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Research Article

Fermatean Picture Fuzzy Continuity in Fermatean Picture Fuzzy Topological Spaces

V. Chitra[1], B. Maheswari*[2] and S. Jafari[3]

[1] *Department of Mathematics, Nallamuthu Gounder Mahalingam College (affiliated with Bharathiar University), Pollachi 642001, Tamil Nadu, India*
[2] *Department of Mathematics, Dr. Mahalingam College of Engineering and Technology (affiliated with Anna University), Pollachi 642003, Tamil Nadu, India*
[3] *College of Vestsjaelland South & Mathematical and Physical Science Foundation, 4200 Slagelse, Denmark*
***Corresponding author:** mahes.sint@gmail.com

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Abstract. The purpose of this paper is to define a new Picture fuzzy continuous mapping in Fermatean picture fuzzy topological spaces and investigate its fundamental properties. In addition, we explore a key topological concept FPF-β connectedness within the framework of Fermatean picture fuzzy topology.

Keywords. Fermatean picture fuzzy set, Fermatean picture fuzzy topological spaces, Fermatean picture fuzzy continuous mapping, FPF-β connectedness

Mathematics Subject Classification (2020). 54A40, 54C05, 54D05

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1. Introduction

Fuzzy set theory, introduced by Zadeh [19] in 1965, revolutionized the way uncertainty is addressed in Mathematics and Engineering. By enabling partial membership rather than binary classification, fuzzy sets became a foundation for dealing with imprecision in real-world scenarios. Building on this concept, Atanassov [2, 3] developed *intuitionistic fuzzy sets* (IFS), adding degrees of membership and non-membership to enhance decision-making under uncertainty. This extension found utility in areas like expert systems and medical diagnosis. Atanassov and Gargov [1] further advanced the field with interval-valued intuitionistic fuzzy sets, which offered a more nuanced approach to managing uncertainty.

Neutrosophic sets, introduced later, take fuzziness a step further by incorporating truth, indeterminacy and falsity as independent components. Broumi and Smarandache [4, 5] contributed significantly to this domain by defining correlation coefficients and similarity measures for neutrosophic sets, enabling their application in practical problems like pattern recognition and decision analysis. Neutrosophic structures have also been applied in algebra, as exemplified by Jun and colleagues [12, 14] studies on cubic structures in BCK/BCI algebras.

Interval neutrosophic sets and their logic, as proposed by Wang *et al*. [16] bridge theoretical advancements and computational applications, particularly in artificial intelligence and machine learning. Similarly, Pythagorean fuzzy sets, introduced by Yager [17], provide an even broader framework, allowing the square sum of membership and non-membership degrees to be less than or equal to one. Peng and Yang [15], and Yang and Li [18] extended this concept, deriving novel results that strengthened the mathematical foundation of Pythagorean fuzzy sets.

Building upon fuzzy and neutrosophic theories, picture fuzzy sets were introduced by Cuong [7, 8] and Jun [13] as a novel approach to handling computational intelligence problems. These sets extend intuitionistic fuzzy logic by adding a fourth parameter-neutrality to model real-world scenarios more effectively. Recent studies have refined the arithmetic operations and aggregation operators of picture fuzzy sets, showcasing their utility in multi-criteria decision-making (Cuong [9], Yang and Li [18]).

Fermatean fuzzy sets and their topological applications are another significant advancement in the fuzzy logic domain. Ibrahim [10] and Ibrahim *et al*. [11] explored Fermatean fuzzy topological spaces, providing theoretical insights and applications in optimization and topology. Now in our research work [6], we have introduced a Fermatean picture fuzzy sets, combining the strengths of both Fermatean and Picture fuzzy paradigms for addressing nonlinear variational problems.

This paper introduces a new type of mapping in the field of Fermatean picture fuzzy topological spaces. The purpose of this work is to define Fermatean picture fuzzy continuous mapping within this framework and examine its fundamental properties. Furthermore, we investigate about FPF-β connectedness, offering new perspectives and tools for understanding the behavior of Fermatean picture fuzzy systems in a topological setting. This exploration bridges the theoretical and applied aspects of fuzzy topology, setting the stage for future advancements in the field.

2. Preliminaries

Definition 2.1. Let X be a universe of discourse, then a fuzzy set A is an object having the following formulation:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X\},$$

where $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is called the membership degree of x in X.

