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Research Article

# New Application of Libera Integral Operator

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**Abstract.** In this paper, we introduce the new subclass $A_j(n,\alpha,\beta)$ of $A(n)$ using $Lf(z)$. Some interesting properties of functions $f(z)\in A(n)$ are studied with illustrative examples.

**Keywords.** Analytic function, Libera integral operator, $\alpha$-spirallike functions of order $\beta$, Subordination, Convex

**Mathematics Subject Classification (2020).** 30C45, 30C80

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## 1. Introduction

Let $A(n)$ denote the class of analytic functions $f(z)$ of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k,\quad n\in\mathbb{N}=\{2,3,4,\ldots\} \tag{1.1}$$

in the open unit disk $U=\{z\in C:|z|<1\}$. For $f(z)\in A(n)$, Libera [2] defined the Libera integral operator $Lf(z)$ given by

$$Lf(z)=\frac{2}{z}\int_0^z f(t)dt. \tag{1.2}$$

Using the above Libera integral operator $Lf(z)$, we introduce the following iterative forms:

$$L_0f(z)=f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k, \tag{1.3}$$

$$L_1f(z)=Lf(z)=\frac{2}{z}\int_0^z f(t)dt=z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)a_kz^k, \tag{1.4}$$

$$L_2 f(z) = L(Lf(z)) = z + \sum_{k=n+1}^{\infty} \left(\frac{2}{k+1}\right)^2 a_k z^k \tag{1.5}$$

and

$$L_j f(z) = L(L_{j-1} f(z)) = z + \sum_{k=n+1}^{\infty} \left(\frac{2}{k+1}\right)^j a_k z^k \tag{1.6}$$

for $j = 1, 2, 3, \ldots$.

With the above integral operator $L_j f(z)$, we say that $f(z) \in A_j(n, \alpha, \beta)$ if $f(z) \in A(n)$ satisfies

$$Re\left(e^{i\alpha} \frac{L_{j-1} f(z)}{L_j f(z)}\right) > \beta \cos, \alpha \quad z \in U, \tag{1.7}$$

for some real $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ and $\beta$ $(0 \le \beta < 1)$.

Noting that

$$\frac{L_{j-1} f(z)}{L_j f(z)} = \frac{1}{2}\left(1 + \frac{z\left(L_j f(z)\right)'}{L_j f(z)}\right), \quad j \in \mathbb{N} \tag{1.8}$$

we know that

$$Re\left(e^{i\alpha} \frac{L_{j-1} f(z)}{L_j f(z)}\right) = Re\left\{\frac{e^{i\alpha}}{2}\left(1 + \frac{z(L_j f(z))'}{L_j f(z)}\right)\right\} > \beta \cos\alpha, \quad z \in U, \tag{1.9}$$

for $f(z) \in A_j(n, \alpha, \beta)$. Therefore, if we consider a function $f(z) \in A(n)$ given by

$$L_j f(z) = z(1 - z^n)^t \tag{1.10}$$

with

$$t = -\frac{4}{n}(1 - \beta) e^{-i\alpha} \cos\alpha, \tag{1.11}$$

then we have that

$$\begin{aligned} Re\left(e^{i\alpha} \frac{L_{j-1} f(z)}{L_j f(z)}\right) &= Re\left\{\frac{e^{i\alpha}}{2}\left(2 - tn \frac{z^n}{1 - z^n}\right)\right\} \\ &= \cos\alpha + 2(1 - \beta)\cos\alpha\, Re\left(\frac{z^n}{1 - z^n}\right) \\ &> \cos\alpha - (1 - \beta)\cos\alpha \\ &= \beta\cos\alpha, \quad z \in U. \end{aligned} \tag{1.12}$$

Thus, we say that $f(z) \in A_j(n, \alpha, \beta)$ for $f(z) \in A(n)$ given by (1.10).

