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Research Article

# Philos-Type Criteria for Testing the Oscillatory Performance of Solutions to Differential Equations With a Natural Argument

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**Abstract.** The oscillatory performance of nonlinear delay differential equation solutions $(b\psi(u)[w']^r)' + qu^\beta(\varrho) = 0$ is examined in this work. The canonical case is considered, and Philos-type criteria are established to test the oscillation of all solutions. The existence of $\phi$ increases the difficulty in obtaining the asymptotic and monotonic properties of the solutions and also increases the possibility of applying the results to a broader range of special cases. The results obtained in this study represent an extension and generalization of earlier findings in the literature.

**Keywords.** Differential equation, Neutral delay, Second-order equations, Oscillation theory

**Mathematics Subject Classification (2020).** 34C10, 34K11



## 1. Introduction

The qualitative theory of differential equations is concerned with analyzing the characteristics of their solutions. These characteristics include oscillation, stability, periodicity, bifurcation, synchronization, symmetry, among others. A specialized area within this field, called oscillation theory, focuses on establishing criteria that determine whether the solutions to differential equations exhibit oscillatory or non-oscillatory behavior. Recent decades have seen considerable interest and research focused on investigating the oscillation conditions of particular *Functional Differential Equations* (FDEs), see Agarwal *et al*. [2], Erbe [7], Györi and Ladas [12], Moaaz and Albalawi [21], Palanisamy *et al*. [22], and Santra *et al*. [26].

A *Neutral Delay Differential Equation* (NDDE) is a type of FDE where the highest derivative of the unknown function appears both with and without a delay. During the last thirty years, the investigation of neutral differential equations has attracted considerable interest, becoming a significant field of study owing to its applications and analytical challenges, see, e.g., Agarwal *et al*. [1], Baculíková and Džurina [3], Candan [5], Dong [6], Grace *et al*. [9, 10], Liu and Bai [17], Liu *et al*. [18], Meng and Xu [20], Tunç and Grace [27], Xu and Meng [28, 29], Ye and Xu [31], and Zhang and Wang [32]. There is a wide range of applications for these types of equations, which include the analytical modeling of vibrating masses coupled with an elastic bar, systems for automatic control, mixing of liquids, and dynamics of populations, see Hale [13, 14]. Specifically, second-order NDDEs are highly valuable in biology for describing how the human body maintains balance and in robotics for designing bipedal robots. These applications highlight the relevance of these equations in understanding complex systems in both technological and biological contexts, see MacDonald [19].

This paper studies the second-order non-linear NDDE of the form

$$(b(\eta)\psi(u(\eta))[w'(\eta)]^r)' + q(\eta)u^\beta(\varrho(\eta)) = 0, \tag{1.1}$$

$\eta \geq \eta_0$, where $w(\eta) = u(\eta) + p(\eta)u(\tau(\eta))$. Hereafter, it will be assumed without further mention that

(A1) $b \in C^1([\eta_0,\infty),\mathbb{R})$, and $r,\beta \in \mathbb{Q}^+$ are quotients of odd numbers;

(A2) $p,q \in C^1([\eta_0,\infty),[0,\infty))$ and $p(\eta) \leq p_0 < 1$;

(A3) $\tau,\varrho \in C([\eta_0,\infty),\mathbb{R})$, $\tau(\eta) \leq \eta$, $\varrho(\eta) \leq \eta$, $\lim_{\eta\to\infty} \tau(\eta) = \infty$ and $\lim_{\eta\to\infty} \varrho(\eta) = \infty$;

(A4) $\psi \in C^1(\mathbb{R},(k,K])$, where $k$ and $K$ are positive constants and $\delta = \sqrt[r]{K/k}$.

In this context, a solution of (1.1) is defined as a real-valued function $u \in C^1([\eta_u,\infty)$, $\eta_u \geq \eta_0$, that satisfies (1.1) on $[\eta_u,\infty)$, and has the properties $b\psi(u)[w']^r \in C^1([\eta_u,\infty),\mathbb{R})$ and $\sup\{|y(\eta)| : \eta \geq \eta_*\} > 0$, for all $\eta_* \geq \eta_u$. An oscillatory solution of (1.1) is defined as having arbitrary large zeros. If not, it is classified as non-oscillatory. In addition, we study the canonical case of (1.1), that is,

$$\int_{\eta_0}^{\infty} b^{-1/r}(\ell)\,\mathrm{d}\ell = \infty. \tag{1.2}$$

In 1987, a new class of functions $\mathbb{P}$ is introduced by Philos [23] with the aim of extending and generalizing the results initially established by Kamenev [15]. Following Philos, it is necessary to define

$$D_0 = \{(\eta,\ell) : \eta > \ell > \eta_0\}$$

and

$$D = \{(\eta,\ell) : \eta \geq \ell \geq \eta_0\}.$$

The class $\mathbb{P}$ is said to include a function $H \in C([\eta_0, \infty), \mathbb{R})$, or $H \in \mathbb{P}$, if

(i) $H(\eta,\eta) = 0$, for $\eta \geq \eta_0$, $H(\eta,\ell) > 0$ on $D_0$.