Definition 2.2 ([6])**.** A *Fermatean picture fuzzy* (FPF) set $\mathcal{A}$ in a universe U is an object of the form,

$$\mathcal{A} = \{\langle x, \langle x, \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)\rangle \mid x \in U\rangle\},$$

where $\alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of x in $\mathcal{A}$ and the following conditions are satisfied

$$0 \le \alpha_{\mathcal{A}}(x), \beta_{\mathcal{A}}(x), \gamma_{\mathcal{A}}(x) \le 1,$$

$$0 \le \alpha^3_{\mathcal{A}}(x) + \beta^3_{\mathcal{A}}(x) + \gamma^3_{\mathcal{A}}(x) \le 1, \quad \text{for all } x \in U.$$

Then, $\forall\ x \in U$, $\pi_{\mathcal{A}}(x) = 1 - \alpha^3_{\mathcal{A}}(x) + \beta^3_{\mathcal{A}}(x) + \gamma^3_{\mathcal{A}}(x)$ is called the degree of refusal membership of x in $\mathcal{A}$.

When dealing with human opinions that involve multiple types of responses such as 'yes', 'abstain', 'no', and 'refusal', Fermatean picture fuzzy Sets offer a suitable mathematical framework to handle the complexity and uncertainty inherent in such scenarios. For an example, in feedback mechanisms for products or services, users might express satisfaction (yes), dissatisfaction (no), neutrality (abstain) or refuse to provide feedback.

Definition 2.3 ([6])**.** Let X be a non-empty set and the FPF sets A and B be in the forms $A = \{\langle x, \langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x \in X\rangle\}$ and $B = \{\langle x, \langle x, \alpha_B(x), \beta_B(x), \gamma_B(x)\rangle \mid x \in X\rangle\}$,

(i) $(A) \subseteq (B)$ iff $\alpha_A(x) \le \alpha_B(x)$, $\beta_A(x) \le \beta_B(x)$ and $\gamma_A(x) \ge \gamma_B(x)$,

(ii) $(A) = (B)$ iff $\alpha_A(x) = \alpha_B(x)$, $\beta_A(x) = \beta_B(x)$ and $\gamma_A(x) = \gamma_B(x)$, $\forall\ x \in X$,

(iii) $A \cap B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle \mid x \in X\}$, where

  (a) $\alpha_{A\cap B}(x) = \min\{\alpha_A(x), \alpha_B(x)\}$,
  (b) $\beta_{A\cap B}(x) = \min\{\beta_A(x), \beta_B(x)\}$,
  (c) $\gamma_{A\cap B}(x) = \max\{\gamma_A(x), \gamma_B(x)\}$,

(iv) $A \cup B = \{\langle x, \alpha_{AB}(x), \beta_{AB}(x), \gamma_{AB}(x)\rangle \mid x \in X\}$, where

  (a) $\alpha_{A\cup B}(x) = \max\{\alpha_A(x), \alpha_B(x)\}$,
  (b) $\beta_{A\cup B}(x) = \min\{\beta_A(x), \beta_B(x)\}$,
  (c) $\gamma_{A\cup B}(x) = \min\{\gamma_A(x), \gamma_B(x)\}$.

Definition 2.4 (Images of Fermatean Picture Fuzzy Set [6])**.** Let X and Y be two non-empty set and $f : X \to Y$ be a function. If $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle \mid x \in X\}$ is an FPF set in X, then the image of A under f denoted by $f(A)$ is the FPF set in Y defined by

$$f(A) = \{\langle y, f(\alpha_A)(y), f(\beta_A)(y), 1 - f(1 - (\gamma_A)(y))\rangle \mid y \in Y\},$$

$$f(\alpha_A)(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \alpha_A(x), & \text{if } f^{-1}(y) \ne \phi, \\ 0, & \text{otherwise,} \end{cases}$$

$$f(\beta_A)(y) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \beta_A(x), & \text{if } f^{-1}(y) \ne \phi, \\ 0, & \text{otherwise,} \end{cases}$$

$$1 - f(1 - (\gamma_A)(y)) = \begin{cases} \inf\limits_{x\in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \ne \phi, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.5 (Preimages of Fermatean Picture Fuzzy Set [6]). Let X and Y be two non-empty set and $f: X \to Y$ be a function.
If $B = \{\langle y, \alpha_B(y), \beta_B(y), \gamma_B(y)\rangle \mid y \in Y\}$ is an FPF set in Y, then the preimage of B under f denoted by $f^{-1}(B)$ is the FPF set in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\alpha_B)(x), f^{-1}(\beta_B)(x), f^{-1}(\gamma_A)(x)\rangle \mid x \in X\},$$

where $f^{-1}(\alpha_B)(x) = \alpha_B(f(x))$, $f^{-1}(\beta_B)(x) = \beta_B(f(x))$, $f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$.