**Remark 1.1.** Recently, Guney and Owa[1] considered the subclass $S(n, \alpha, \beta)$ and $C(n, \alpha, \beta)$ of $A(n)$ consisting of $f(z)$ which satisfies

$$Re\left(e^{i\alpha} \frac{z f'(z)}{f(z)}\right) > \beta \cos\alpha, \quad z \in U \tag{1.13}$$

and

$$Re\left(e^{i\alpha}\left(1 + \frac{z f''(z)}{f'(z)}\right)\right) > \beta \cos\alpha, \quad z \in U \tag{1.14}$$

respectively, for some real $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ and $\beta$ $(0 \le \beta < 1)$.
A function $f(z)$ in the class $S(n, \alpha, \beta)$ is said to be $\alpha$-spirallike of order $\beta$ in $U$.

[1] H. O. Güney and S. Owa, Some properties for subordinations of $\alpha$-spirallike functions of order $\beta$, *Applied Mathematics E-Notes* (accepted for publication).

## 2. Conditions for the Class $A_j(n,\alpha,\beta)$

**Theorem 2.1.** *If $f(z)\in A(n)$ satisfies*

$$\left|\frac{z(L_jf(z))'}{L_jf(z)}-1\right|<2(1-\beta)\cos\alpha,\quad z\in U, \tag{2.1}$$

*for some real $\alpha$ $\left(|\alpha|<\frac{\pi}{2}\right)$ and $\beta$ $(0\le\beta<1)$, then the subclass $f(z)\in A_j(n,\alpha,\beta)$.*

*Proof.* Let us consider a function $w(z)$ given by

$$\frac{z(L_jf(z))'}{L_jf(z)}=1+2(1-\beta)\cos\alpha w(z), \tag{2.2}$$

for $f(z)$ which satisfies (2.1). Then $w(z)$ is analytic in $U$, $w(0)=0$, and $|w(z)|<1$ $(z\in U)$. For such $w(z)$, we know that

$$\frac{L_{j-1}f(z)}{L_jf(z)}=\frac{1}{2}\left(1+\frac{z(L_jf(z))'}{L_jf(z)}\right)=1+(1-\beta)\cos\alpha w(z) \tag{2.3}$$

and

$$\begin{aligned}Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}\right)&=\cos\alpha+(1-\beta)\cos\alpha Re(e^{i\alpha}w(z))\\&\ge\cos\alpha-(1-\beta)\cos\alpha|e^{i\alpha}w(z)|\\&>\beta\cos\alpha,\quad z\in U.\end{aligned} \tag{2.4}$$

This gives us that $f(z)\in A_j(n,\alpha,\beta)$. □

Next, we consider the following theorem.

**Theorem 2.2.** *If $f(z)\in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty}\left[\frac{2}{k+1}\right]^j\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k|\le 1, \tag{2.5}$$

*for some real $\alpha$ $\left(|\alpha|<\frac{\pi}{2}\right)$ and $\beta$ $(0\le\beta<1)$, then the subclass $f(z)\in A_j(n,\alpha,\beta)$.*

*Proof.* For $f(z)\in A(n)$, we know that

$$\left|\frac{z(L_jf(z))'}{L_jf(z)}-1\right|=\left|\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)a_kz^k}{1+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^ja_kz^k}\right|<\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)|a_k|}{1-\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j|a_k|}. \tag{2.6}$$

Therefore, using (2.1), if $f(z)$ satisfies

$$\frac{\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j(k-1)|a_k|}{1-\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j|a_k|}\le 2(1-\beta)\cos\alpha, \tag{2.7}$$

then $f(z)\in A_j(n,\alpha,\beta)$. Further, we say that the inequality (2.7) is the same as (2.5).

We consider a function $f(z)\in A(n)$ given by

$$f(z)=z+\sum_{k=n+1}^{\infty}A_kz^k \tag{2.8}$$

with

$$A_k=\frac{(n+1)e^{i\theta}}{\left(\frac{2}{k+1}\right)^j\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)k(k+1)},\quad 0\le\theta\le 2\pi. \tag{2.9}$$

For such $f(z)$, we have

$$\sum_{k=n+1}^{\infty}\left[\frac{2}{k+1}\right]^{j}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|A_k| = \sum_{k=n+1}^{\infty}\frac{n+1}{k(k+1)}$$
$$= (n+1)\sum_{k=n+1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1. \tag{2.10}$$

Therefore, a function $f(z)$, given by (2.8), satisfies the equality in (2.5). □

## 3. Coefficient Problems for the Class $A_j(n,\alpha,\beta)$

We consider coefficient problems of $f(z)\in A(n)$ for the class $A_j(n,\alpha,\beta)$. To consider the above problems, we need the following lemma by Carathéodory [1].