(ii) $H(\eta,\ell)$ has a continuous and nonpositive partial derivative $\partial H/\partial \ell$ on $D_0$ such that the condition

$$\frac{\partial H}{\partial \ell} = -h(\eta,\ell)[H(\eta,\ell)]^{r/(r+1)},$$

for all $(\eta,\ell) \in D_0$, satisfies for some $h \in C(D,\mathbb{R})$.

Afterward, differential equations of various types and orders have been studied using Philos class $H$.

In 1985, Grammatikopoulos *et al.* [11] studied the neutral equation

$$(u(\eta) + p(\eta)u(\eta - \tau))'' + q(\eta)u(\eta - \varrho) = 0,$$

and they showed that it oscillates if $0 \leq p(\eta) \leq 1$ and $\int^{\infty} q(\ell)(1 - p(\ell - \varrho))\mathrm{d}\ell = \infty$.

For the equation

$$(u(\eta) + p(\eta)u(\eta - \tau))'' + q(\eta)f(u(\eta - \varrho)) = 0,$$

Xu and Xia [30] proved that it oscillates provided that $0 \leq p(\eta) < \infty$ and $q(\eta) \geq M > 0$.

In 2011, new oscillation criteria were presented by Baculikova and Džurina [4] for

$$(b(\eta)[u(\eta) + p(\eta)u(\tau(\eta))]')' + q(\eta)u(\varrho(\eta)) = 0,$$

where $0 \leq p(\eta) \leq p_0 < \infty$ and $\tau \circ \varrho = \varrho \circ \tau$.

By using comparison principles, Baculikova and Džurina [3] studied the equation

$$(b(\eta)[w'(\eta)]^r)' + q(\eta)u^{\beta}(\varrho(\eta)) = 0,$$

where $w(\eta) = u(\eta) + p(\eta)u(\tau(\eta))$, and they showed that it oscillates provided $0 \leq p(\eta) \leq p_0 < \infty$, $\varrho'(\eta) \geq 0$, $\tau'(\eta) \geq \tau_0 > 0$ and $\tau \circ \varrho = \varrho \circ \tau$.

Li *et al.* [16] studied the equation

$$(b(\eta)[(u(\eta) - p(\eta)u(\tau(\eta)))']^r)' + q(\eta)u^{\beta}(\varrho(\eta)) = 0,$$

where $0 \leq p(\eta) \leq p_0 < 1$, and derived oscillation criteria.

In [8], a number of oscillation-related results were obtained for

$$(b(\eta)[(u(\eta) - p(\eta)u(\tau(\eta)))']^r)' + q(\eta)u^{\beta}(\varrho(\eta)) = 0.$$

Rogovchenko [24] employed the Philos class $H$ to analyze the oscillatory behavior of the delay equation

$$(b(\eta)\psi(u(\eta))u'(\eta))' + q(\eta)f(u(\varrho(\eta))) = 0.$$

Concerning the neutral equation,

$$(b(\eta)\psi(u(\eta))w'(\eta))' + q(\eta)f(u(\varrho(\eta))) = 0,$$

which represents a more general case, Şahiner in [25] demonstrated that it oscillates by utilizing three frequently referenced sets of conditions in the literature.

This work focuses on studying the super-linear case and examining the oscillatory behavior of the NDDE (1.1). The criteria established herein guarantee that every solution of (1.1) must be oscillatory.

## 2. Auxiliary Lemmas

For the sake of convenience, $r$ and $\lambda$ are defined as $r := r^r/(r+1)^{r+1}$ and

$$\lambda_c(\eta) := \int_c^\eta b^{-1/r}(\ell)\,\mathrm{d}\ell. \tag{2.1}$$

In preparation for the main results, the following lemmas are introduced:

**Lemma 2.1.** *Assume that $N(\theta) = c_1\theta - c_2\theta^{1+1/r}$, where $c_1$ and $c_2 > 0$. Then $N$ has the maximum value at $\theta_{\max} := (rc_1/((r+1)c_2))^r$ and $N(\theta) \le N(\theta_{\max}) = rc_1^{r+1}c_2^{-r}$, for $\theta \in \mathbb{R}$.*