Definition 2.6 ([6]). Let X be a non-empty set and τ be a family of Fermatean picture fuzzy ($\mathcal{F}$) subset of X. If

(i) $1_X, 0_X \in \tau$,

(ii) for any $\mathcal{F}_1, \mathcal{F}_2 \in \tau$, we have $\mathcal{F}_1 \cap \mathcal{F}_2 \in \tau$,

(iii) for any $\{\mathcal{F}_i\}_{i \in I} \subset \tau$, we have $\bigcup\limits_{i \in I} \mathcal{F}_i \in \tau$, where I is an arbitrary index set then τ is called a Fermatean picture fuzzy topology on X.

Fermatean Picture Fuzzy Topological Space (FPFTS) is defined as the pair (X, τ). Every element in τ is referred to as an open Fermatean picture fuzzy subset. A closed Fermatean picture fuzzy set is the complement of an open Fermatean picture fuzzy subset.

Definition 2.7 ([6]). $A = \{x, \alpha_{\mathcal{F}}(x), \beta_{\mathcal{F}}(x), (\gamma_{\mathcal{F}}(x)) : x \in X\}$ be a FPFS in X and (X, τ) be a FPFTS. Fermatean picture fuzzy closure and interior are defined by

(i) $\text{FPFcl}(A) = \cap\{H : H$ is closed Fermatean picture fuzzy set in X and $\mathcal{F} \subset H\}$.

(ii) $\text{FPFint}(A) = \cup\{G : G$ is open Fermatean picture fuzzy set in X and $G \subset \mathcal{F}\}$.

Definition 2.8 ([6]). Let (X, τ) be a Fermatean picture fuzzy topological space and $\mathcal{F}_1, \mathcal{F}_2$ be Fermatean picture fuzzy sets in X. Then, the following properties hold:

(i) $\text{FPFint}(\mathcal{F}_1) \subset \mathcal{F}_1$ and $\mathcal{F}_1 \subset \text{FPFcl}(\mathcal{F}_1)$.

(ii) If $\mathcal{F}_1 \subset \mathcal{F}_2$, then $\text{FPFint}(\mathcal{F}_1) \subset \text{FPFint}(\mathcal{F}_2)$ and $\text{FPFcl}(\mathcal{F}_1) \subset \text{FPFcl}(\mathcal{F}_2)$.

(iii) $\mathcal{F}_1$ is open Fermatean picture fuzzy if and only if $\mathcal{F}_1 = \text{FPFint}(\mathcal{F}_1)$.

(iv) $\mathcal{F}_1$ is closed Fermatean picture fuzzy if and only if $\mathcal{F}_1 = \text{FPFcl}(\mathcal{F}_1)$.

(v) $\text{FPFint}(\mathcal{F}_1) \cup \text{FPFint}(\mathcal{F}_2) \subset \text{FPFint}(\mathcal{F}_1 \cup \mathcal{F}_2)$.

(vi) $\text{FPFcl}(\mathcal{F}_1 \cap \mathcal{F}_2) \subset \text{FPFcl}(\mathcal{F}_1) \cap \text{FPFcl}(\mathcal{F}_2)$.

(vii) $\text{FPFint}(\mathcal{F}_1 \cap \mathcal{F}_2) = \text{FPFint}(\mathcal{F}_1) \cap \text{FPFint}(\mathcal{F}_2)$.

(viii) $\text{FPFcl}(\mathcal{F}_1) \cup \text{FPFcl}(\mathcal{F}_2) = \text{FPFcl}(\mathcal{F}_1 \cup \mathcal{F}_2)$.

Remark 2.9. Let (X, τ) be a Fermatean picture fuzzy topological space and $\mathcal{F}$ be a Fermatean picture fuzzy set in (X, τ). Then, the ensuing characteristics are true:

(i) $\text{FPFcl}(\mathcal{F}^c) = \text{FPFint}(\mathcal{F})^c$.