**Lemma 3.1.** *If $p(z)$ given by*

$$p(z) = 1+\sum_{k=1}^{\infty}c_k z^k \tag{3.1}$$

*is analytic in $U$ and $\mathrm{Re}\,p(z)>\alpha$ $(z\in U)$ for $0\le\alpha<1$, then*

$$|c_k|\le 2(1-\alpha),\quad k=1,2,3,\ldots. \tag{3.2}$$

*The equality in (3.2) is given by*

$$p(z) = \frac{1+(1-2\alpha)z}{1-z}. \tag{3.3}$$

With Lemma 3.1, we have

**Lemma 3.2.** *If $p(z)$ given by*

$$p(z) = 1+\sum_{k=n}^{\infty}c_k z^k,\quad n=1,2,3,\ldots \tag{3.4}$$

*is analytic in $U$ and $\mathrm{Re}\,p(z)>\alpha$, $z\in U$ for $0\le\alpha<1$, then*

$$|c_k|\le 2(1-\alpha),\quad k=n,n+1,n+2,\ldots. \tag{3.5}$$

*The equality in (3.5) is given by*

$$p(z) = \frac{1+(1-2\alpha)z^n}{1-z^n}. \tag{3.6}$$

Using Lemma 3.2, we show

**Theorem 3.1.** *If $f(z)\in A_j(n,\alpha,\beta)$, then*

$$|a_{n+1}|\le\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\left(\frac{n+2}{2}\right)^{j} \tag{3.7}$$

*and*

$$|a_k|\le\frac{4|e^{i\alpha}-\beta\cos\alpha|}{k-1}\left(\frac{k+1}{2}\right)^{j}\prod_{l=n}^{k-2}\left(1+\frac{4|e^{i\alpha}-\beta\cos\alpha|}{l}\right),\quad k=n+2,n+3,n+4,\ldots. \tag{3.8}$$

*Inequalities (3.7) and (3.8) hold for function $f(z)\in A(n)$ such that*

$$L_j f(z) = \frac{z}{(1-z^{n-1})^t} \tag{3.9}$$

*with*

$$t=\frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1}. \tag{3.10}$$

*Proof.* We define a function $p(z)$ by

$$p(z)=\frac{e^{i\alpha}\frac{L_{j-1}f(z)}{L_jf(z)}-\beta\cos\alpha}{e^{i\alpha}-\beta\cos\alpha} \tag{3.11}$$

for $f(z)\in A_j(n,\alpha,\beta)$. Then $p(z)$ is analytic in $U$, $Re\,p(z)>0$ $(z\in U)$, and can be written by

$$p(z)=1+\sum_{k=n+1}^{\infty}c_kz^{k-1}. \tag{3.12}$$

It follows from (3.11) that

$$e^{i\alpha}L_{j-1}f(z)-\beta\cos\alpha L_jf(z)=(e^{i\alpha}-\beta\cos\alpha)p(z)L_jf(z). \tag{3.13}$$

We also note that

$$e^{i\alpha}L_{j-1}f(z)-\beta\cos\alpha L_jf(z)=(e^{i\alpha}-\beta\cos\alpha)z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^j\left(\frac{k+1}{2}e^{i\alpha}-\beta\cos\alpha\right)a_kz^k \tag{3.14}$$

and

$$(e^{i\alpha}-\beta\cos\alpha)p(z)L_j(z)=(e^{i\alpha}-\beta\cos\alpha)\left(1+\sum_{k=n+1}^{\infty}c_kz^{k-1}\right)\left(z+\sum_{k=n+1}^{\infty}\left(\frac{2}{k+1}\right)^ja_kz^k\right). \tag{3.15}$$