**Lemma 2.2.** *Suppose that $x$ is a positive solution. Then*

(P1) *$w$ and $w'$ are positive and $(b\psi(u)[w']^r)' < 0$ eventually.*

(P2) *$w > \frac{1}{\delta} b^{1/r} w' \lambda_{\eta_1}$ and $(w/\lambda_{\eta_1}^\delta)' < 0$.*

*Proof.* Let $u$ be an eventually positive solution of (1.1). Then, there is a $\eta_1 \ge \eta_0$ such that

$$u(\tau(\eta)) > 0 \text{ and } u(\varrho(\eta)) > 0, \quad \text{for } \eta \ge \eta_1.$$

Based on the definition of $w$, it can be concluded that $w(\eta) > 0$, for $\eta \ge \eta_1$. It follows from (1.1) and (A2) that

$$(b\psi(u)[w']^r)' = -qu^\beta(\varrho) \le 0. \tag{2.2}$$

Thus, $(b\psi(u)[w']^r)' \le 0$ and $w'$ is of a constant sign.

Suppose $w' < 0$ for $\eta \ge \eta_2$. Now,

$$\begin{aligned} b(\eta)\psi(u(\eta))[w'(\eta)]^r &\le b(\eta_2)\psi(u(\eta_2))[w'(\eta_2)]^r \\ &= -m_0 < 0. \end{aligned}$$

So,

$$[w']^r \le \frac{-m_0}{b\psi(u)}.$$

Also, $\psi(u) \le K$; this implies that

$$[w']^r \le \frac{-m_0}{K} \cdot \frac{1}{b}.$$

Thus,

$$w' \le -\sqrt[r]{\frac{m_0}{K}} \cdot \frac{1}{b^{1/r}}. \tag{2.3}$$

Integrating (2.3) leads to

$$w(\eta) \le w(\eta_2) - \sqrt[r]{\frac{m_0}{K}} \int_{\eta_2}^\eta \frac{1}{b^{1/r}(\ell)}\,\mathrm{d}\ell. \tag{2.4}$$

It follows from (1.2) that a contradiction is obtained.

Thus,

$$w'(\eta) > 0, \quad \text{for } \eta \ge \eta_2.$$

Now, it follows that

$$w' = \frac{(b\psi(u))^{1/r} w'}{b^{1/r}\psi^{1/r}(u)}, \quad \text{for } \eta \ge \eta_2.$$

Since $\psi(u) \le K$, then

$$\frac{(b\psi(u))^{1/r}w'}{b^{1/r}\psi^{1/r}(u)} \ge \frac{(b\psi(u))^{1/r}w'}{\sqrt[r]{K}b^{1/r}}, \quad \text{for } \eta \ge \eta_2.$$

Now,

$$w' \ge \frac{(b\psi(u))^{1/r}w'}{\sqrt[r]{K}b^{1/r}}, \quad \text{for } \eta \ge \eta_2. \tag{2.5}$$

Integrating (2.5) leads to

$$w(\eta) \ge w(\eta_2) + \int_{\eta_2}^{\eta} \frac{(b(\ell)\psi(u(\ell)))^{1/r}w'(\ell)}{\sqrt[r]{K}b^{1/r}(\ell)}\,\mathrm{d}\ell.$$

Utilizing the properties in (P1), one obtains

$$w(\eta) \ge \frac{1}{\sqrt[r]{K}}(b(\eta)\psi(u(\eta)))^{1/r}w'(\eta)\int_{\eta_2}^{\eta}\frac{1}{b^{1/r}(\ell)}\,\mathrm{d}\ell.$$

Since $\psi(u) > k$, then

$$\begin{aligned} w(\eta) &> \frac{\sqrt[r]{k}}{\sqrt[r]{K}}b^{1/r}(\eta)w'(\eta)\int_{\eta_2}^{\eta}\frac{1}{b^{1/r}(\ell)}\mathrm{d}\ell \\ &= \frac{1}{\delta}b^{1/r}(\eta)w'(\eta)\lambda_{\eta_1}(\eta). \end{aligned}$$

Now,

$$\begin{aligned} \left(\frac{w}{\lambda_{\eta_1}^{\delta}}\right)' &= \frac{w'\lambda_{\eta_1}^{\delta} - \delta\lambda_{\eta_1}^{\delta-1}b^{-1/r}w}{\lambda_{\eta_1}^{2\delta}} \\ &= \frac{\lambda_{\eta_1}w' - \delta b^{-1/r}w}{\lambda_{\eta_1}^{\delta+1}} \\ &< 0. \end{aligned}$$

The proof is thus concluded. □

## 3. Main Results

In what comes next, it is assumed that all functional inequalities hold eventually, meaning they are satisfied for sufficiently large values of $\eta$. This assumption does not lead to a loss of generality; specifically, any nonoscillatory solution $u$ of (1.1) can be regarded as eventually positive. The first oscillation result is now introduced, and its proof is established.