(ii) $\text{FPFint}(\mathcal{F}^c) = \text{FPFcl}(\mathcal{F})^c$.

(iii) $\text{FPFcl}(\mathcal{F}^c)^c = \text{FPFint}(\mathcal{F})$.

(iv) $\text{FPFint}(\mathcal{F}^c)^c = \text{FPFcl}(\mathcal{F})$.

Definition 2.10 ([6])**.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two Fermatean picture fuzzy subsets in a FPFTS. Then, if there is an open Fermatean picture fuzzy subset A, such as $\mathcal{F}_1 \subset A \subset \mathcal{F}_2$, then $\mathcal{F}_2$ is said to be a neighbourhood of $\mathcal{F}_1$.

Definition 2.11 ([6])**.** Let $g : X \to Y$ be a function and (X,τ_1), (Y,τ_2) two Fermatean picture fuzzy topological spaces. If there exists a neighbourhood U of $\mathcal{F}_1$ such that $g[U] \subset V$ for every neighbourhood V of $g[\mathcal{F}_1]$ and for any Fermatean picture fuzzy subset $\mathcal{F}_1$ of X, then g is said to be Fermatean picture fuzzy continuous.

3. Fermatean Picture Fuzzy Continuity

Definition 3.1. Let $(X,\mathcal{F}_p)$ be a *Fermatean Picture Fuzzy Topological Space* (FPFTS) and a *Fermatean Picture Fuzzy* (FPF) subset A on a non-empty set X is called *Fermatean Picture Fuzzy Open* (FPF-O) set if for each element $x \in A$, there exists an open neighborhood around x within A, where

(i) the degrees of positive membership $(m(x))$ neutral membership $(n(x))$ and negative membership $(l(x))$ are consistently high within this neighborhood for each element,

(ii) the hesitation degree $h(x)$ is also appropriately accounted for, ensuring the Fermatean condition $m^3 + n^3 + l^3 + h^3 \le 1$.

Definition 3.2. An FPF subset A on a non-empty set X is called *FPF Pre-Open* (FPF-PO) set if there exists an FPF subset $B \subseteq X$ such that B is FPF-O set in X and $B \subseteq A \subseteq \text{FPFcl}(B)$.

In other words, an FPF pre-open set is an 'almost open' set where each element's fuzzy membership values (positive, neutral, negative, and hesitation) are such that they satisfy a looser neighborhood condition around each point in A, relative to the open sets in the FPFTS.

Theorem 3.3. *Every FPF-O set is FPF-PO set.*

Proof. By definition, an FPF subset $A \subseteq X$ is FPF-PO set if there exists an FPF-O set $B \subseteq X$ such that $B \subseteq A \subseteq \text{FPFcl}(B)$. For an FPF-O set A, we can choose $B = A$. Clearly, $B = A \subseteq A \subseteq \text{FPFcl}(A)$ for any subset A. Thus, every FPF-O set A satisfies the definition of FPF-PO set, making A is FPF-PO set in X. □

Theorem 3.4. *If $\{A_i\}_{i\in I}$ is a collection of FPF-PO sets in FPFTS $(X,\mathcal{F}_p)$, then the union $\bigcup_{i\in I} A_i$ is also an FPF-PO set in X.*

Proof. Since each A_i is FPF-PO set, for each A_i there exists an open set $B_i \subseteq X$ such that $B_i \subseteq A_i \subseteq \text{FPFcl}(B_i)$. Consider the union $\bigcup_{i\in I} A_i$. We have, the union of FPF-O sets $\bigcup_{i\in I} B_i$ is FPF-O. Also $\bigcup_{i\in I} B_i \subseteq \bigcup_{i\in I} A_i \subseteq \text{FPFcl}(\bigcup_{i\in I} B_i)$.

Therefore, $\bigcup_{i\in I} A_i$ satisfies the condition for being FPF-PO, making it as FPF-PO set in X. □

Theorem 3.5. *The intersection of two FPF-PO sets in FPFTS X is not necessarily FPF-PO.*

Proof. In general, the intersection of two pre-open sets does not need to satisfy the conditions required for pre-openness. Specifically, if we take two FPF-PO sets A and C with corresponding open subsets $B \subseteq A$ and $D \subseteq C$, then it is not always possible to find a single open subset $E \subseteq A \cap C$ such that $E \subseteq A \cap C \subseteq \text{FPFcl}(E)$. □

Example 3.6. Consider $X = \{a,b,c\}$, with the FPF set $F = \{(x,\mu(x),\nu(x),\pi(x)) \mid x \in X\}$ in FPFTS, let the parameters be as follows:

- for a: $(\mu(a),\nu(a),\pi(a)) = (0.8,0.1,0.1)$,
- for b: $(\mu(b),\nu(b),\pi(b)) = (0.6,0.3,0.1)$,
- for c: $(\mu(c),\nu(c),\pi(c)) = (0.5,0.4,0.1)$.