Let us consider the coefficients of $z^k$ in analytic in (3.14) and (3.15). Then, we have that

$$\begin{aligned}\left(\frac{2}{k+1}\right)^j\left(\frac{k-1}{2}\right)a_k=(e^{i\alpha}-\beta\cos\alpha)\Bigg\{&c_k+\left(\frac{2}{n+2}\right)^jc_{k-n}a_{n+1}+\left(\frac{2}{n+3}\right)^jc_{k-n-1}a_{n+2}\\&+\left(\frac{2}{n+4}\right)^jc_{k-n-2}a_{n+3}+\ldots+\left(\frac{2}{k-n}\right)^jc_{n+2}a_{k-n-1}\\&+\left(\frac{2}{k-n+1}\right)^jc_{n+1}a_{k-n}\Bigg\}.\end{aligned} \tag{3.16}$$

Since

$$|c_k|\le 2,\quad k=n+1,n+2,n+3,\ldots \tag{3.17}$$

by Lemma 3.2, we obtain that

$$\begin{aligned}\left(\frac{2}{k+1}\right)^j\left(\frac{k-1}{2}\right)|a_k|\le 2|e^{i\alpha}-\beta\cos\alpha|\Bigg\{&1+\left(\frac{2}{n+2}\right)^j|a_{n+1}|+\left(\frac{2}{n+3}\right)^j|a_{n+2}|\\&+\left(\frac{2}{n+4}\right)^j|a_{n+3}|+\ldots+\left(\frac{2}{k-n}\right)^j|a_{k-n-1}|+\left(\frac{2}{k-n-1}\right)^j|a_{k-n}|\Bigg\}.\end{aligned} \tag{3.18}$$

If $k=n+1$, then

$$\left(\frac{2}{n+2}\right)^j\left(\frac{n}{2}\right)|a_{n+1}|\le 2|e^{i\alpha}-\beta\cos\alpha| \tag{3.19}$$

or

$$|a_{n+1}|\le\frac{4|e^{i\alpha}-\beta\cos\alpha|}{n}\left(\frac{n+2}{2}\right)^j. \tag{3.20}$$

If $k = n+2$, then

$$\left(\frac{2}{n+3}\right)^j \left(\frac{n+1}{2}\right) |a_{n+2}| \le 2|e^{i\alpha} - \beta\cos\alpha| \left\{1 + \left(\frac{2}{n+2}\right)^j |a_{n+1}|\right\}$$

and

$$|a_{n+2}| \le \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+1} \left(\frac{n+3}{2}\right)^j \left\{1 + \left(\frac{2}{n+2}\right)^j \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n} \left(\frac{n+2}{2}\right)^j\right\}$$

$$= \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+1} \left(\frac{n+3}{2}\right)^j \left(1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n}\right). \tag{3.21}$$

Further, if $k = n+3$, then

$$\left(\frac{2}{n+4}\right)^j \left(\frac{n+2}{2}\right) |a_{n+3}| \le 2|e^{i\alpha} - \beta\cos\alpha| \left\{1 + \left(\frac{2}{n+2}\right)^j |a_{n+1}| + \left(\frac{2}{n+3}\right)^j |a_{n+2}|\right\} \tag{3.22}$$

and

$$|a_{n+3}| \le \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+2} \left(\frac{n+4}{2}\right)^j \left\{1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n} + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+1} \left(1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n}\right)\right\}$$

$$= \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+2} \left(\frac{n+4}{2}\right)^j \left(1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n}\right) \left(1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{n+1}\right). \tag{3.23}$$

Therefore, the theorem holds for $k = n+1, n+2, n+3, \ldots$. By applying mathematical induction, we obtain that

$$|a_k| \le \frac{4|e^{i\alpha} - \beta\cos\alpha|}{k-1} \left(\frac{k+1}{2}\right)^j \prod_{l=n}^{k-2} \left(1 + \frac{4|e^{i\alpha} - \beta\cos\alpha|}{l}\right), \tag{3.24}$$

for if $k = n+1, n+2, n+3, \ldots$.