**Theorem 3.1.** *Suppose that $\varrho'(\eta) \ge 0$, $r \le \beta$, and there exists a function $\rho \in C^1(I,\mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty}\frac{1}{H(\eta,\eta_0)}\int_{\eta_0}^{\eta}\left[H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta} - \frac{Kr^r\rho(\ell)b(\varrho(\ell))}{M^{\beta-r}\beta^r(\varrho'(\ell))^r}\frac{[Q(\eta,\ell)]^{r+1}}{(r+1)^{r+1}}\right]\mathrm{d}\ell = \infty, \tag{3.1}$$

*where*

$$Q(\eta,\ell) = \frac{\rho'(\ell)}{\rho(\ell)}H(\eta,\ell)^{1/(r+1)} - h(\eta,\ell). \tag{3.2}$$

*Hence, every solution of* (1.1) *oscillates.*

*Proof.* Assume, contrary to the claim, that $u$ is an eventually positive solution of (1.1). Besides, there is $\eta_1 \geq \eta_0$ such that $u(\tau(\eta)) > 0$ and $u(\varrho(\eta)) > 0$, for $\eta \geq \eta_1$.

According to the definition of $w$,

$$\begin{aligned} u &= w - pu(\tau) \\ &\geq w - pw(\tau) \\ &\geq w - pw. \end{aligned}$$

Thus,

$$u \geq w(1-p). \tag{3.3}$$

From eq. (1.1) and using (3.3), it can be deduced that

$$\begin{aligned} (b\psi(u)[w']^r)' &= -qu^\beta(\varrho) \\ &\leq -qw^\beta(\varrho)(1-p(\varrho(\eta)))^\beta. \end{aligned} \tag{3.4}$$

The function $\varphi$ is defined as

$$\varphi = \rho \frac{b\psi(u)[w']^r}{w^\beta(\varrho)} > 0. \tag{3.5}$$

Now,

$$\varphi'(\eta) = \frac{\rho'(\eta)}{\rho(\eta)}\varphi(\eta) + \rho(\eta)\frac{(b\psi(u)[w']^r)'}{w^\beta(\varrho)} - \beta\rho(\eta)\frac{b\psi(u)[w']^r w'(\varrho)\varrho'}{w^{\beta+1}(\varrho)}.$$

Using (3.4), one obtains

$$\varphi'(\eta) \leq \frac{\rho'(\eta)}{\rho(\eta)}\varphi(\eta) - \rho(\eta)q(1-p(\varrho))^\beta - \beta\rho(\eta)\frac{b\psi(u)[w']^r w'(\varrho)\varrho'}{w^{\beta+1}(\varrho)}. \tag{3.6}$$

Since $(b(\eta)\psi(u(\eta))[w'(\eta)]^r)' \leq 0$, then

$$b(\eta)(\varrho)\psi(u(\varrho))[w'(\varrho)]^r \geq b\psi(u)[w']^r.$$

This leads to

$$w'(\varrho) \geq \frac{b^{1/r}\psi^{1/r}(u)w'}{b^{1/r}(\varrho)\psi^{1/r}(u(\varrho))}. \tag{3.7}$$

Using (3.7) and (A4), (3.6) becomes

$$\varphi'(\eta) \leq \frac{\rho'}{\rho}\varphi - \rho q(1-p(\varrho))^\beta - \frac{\beta}{K^{1/r}}\rho\frac{(b\psi(u)[w']^r)^{1+1/r}}{w^{\beta+1}(\varrho)}\frac{\varrho'}{b^{1/r}(\varrho)}.$$

Now, one can write

$$(w(\varrho))^{\beta+1} = \frac{(w(\varrho))^{\beta+(\beta/r)}}{(w(\varrho))^{(\beta/r)-1}}. \tag{3.8}$$

Back to the last inequality and using (3.8), it is concluded that

$$\begin{aligned} \varphi' &\leq \frac{\rho'}{\rho}\varphi - \rho q(1-p(\varrho))^\beta - \frac{\beta}{K^{1/r}}\left[\rho\frac{b\psi(u)[w']^r}{w^\beta(\varrho)}\right]^{1+1/r}\frac{\varrho'(w(\varrho))^{(\beta/r)-1}}{\rho^{1/r}b^{1/r}(\varrho)} \\ &= \frac{\rho'}{\rho}\varphi - \rho q(1-p(\varrho))^\beta - \frac{\beta}{K^{1/r}}\frac{\varrho'}{\rho^{1/r}b^{1/r}(\varrho)}(w(\varrho))^{(\beta/r)-1}\varphi^{1+1/r}. \end{aligned} \tag{3.9}$$

