu(t)$ governing each transaction.

The first main result (Theorem 2.1) shows that when every pair of agents is repeatedly reachable through the social graph and all transactions are moderately conservative—namely $\sup_{t\geq 0} |\mu(t) - 1/2| < 1/2$, so that $\mu(t) \in (0,1)$ at all times—the system converges to equal wealth. In this regime every agent eventually holds the average of the total money in the system. Thus, sufficiently frequent connectivity of the social graph, together with non-extreme update intensities, guarantees full wealth equalization.

The second main result (Theorem 2.2) describes a fundamentally different regime. When the update parameters are extreme in the sense that $\inf_{t\geq 0} |\mu(t) - 1/2| \geq 1/2$, a transaction may push the richer agent to become much richer or even poorer after interacting. In this setting, the individual wealths do not necessarily converge, but the ordering of agents' wealths becomes asymptotically stable, provided that the social graph becomes complete after some time. That is, $m_{(i)}(t)$ converges as $t \to \infty$ for all $i \in [n]$, even though each agent's wealth may continue to fluctuate.

Together, these results clarify how network connectivity and transaction intensity jointly determine the long-run structure of wealth in this model. Moderate interactions lead to full equality, whereas extreme interactions do not force individual wealths to converge but instead stabilise the inequality pattern by fixing the ordering of agents' wealths.

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Competing Interests

The author declares that she has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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