Finally, we consider a function $p(z)$ given by

$$p(z) = \frac{1+z^{n-1}}{1-z^{n-1}}. \tag{3.25}$$

Then, we see that

$$\frac{L_{j-1}f(z)}{L_j f(z)} = \frac{1 + (1 - 2\beta\cos\alpha e^{-i\alpha})z^{n-1}}{1-z^{n-1}} \tag{3.26}$$

and that

$$\frac{(L_j f(z))'}{L_j f(z)} = \frac{1}{z} + \frac{4(1-\beta\cos\alpha e^{-i\alpha})z^{n-2}}{1-z^{n-1}} \tag{3.27}$$

by (1.8). Therefore, we have

$$\log(L_j f(z)) = \log z - \frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1} \log(1-z^{n-1}) \tag{3.28}$$

and

$$L_j f(z) = \frac{z}{(1-z^{n-1})^t} \tag{3.29}$$

with

$$t = \frac{4(1-\beta\cos\alpha e^{-i\alpha})}{n-1}. \tag{3.30}$$

□

## 4. Subordination Problems

Let functions $f(z)$ and $g(z)$ be analytic in $U$. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, $z \in U$, if there exists a function $w(z)$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ and such that $f(z) = g(w(z))$ (see, Miller and Mocanu [3]).

For subordinations, we have

**Theorem 4.1.** *If $f(z) \in A(n)$ satisfies*

$$\frac{z(L_j f(z))'}{L_j f(z)} \prec \frac{1-(1-4(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}, \quad z \in U \tag{4.1}$$

*for some real $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ and $\beta$ $(0 \le \beta < 1)$, then $f(z) \in A_j(n,\alpha,\beta)$, $j = 1,2,3,\ldots$.*

*Proof.* We note that

$$Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_j f(z)}\right) = Re\left\{\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right)\right\} > \beta\cos,\alpha \quad z \in U \tag{4.2}$$

for $f(z) \in A_j(n,\alpha,\beta)$. Since

$$\begin{aligned}\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right) &\prec \frac{e^{i\alpha}}{2}\left\{2\frac{1-(1-2(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}\right\}\\ &= e^{i\alpha}\left\{\frac{1-(1-2(1-\beta)\cos\alpha e^{-i\alpha})z^n}{1-z^n}\right\},\end{aligned} \tag{4.3}$$

we have that

$$\begin{aligned}Re\left(e^{i\alpha}\frac{L_{j-1}f(z)}{L_j f(z)}\right) &= Re\left\{\frac{e^{i\alpha}}{2}\left(1+\frac{z(L_j f(z))'}{L_j f(z)}\right)\right\}\\ &= Re\left\{e^{i\alpha}+2(1-\beta)\cos\alpha\frac{z^n}{1-z^n}\right\}\\ &> \cos\alpha-(1-\beta)\cos\alpha = \beta\cos\alpha, \quad z \in U.\end{aligned} \tag{4.4}$$

This completes the proof of the theorem.

A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{4.5}$$

is analytic, univalent and convex in $U$, we have

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z), \quad z \in U \tag{4.6}$$

with $a_1 = 1$. For such subordinating factor sequences, Wilf [4] gave the following result. □

**Lemma 4.1.** *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left(1+2\sum_{k=1}^{\infty} b_k z^k\right) > 0, \quad z \in U. \tag{4.7}$$

*For functions $f(z)$ and $g(z)$ given by*

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A(n) \tag{4.8}$$

*and*

$$g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \in A(n) \tag{4.9}$$

*the convolution* $(f * g)(z)$ *is given by*

$$(f * g)(z) = z + \sum_{k=n+1}^{\infty} a_k b_k z^k. \tag{4.10}$$

*By the relation* (1.8)*, we know that*

$$L_{j-1} f(z) = \frac{1}{2}\left(z L_j f(z)\right)', \quad j \in \mathbb{N}. \tag{4.11}$$

*We specially consider the case* $j = 0$ *in* (4.11)*. Then, we have*

$$L_{-1} f(z) = \frac{1}{2}(z f(z))' = z + \sum_{k=n+1}^{\infty} \left(\frac{k+1}{2}\right) a_k z^k. \tag{4.12}$$

*With* (4.12)*, we introduce the subclass* $A_0(n, \alpha, \beta)$ *of* $A(n)$ *by*

$$Re\left(e^{i\alpha}\frac{L_{-1} f(z)}{L_0 f(z)}\right) = Re\left(e^{i\alpha}\left(1 + \frac{z f'(z)}{f(z)}\right)\right) > \beta \cos\alpha, \quad z \in U \tag{4.13}$$

*for some real* $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ *and* $\beta$ $(0 \le \beta < 1)$.