Since $w \geq M$ for all $\eta \geq \eta_2$ and letting $r \leq \beta$, then

$$(w(\varrho))^{(\beta/r)-1} \geq M^{(\beta/r)-1}. \tag{3.10}$$

Using (3.10) and substituting in (3.9), it follows that

$$\varphi' \le \frac{\rho'}{\rho}\varphi - \rho q(1-p(\varrho))^{\beta} - \frac{\beta M^{(\beta/r)-1}}{K^{1/r}}\frac{\varrho'}{\rho^{1/r}b^{1/r}(\varrho)}\varphi^{1+1/r}. \tag{3.11}$$

Multiplying (3.11) by $H(\eta,\ell)$ and integrating, one obtains

$$\int_{\eta_2}^{\eta} H(\eta,\ell)\varphi'(\ell)\mathrm{d}\ell \le \int_{\eta_2}^{\eta} H(\eta,\ell)\frac{\rho'(\ell)}{\rho(\ell)}\varphi(\ell)\mathrm{d}\ell - \int_{\eta_2}^{\eta} H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta}\mathrm{d}\ell - \frac{\beta M^{(\beta/r)-1}}{K^{1/r}}\int_{\eta_2}^{\eta} H(\eta,\ell)\frac{\varrho'(\ell)}{\rho^{1/r}(\ell)b^{1/r}(\varrho(\ell))}\varphi^{1+1/r}(\ell)\,\mathrm{d}\ell. \tag{3.12}$$

Then,

$$\int_{\eta_2}^{\eta} H(\eta,\ell)\varphi'(\ell)\mathrm{d}\ell = -H(\eta,\eta_2)\varphi(\eta_2) + \int_{\eta_2}^{\eta} h(\eta,\ell)[H(\eta,\ell)]^{r/(r+1)}\varphi(\ell)\,\mathrm{d}\ell.$$

Returning to (3.12), it follows that

$$\begin{aligned}\int_{\eta_2}^{\eta} H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta}\mathrm{d}\ell \le H(\eta,\eta_2)\varphi(\eta_2) &- \int_{\eta_2}^{\eta} h(\eta,\ell)[H(\eta,\ell)]^{r/(r+1)}\varphi(\ell)\,\mathrm{d}\ell \\ &+ \int_{\eta_2}^{\eta} H(\eta,\ell)\frac{\rho'(\ell)}{\rho(\ell)}\varphi(\ell)\,\mathrm{d}\ell \\ &- \frac{\beta M^{(\beta/r)-1}}{K^{1/r}}\int_{\eta_2}^{\eta} H(\eta,\ell)\frac{\varrho'(\ell)}{\rho^{1/r}(\ell)b^{1/r}(\varrho(\ell))}\varphi^{1+1/r}(\ell)\,\mathrm{d}\ell,\end{aligned}$$

or

$$\begin{aligned}\int_{\eta_2}^{\eta} H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta}\mathrm{d}\ell \le &-\frac{\beta M^{(\beta/r)-1}}{K^{1/r}}\int_{\eta_2}^{\eta} H(\eta,\ell)\frac{\varrho'(\ell)}{\rho^{1/r}(\ell)b^{1/r}(\varrho(\ell))}\varphi^{1+1/r}(\ell)\,\mathrm{d}\ell \\ &+ \int_{\eta_2}^{\eta}[H(\eta,\ell)]^{r/(r+1)}\left[-h(\eta,\ell)+\frac{\rho'(\ell)}{\rho(\ell)}[H(\eta,\ell)]^{1/(r+1)}\right]\varphi(\ell)\,\mathrm{d}\ell \\ &+ H(\eta,\eta_2)\varphi(\eta_2).\end{aligned}$$