The subsets $U = \{a,b\}$ and $V = \{b,c\}$ are FPF-PO sets but $U \cap V$ is not an FPF-PO set.

Definition 3.7. A subset $A \subseteq X$ in a FPFTS is called *Fermatean Picture Fuzzy Semi-Open* (FPF-SO) set if $A \subseteq \text{FPFcl}(\text{FPFint}(A))$.

Example 3.8. Consider $X = \{a,b,c\}$, with the FPF set $A = \{(x,\mu(x),\nu(x),\pi(x)) \mid x \in X\}$ in FPFTS, let the parameters be as follows:

- for $a : (\mu(a),\nu(a),\pi(a)) = (0.8,0.1,0.1)$,
- for $b : (\mu(b),\nu(b),\pi(b)) = (0.7,0.2,0.1)$,
- for $c : (\mu(c),\nu(c),\pi(c)) = (0.5,0.4,0.1)$,

where $A \subseteq \text{FPFcl}(\text{FPFint}(A))$, making it FPF-SO set.

Definition 3.9. A subset $A \subseteq X$ in a FPFTS is called a *Fermatean Picture Fuzzy β-Open* (FPF-βO) set if there exists n FPF-O set O such that $O \subseteq A \subseteq \text{FPFcl}(O)$.

Definition 3.10. For any FPFTS $(X,\mathcal{F}_p)$, we have the following:

(i) every FPF-O set is an FPF-PO set,

(ii) every FPF-SO set is an FPF-PO set,

(iii) every FPF-O set is an FPF-SO set,

(iv) every FPF-βO set is an FPF-PO set.

The converse of the above statements need not be true which can be seen from the following examples.

Definition 3.11. Let $(X,\mathcal{F}_p)$ and $(Y,\mathcal{H}_p)$ be two FPFTS. Then, a bijective mapping $f:(X,\mathcal{F}_p) \to (Y,\mathcal{H}_p)$ is called:

(i) *Fermatean Picture Fuzzy Continuous* mapping (in short FPF-C mapping) if and only if $f^{-1}(L)$ is an FPF-O set in X, whenever L is an FPF-O set in Y.

(ii) *Fermatean Picture Fuzzy Semi Continuous* mapping (in short FPF-SC mapping) if and only if $f^{-1}(L)$ is an FPF-SO set in X, whenever L is an FPF-O set in Y.

(iii) *Fermatean Picture Fuzzy Pre Continuous* mapping (in short FPF-PC mapping) if and only if $f^{-1}(L)$ is an FPF-PO set in X, whenever L is FPF-O set in Y.

(iv) *Fermatean Picture Fuzzy β-Continuous* mapping (in short FPF-βC mapping) if and only if $f^{-1}(L)$ is an FPF-βO set in X, whenever L is an FPF-O set in Y.

Theorem 3.12. *Let $(X,\mathcal{F}_p)$ and $(Y,\mathcal{H}_p)$ be two FPFTS. Then, every FPF-C mapping from $(X,\mathcal{F}_p)$ to $(Y,\mathcal{H}_p)$ is an FPF-PC mapping (FPF-SC mapping).*

Proof. Let L be an FPF-O set in $(Y,\mathcal{H}_p)$. Since, $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-C mapping, so $f^{-1}(L)$ is an FPF-O set in $(X,\mathcal{F}_p)$. Every FPF-O set is an FPF-PO set (FPF-SO set). Therefore, $f^{-1}(L)$ is an FPF-PO set (FPF-SO set) in $(X,\mathcal{F}_p)$. Hence, $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-PC mapping (FPF-SC mapping). □

Example 3.13. Let $X=\{x_1,x_2\}$ and $Y=\{y_1,y_2\}$,

$$F_X=\{\langle x_1,0.8,0.1,0.1\rangle,\langle x_2,0.5,0.4,0.1\rangle\},\quad F_Y=\{\langle y_1,0.9,0.1,0\rangle,\langle y_2,0.6,0.3,0.1\rangle\},$$