It is clear that Theorem 2.2 gives us the following lemma.

**Lemma 4.2.** *If* $f(z) \in A(n)$ *satisfies*

$$\sum_{k=n+1}^{\infty} \left(1 + \frac{k-1}{2(1-\beta)} \sec\alpha\right) |a_k| \le 1 \tag{4.14}$$

*for some real* $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ *and* $\beta$ $(0 \le \beta < 1)$*, then* $f(z) \in A_0(n, \alpha, \beta)$.

Now, we derive

**Theorem 4.2.** *If* $f(z) \in A(n)$ *be satisfy the coefficient inequality* (4.14) *and* $g(z) \in A(1)$ *be convex in* $U$*. Then*

$$\frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} (f * g)(z) \prec g(z), \quad z \in U. \tag{4.15}$$

*In particular*

$$Re\, f(z) > -\frac{4(1-\beta) + n\sec\alpha}{2(1-\beta) + n\sec\alpha}, \quad z \in U. \tag{4.16}$$

*The constant in* (4.15) *can not be replace by any lager one.*

*Proof.* Let us consider a function $f(z) \in A(n)$ given by (4.8) and a function $g(z) \in A(n)$ given by (4.9). Then, we know

$$\frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} (f * g)(z) = \frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} \left(z + \sum_{k=n+1}^{\infty} a_k b_k z^k\right) \tag{4.17}$$

It follows from Lemma 4.1,

$$Re\left\{1 + 2\sum_{k=1}^{\infty} \frac{2(1-\beta) + n\sec\alpha}{2\{4(1-\beta) + n\sec\alpha\}} a_k z^k\right\}$$

$$= Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}\left(z+\sum_{k=n+1}^{\infty}a_k z^k\right)\right\}>0, \quad z\in U. \tag{4.18}$$

We see that

$$\begin{aligned}
&Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}\left(z+\sum_{k=n+1}^{\infty}a_k z^k\right)\right\}\\
&= Re\left\{1+\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}z+\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}\sum_{k=n+1}^{\infty}\left(1+\frac{n}{2(1-\beta)}\sec\alpha\right)a_k z^k\right\}\\
&\geq 1-\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}|z|-\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}\sum_{k=n+1}^{\infty}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k||z|^k\\
&\geq 1-\frac{2(1-\beta)+n\sec\alpha}{4(1-\beta)+n\sec\alpha}|z|-\frac{2(1-\beta)}{4(1-\beta)+n\sec\alpha}|z|^{n+1}\\
&\geq 1-|z|>0, \quad z\in U.
\end{aligned} \tag{4.19}$$

This implies the subordination (4.15).

Next, we take $g(z)$ given by

$$g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k\in A(1). \tag{4.20}$$

Then, $g(z)$ satisfies

$$Re\left(1+\frac{zg''(z)}{g'(z)}\right)=Re\left(\frac{1+z}{1-z}\right)>0, \quad z\in U. \tag{4.21}$$

Thus $g(z)$ is convex in $U$. Further, we take the function $f(z)\in A(n)$ given by

$$f(z)=z+\sum_{k=n+1}^{\infty}a_k z^k, \quad z\in U \tag{4.22}$$

with

$$a_k=\frac{2(n+1)(1-\beta)e^{i\theta}}{(2(1-\beta)+(k-1)\sec\alpha)k(k+1)}, \quad 0\leq\theta\leq 2\pi. \tag{4.23}$$

Then, $f(z)$ satisfies

$$\sum_{k=n+1}^{\infty}\left(1+\frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k|=\sum_{k=n+1}^{\infty}\frac{n+1}{k(k+1)}=(n+1)\sum_{k=n+1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1. \tag{4.24}$$

This function, $f(z)\in A(n)$ satisfies (4.15).