Using Lemma 2.1 with $\theta=\varphi$,

$$c_1 = [H(\eta,\ell)]^{r/(r+1)}\left[-h(\eta,\ell)+\frac{\rho'(\ell)}{\rho(\ell)}[H(\eta,\ell)]^{1/(r+1)}\right] = [H(\eta,\ell)]^{r/(r+1)}Q(\eta,\ell)$$

and

$$c_2 = \frac{\beta M^{(\beta/r)-1}}{K^{1/r}}H(\eta,\ell)\frac{\varrho'(\ell)}{\rho^{1/r}(\ell)b^{1/r}(\varrho(\ell))},$$

one obtains

$$\begin{aligned}H(\eta,\eta_2)\varphi(\eta_2) \ge &\int_{\eta_2}^{\eta} H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta}\,\mathrm{d}\ell \\ &- \int_{\eta_2}^{\eta}\frac{r^r}{(r+1)^{r+1}}[[H(\eta,\ell)]^{r/(r+1)}Q(\eta,\ell)]^{r+1}\frac{KM^{r-\beta}}{\beta^r}\frac{\rho(\ell)b(\varrho(\ell))}{(\varrho'(\ell))^r}\frac{1}{[H(\eta,\ell)]^r}\,\mathrm{d}\ell\end{aligned}$$

or

$$\varphi(\eta_2) \ge \frac{1}{H(\eta,\eta_2)}\int_{\eta_2}^{\eta}\left[H(\eta,\ell)\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^{\beta} - \frac{Kr^r\rho(\ell)b(\varrho(\ell))}{M^{\beta-r}\beta^r(\varrho'(\ell))^r}\frac{[Q(\eta,\ell)]^{r+1}}{(r+1)^{r+1}}\right]\mathrm{d}\ell,$$

which contradicts assumption (3.1).

The proof is thus concluded. □

**Theorem 3.2.** *Suppose that $\varrho'(\eta) \geq 0$, $r \leq \beta$, and there exists a function $\rho \in C^1(I, \mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left[\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^\beta - \frac{KM^{r-\beta}r^r}{\beta^r(r+1)^{r+1}} \frac{(\rho'(\ell))^{r+1}}{(\rho(\ell))^r} \frac{b(\varrho(\ell))}{(\varrho'(\ell))^r}\right] \mathrm{d}\ell = \infty. \tag{3.13}$$

*Hence, every solution of* (1.1) *oscillates.*

*Proof.* By following the approach used in the proof of Theorem 3.1, one obtains

$$\varphi' \leq \frac{\rho'}{\rho}\varphi - \rho q(1-p(\varrho))^\beta - \frac{\beta M^{(\beta/r)-1}}{K^{1/r}} \frac{\varrho'}{\rho^{1/r} b^{1/r}(\varrho)} \varphi^{1+1/r}.$$

Using Lemma 2.1 with $\theta = \varphi$, $c_1 = \rho'/\rho$ and

$$c_2 = \frac{\beta M^{(\beta/r)-1}}{K^{1/r}} \frac{\varrho'}{\rho^{1/r} b^{1/r}(\varrho)},$$

it can be concluded that

$$\varphi' \leq -\rho q(1-p(\varrho))^\beta + \frac{KM^{r-\beta}r^r}{\beta^r(r+1)^{r+1}} \left(\frac{\rho'}{\rho}\right)^{r+1} \frac{\rho b(\varrho)}{(\varrho')^r}. \tag{3.14}$$

Integrating (3.14) leads to

$$\varphi(\eta) - \varphi(\eta_2) \leq \int_{\eta_2}^{\eta} \left[-\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^\beta + \frac{KM^{r-\beta}r^r}{\beta^r(r+1)^{r+1}} \frac{(\rho'(\ell))^{r+1}}{\left(\rho(\ell)\right)^r} \frac{b(\varrho(\ell))}{(\varrho'(\ell))^r}\right] \mathrm{d}\ell.$$

Thus,

$$\varphi(\eta_2) \geq \int_{\eta_2}^{\eta} \left[\rho(\ell)q(\ell)(1-p(\varrho(\ell)))^\beta - \frac{KM^{r-\beta}r^r}{\beta^r(r+1)^{r+1}} \frac{(\rho'(\ell))^{r+1}}{(\rho(\ell))^r} \frac{b(\varrho(\ell))}{(\varrho'(\ell))^r}\right] \mathrm{d}\ell,$$

which contradicts assumption (3.13).