$f:X\to Y$ defined by $f(x_1)=y_1$ and $f(x_2)=y_2$,

$$f^{-1}(L)=\{x_1,x_2\}.$$

Since $\{x_1,x_2\}$ can be defined as an FPF-O sets in X,we conclude that f is an FPF-C mapping. $f^{-1}(L)$ must not only be FPF-O set but also FPF-PO set. Since every FPF-O set is also an FPF-PO set, we know $f^{-1}(L)=\{x_1,x_2\}$ is an FPF-PO set.
Since f meets the conditions for both FPF-C mapping and FPF-PC mapping, we conclude that $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is indeed an FPF-PC mapping.

Theorem 3.14. *Let $(X,\mathcal{F}_p)$ and $(Y,\mathcal{H}_p)$ be two FPFTS. Then, every FPF-SC mapping (FPF-PC mapping) from $(X,\mathcal{F}_p)$ to $(Y,\mathcal{H}_p)$ is an FPF-βC mapping.*

Proof. Let L be an FPF-O set in $(Y,\mathcal{H}_p)$. Since, $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-SC mapping (FPF-PC mapping), so $f^{-1}(L)$ is an FPF-SO set (FPF-PO set) in $(X,\mathcal{F}_p)$. It is known that, every FPF-SO set (FPF-PO set) is an FPF-βO set. Therefore, $f^{-1}(L)$ is an FPF-βO set in $(X,\mathcal{F}_p)$. Hence, $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-βC mapping. □

Example 3.15. Let $X=\{x_1,x_2\}$ and $Y=\{y_1,y_2\}$,

$$F_X=\{\langle x_1,0.9,0.1,0\rangle,\langle x_2,0.4,0.5,0.1\rangle\},\quad F_Y=\{\langle y_1,0.8,0.2,0\rangle,\langle y_2,0.3,0.6,0.1\rangle\},$$

$f:X\to Y$ defined by $f(x_1)=y_1$ and $f(x_2)=y_2$.
Let $L=\{y_1\}$ be an FPF-O set in Y. The preimage under f is $f^{-1}(L)=\{x_1\}$.
Since x_1 is included in F_X with a high membership degree, $f^{-1}(L)$ is an FPF-SO set in X. It follows that it is also an FPF-βO set. By definition, every FPF-SO set is an FPF-βO set. Since $f^{-1}(L)$ is an FPF-βO set in X, we conclude that $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-βC mapping.

Theorem 3.16. *Let $(X,\mathcal{F}_p)$ and $(Y,\mathcal{H}_p)$ be two FPFTS. Then, every FPF-C mapping from $(X,\mathcal{F}_p)$ to $(Y,\mathcal{H}_p)$ is an FPF-βC mapping.*

Proof. Let L be an FPF-O set in $(Y,\mathcal{H}_p)$. Since,$f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-C mapping, so $f^{-1}(L)$ is an FPF-O set in $(X,\mathcal{F}_p)$. It is known that every FPF-O set is an FPF-βO set. Therefore, $f^{-1}(L)$ is an FPF-βO set in $(X,\mathcal{F}_p)$. Hence, $f:(X,\mathcal{F}_p)\to(Y,\mathcal{H}_p)$ is an FPF-βC mapping. □

Example 3.17. Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$,

$F_X = \{\langle x_1, 0.8, 0.1, 0.1\rangle, \langle x_2, 0.7, 0.2, 0.1\rangle\}$,

$F_Y = \{\langle y_1, 0.85, 0.1, 0.05\langle,\rangle y_2, 0.75, 0.2, 0.05\rangle\}$,

$f : X \to Y$ defined by $f(x_1) = y_1$ and $f(x_2) = y_2$.
Since f is an FPF-C mapping, the image of any FPF-O set in X is also an FPF-βO set in Y. Therefore, f is an FPF-βC mapping.