Further, we see that

$$\begin{aligned}
&\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}(f*g)(z)\right\}\\
&=\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}f(z)\right\}\\
&=\min_{|z|\leqslant 1}Re\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}\left(z+\sum_{k=n+1}^{\infty}a_k z^k\right)\right\}\\
&=-\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}\left(1+\sum_{k=n+1}^{\infty}|a_k|\right)
\end{aligned}$$

$$= -\left\{\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}} + \frac{(n+1)(1-\beta)}{4(1-\beta)+n\sec\alpha}\sum_{k=n+1}^{\infty}\frac{1}{k(k+1)}\right\}$$
$$= -\frac{1}{2}$$

with (4.23). Since the function $g(z) \in A(1)$ given by (4.2) satisfies $Re\, g(z) > -\frac{1}{2}$, $z \in U$, the constant $\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}$ is the best possible. □

**Remark 4.1.** We tried to get a suitable function $g(z) \in A(n)$ in Theorem 4.2. But we did not get any good $g(z) \in A(n)$ in our paper. Hence, we consider the convex function $g(z) \in A(1)$ defined in (4.20).

## 5. Appendix

We consider Libera integral operator $L_j f(z)$ for $f(z) \in A(n)$. Further, we introduce $L_{-1} f(z)$ by

$$L_{-1}f(z) = \frac{1}{2}(zf(z))' = z + \sum_{k=n+1}^{\infty}\left(\frac{k+1}{2}\right)a_k z^k. \tag{5.1}$$

In order to use (5.1)), we introduce

$$L_{-2}f(z) = \frac{1}{2}(zL_{-1}f(z))' = z + \sum_{k=n+1}^{\infty}\left(\frac{k+1}{2}\right)^2 a_k z^k \tag{5.2}$$

and

$$L_{-j-1}f(z) = \frac{1}{2}(zL_{-j}f(z))' = z + \sum_{k=n+1}^{\infty}\left(\frac{k+1}{2}\right)^{j+1} a_k z^k \tag{5.3}$$

for $j = 0, 1, 2, \ldots$. For such $L_{-j}f(z)$, we consider a subclass $A_{-j}(n, \alpha, \beta)$ of $A(n)$ by

$$Re\left(e^{i\alpha}\frac{L_{-j-1}f(z)}{L_{-j}f(z)}\right) > \beta\cos\alpha, \quad z \in U \tag{5.4}$$

for some real $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ and $\beta$ $(0 \le \beta < 1)$.

With Theorem 2.2, we can prove the following result for $f(z) \in A_{-j}(n, \alpha, \beta)$.

**Theorem 5.1.** *If $f(z) \in A(n)$ satisfies*

$$\sum_{k=n+1}^{\infty}\left[\frac{k+1}{2}\right]^j\left(1 + \frac{k-1}{2(1-\beta)}\sec\alpha\right)|a_k| \le 1 \tag{5.5}$$

*for some real $\alpha$ $\left(|\alpha| < \frac{\pi}{2}\right)$ and $\beta$ $(0 \le \beta < 1)$, then the subclass $f(z) \in A_{-j}(n, \alpha, \beta)$.*

Using Theorem 5.1, we derive

**Theorem 5.2.** *If $f(z) \in A(n)$ be satisfy the coefficient inequality* (5.5) *and $g(z) \in A(1)$ be convex in $U$. Then*

$$\frac{2(1-\beta)+n\sec\alpha}{2\{4(1-\beta)+n\sec\alpha\}}(f * g)(z) \prec g(z), \quad z \in U. \tag{5.6}$$

*In particular*

$$Re\, f(z) > -\frac{4(1-\beta)+n\sec\alpha}{2(1-\beta)+n\sec\alpha}, \quad z \in U. \tag{5.7}$$

*The constant in* (5.6) *cannot be replaced by any larger one.*

The manner of the proof is the same as the proof of Theorem 4.2.

## 6. Conclusion

Using Libera integral operator $Lf(z)$, we introduce the new subclass $A_j(n,\alpha,\beta)$ of $A(n)$. The coefficient and subordinate problem of class $A_j(n,\alpha,\beta)$ has been solved. Some results obtained by giving detailed proofs of theorems are presented.

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### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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