The proof is thus concluded. □

**Theorem 3.3.** *Suppose that there exists a function $\rho \in C^1(I, \mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty} \int_{\eta_1}^{\eta} \left[\rho(\ell)q(\ell)\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho(\ell))}{\lambda_{\eta_1}^{\delta}(\ell)}\right)^\beta (1-p(\varrho(\ell)))^\beta - \frac{Kr^r}{\beta^r(r+1)^{r+1}} \frac{(\rho'(\ell))^{r+1}}{\rho^r(\ell)} b(\ell)\right] \mathrm{d}\ell = \infty. \tag{3.15}$$

*Hence, every solution of* (1.1) *oscillates.*

*Proof.* Following the steps in the proof of Theorem 3.1, one arrives at

$$(b(\eta)\psi(u(\eta))[w'(\eta)]^r)' \leq -q(\eta)w^\beta(\varrho(\eta))(1-p(\varrho(\eta)))^\beta.$$

The function $\varphi$ is defined as

$$\varphi = \rho \frac{b\psi(u)[w']^r}{w^\beta} > 0. \tag{3.16}$$

Then,

$$\begin{aligned}\varphi' &= \frac{\rho'}{\rho}\varphi + \rho\left[\frac{(b\psi(u)[w']^r)'}{w^{2\beta}} w^\beta - \beta \frac{b\psi(u)[w']^r}{w^{2\beta}} w^{\beta-1} w'\right] \\ &\leq \frac{\rho'}{\rho}\varphi + \rho\left[\frac{-qw^\beta(\varrho)(1-p(\varrho))^\beta}{w^\beta} - \beta\frac{b\psi(u)[w']^{r+1}}{w^{\beta+1}}\right]\end{aligned}$$

$$\leq \frac{\rho'}{\rho}\varphi + \rho\left[-q\left(\frac{w(\varrho)}{w}\right)^{\beta}(1-p(\varrho(\eta)))^{\beta} - \beta b\psi(u)\frac{[w']^{r+1}}{w^{\beta+1}}\right],$$

which by using (3.16) implies that

$$\begin{aligned}\varphi' &\leq \frac{\rho'}{\rho}\varphi + \rho\left[-q\left(\frac{w(\varrho)}{w}\right)^{\beta}(1-p(\varrho(\eta)))^{\beta} - \beta\frac{b\psi(u)\varphi^{\frac{r+1}{r}}}{\rho^{\frac{r+1}{r}}b^{\frac{r+1}{r}}\psi^{\frac{r+1}{r}}(u)}\right] \\ &= \frac{\rho'}{\rho}\varphi - \rho q\left(\frac{w(\varrho)}{w}\right)^{\beta}(1-p(\varrho(\eta)))^{\beta} - \beta\frac{\varphi^{\frac{r+1}{r}}}{\rho^{\frac{1}{r}}b^{\frac{1}{r}}\psi^{\frac{1}{r}}(u)}. \end{aligned} \tag{3.17}$$

Since $\psi(u) \leq K$, (3.17) becomes

$$\varphi' \leq \frac{\rho'}{\rho}\varphi - \rho q\left(\frac{w(\varrho)}{w}\right)^{\beta}(1-p(\varrho(\eta)))^{\beta} - \frac{\beta}{K^{\frac{1}{r}}}\frac{\varphi^{\frac{r+1}{r}}}{\rho^{\frac{1}{r}}b^{\frac{1}{r}}}. \tag{3.18}$$

It follows from Lemma 2.2 that

$$\left(\frac{w}{\lambda_{\eta_1}^{\delta}}\right)' < 0,$$

which leads to

$$\left(\frac{w(\varrho)}{w}\right)^{\beta} \geq \left(\frac{\lambda_{\eta_1}^{\delta}(\varrho)}{\lambda_{\eta_1}^{\delta}}\right)^{\beta}.$$

Using this and substituting in (3.18), then

$$\varphi' \leq \frac{\rho'}{\rho}\varphi - \rho q\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho)}{\lambda_{\eta_1}^{\delta}}\right)^{\beta}(1-p(\varrho))^{\beta} - \frac{\beta}{K^{\frac{1}{r}}}\frac{\varphi^{\frac{r+1}{r}}}{\rho^{\frac{1}{r}}b^{\frac{1}{r}}}. \tag{3.19}$$

Using Lemma 2.1 with $\theta = \varphi$, $c_1 = \rho'/\rho$ and

$$c_2 = \frac{\beta}{K^{\frac{1}{r}}}\frac{1}{\rho^{\frac{1}{r}}b^{\frac{1}{r}}}.$$

Eq. (3.19) becomes

$$\begin{aligned}\varphi' &\leq -\rho q\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho)}{\lambda_{\eta_1}^{\delta}}\right)^{\beta}(1-p(\varrho))^{\beta} + \frac{Kr^r}{\beta^r(r+1)^{r+1}}\left(\frac{\rho'}{\rho}\right)^{r+1}\rho b \\ &= -\rho q\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho)}{\lambda_{\eta_1}^{\delta}}\right)^{\beta}(1-p(\varrho))^{\beta} + \frac{Kr^r}{\beta^r(r+1)^{r+1}}\frac{(\rho')^{r+1}}{\rho^r}b. \end{aligned} \tag{3.20}$$

Integrating (3.20) leads to

$$\varphi(\eta) - \varphi(\eta_2) \leq \int_{\eta_2}^{\eta}\left[-\rho(\ell)q(\ell)\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho(\ell))}{\lambda_{\eta_1}^{\delta}(\ell)}\right)^{\beta}(1-p(\varrho(\ell)))^{\beta} + \frac{Kr^r}{\beta^r(r+1)^{r+1}}\frac{(\rho'(\ell))^{r+1}}{\rho^r(\ell)}b(\ell)\right]\mathrm{d}\ell.$$

Thus,

$$\varphi(\eta_2) \geq \int_{\eta_2}^{\eta}\left[\rho(\ell)q(\ell)\left(\frac{\lambda_{\eta_1}^{\delta}(\varrho(\ell))}{\lambda_{\eta_1}^{\delta}(\ell)}\right)^{\beta}(1-p(\varrho(\ell)))^{\beta} - \frac{Kr^r}{\beta^r(r+1)^{r+1}}\frac{(\rho'(\ell))^{r+1}}{\rho^r(\ell)}b(\ell)\right]\mathrm{d}\ell,$$

which contradicts with assumption (3.15).

The proof is complete. □

By replacing $H(\eta,\ell)$ with $(\eta-\ell)^n$, $n\in w^+$, in Theorem 3.1, the following corollary is obtained.

**Corollary 3.1.** *Suppose that $\varrho'\geq 0$, $r\leq\beta$, and there exists a function $\rho\in C^1(I,\mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty}\frac{1}{(\eta-\eta_0)^n}\int_{\eta_0}^{\eta}\left[\rho(\ell)q(\ell)(\eta-\ell)^n(1-p(\varrho(\ell)))^\beta-\frac{Kr^r\rho(\ell)b(\varrho(\ell))}{M^{\beta-r}\beta^r(\varrho'(\ell))^r}\frac{[Q(\eta,\ell)]^{r+1}}{(r+1)^{r+1}}\right]\mathrm{d}\ell=\infty,$$

*where $Q$ is given by the definition in Theorem 3.1. Hence, every solution of (1.1) oscillates.*

By selecting $\rho(\ell)=\lambda^r_{\eta_1}(\varrho(\ell))$, the following oscillation result for (1.1) is derived.

**Corollary 3.2.** *Let $\varrho'(\eta)\geq 0$, $r\leq\beta$, and suppose there exists a function $\rho\in C^1(I,\mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty}\int_{\eta_1}^{\eta}\left[q(\ell)\lambda^r_{\eta_1}(\varrho(\ell))(1-p(\varrho(\ell)))^\beta-\frac{KM^{r-\beta}r^{2r+1}}{\beta^r(r+1)^{r+1}}\frac{\varrho'(\ell)}{\lambda_{\eta_1}(\varrho(\ell))(b(\varrho(\ell)))^{\frac{1}{r}}}\right]\mathrm{d}\ell=\infty.$$

*Hence, every solution of (1.1) oscillates.*

As a special case of (1.1), for

$$(b(\eta)\psi(u(\eta))[w'(\eta)])'+q(\eta)u(\varrho(\eta))=0, \tag{3.21}$$

the criterion that is proposed here is as follows:

**Corollary 3.3.** *Suppose that there exists a function $\rho\in C^1(I,\mathbb{R}^+)$ such that*

$$\limsup_{\eta\to\infty}\int_{\eta_1}^{\eta}\left[\rho(\ell)q(\ell)\left(\frac{\lambda^\delta_{\eta_1}(\varrho(\ell))}{\lambda^\delta_{\eta_1}(\ell)}\right)(1-p(\varrho(\ell)))-\frac{K}{4}\frac{(\rho'(\ell))^2}{\rho(\ell)}b(\ell)\right]\mathrm{d}\ell=\infty,$$

*where $\lambda_{\eta_1}(\eta):=\int_{\eta_1}^{\eta}[1/(b(\ell))]\,\mathrm{d}\ell$. Hence, every solution of (3.21) oscillates.*

## 4. Conclusion

This paper has presented some new theorems that investigate the oscillation of the super-linear equation (1.1) in its canonical case. The Riccati transformation technique was employed as a fundamental tool. In Theorem 3.1 and Theorem 3.2, monotonic constraints are not required on the delay functions, while Theorem 3.3 requires that $\varrho'\geq 0$. It would be intriguing if subsequent research could extend the results for the non-canonical case and also for higher-order equations.

### Competing Interests

The authors declare that they have no competing interests.

### Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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