Theorem 3.18. *If $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ and $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$ be two FPF-C mappings, then the composition mapping $g \circ f : (X, \mathcal{F}_p) \to (Z, \mathcal{K}_p)$ is also an FPF-C mapping.*

Proof. Let $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ and $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$ be two FPF-C mappings. Let L be an FPF-O set in $(Z, \mathcal{K}_p)$. Since, $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$ is an FPF-C mapping, so $g^{-1}(L)$ is an FPF-O set in Y. Since, $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ is an FPF-C mapping, so $f^{-1}((g^{-1}(L)) = ((g \circ f)^{-1}(L))$ is an FPF-O set in X. □

Example 3.19. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$. $f : X \to Y$ and $g : Y \to Z$ defined by $f(x_1) = y_1$, $f(x_2) = y_2$, $g(y_1) = z_1$, $g(y_2) = z_2$,

$F_X = \{\langle x_1, 0.8, 0.1, 0.1\rangle, \langle x_2, 0.7, 0.15, 0.15\rangle\}$,

$F_Y = \{\langle y_1, 0.75, 0.2, 0.05\rangle, \langle y_2, 0.85, 0.1, 0.05\rangle\}$,

$F_Z = \{\langle z_1, 0.85, 0.1, 0.05\rangle, \langle z_2, 0.8, 0.15, 0.05\rangle\}$.

Since both f and g are FPF-C mappings. Their composition $g \circ f$ is also an FPF-C mapping. Thus, the FPFTS structure is preserved under the composition of mappings.

Theorem 3.20. *If $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ is an FPF-βC mapping and $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$, an FPF-C mapping, then the composition mapping $g \circ f : (X, \mathcal{F}_p) \to (Z, \mathcal{K}_p)$ is an FPF-βC mapping.*

Proof. Let $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ be an FPF-βC mapping and $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$ be an FPF-C mapping. Let L be an FPF-O set in $(Z, \mathcal{K}_p)$. Since, $g : (Y, \mathcal{H}_p) \to (Z, \mathcal{K}_p)$ is an FPF-C mapping, so $g^{-1}(L)$ is an FPF-O set in Y. Now, in a FPFTS $(X, \mathcal{F}_p)$, every FPF-PO set (FPF-SO set) is an FPF-βO set, it is clear that $f^{-1}(L)$ is an FPF-O set in $(Y, \mathcal{H}_p)$. Since, $f : (X, \mathcal{F}_p) \to (Y, \mathcal{H}_p)$ is an FPF-βC mapping, so $f^{-1}(g^{-1}(L)) = (g \circ f)^{-1}(L)$ is an FPF-O set in X. Since, every FPF-O set is an FPF-βO set, so $(g \circ f)^{-1}(L)$ is an FPF-βO set in X. Hence, $g \circ f : (X, \mathcal{F}_p) \to (Z, \mathcal{K}_p)$ is an FPF-βC mapping. □

Example 3.21. By using Example 3.19, let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $f : X \to Y$ be defined by

$f(x_1) = y_1, \quad f(x_2) = y_2$.

Let us define a fuzzy neighbourhood V in Y is

$V = \{\langle y_1, 0.8, 0.1, 0.1\rangle, \langle x_2, 0.75, 0.15, 0.1\rangle\}$.

Then find a corresponding U in X such that $U \subseteq f^{-1}(V)$,

$U = \{\langle y_1, 0.8, 0.1, 0.1\rangle, \langle x_2, 0.75, 0.15, 0.1\rangle\}$.

If $U \subseteq f^{-1}(V)$, then f satisfies FPF-β continuity.
Therefore, the composition $g \circ f$ is also an FPF-βC mapping.

Definition 3.22. A subset B is called an FPF-β disconnected subset of an FPFTS $(X,\mathcal{F}_p)$ if there exist FPF-β open sets M, N such that $M \cap B \neq \varphi \neq N \cap B$, $M \cap N \cap B = \varphi$ and $B \subseteq M \cup N$, otherwise B is called an FPF-βconnected subset.

Theorem 3.23. *The union of any family of FPF-β connected sets with a nonempty intersection is FPF-β connected.*

Proof. Take $P = \bigcup_{i \in I} P_i$, where each P_i is FPF-β connected with $\cap P_i \neq \emptyset$. Suppose that P is not FPF-β connected. Then $P = R \cup S$, where R and S are two nonempty disjoint sets such that $(R \cap \text{FPF-}\beta cl(S)) \cup (\text{FPF-}\beta cl(R) \cap S) = \emptyset$. Since P_i is FPF-β connected and $P_i \subseteq P$, we have $P_i \subseteq R$ or $P_i \subseteq S$. Therefore, $\cup P_i \subseteq R$ or, $\cup P_i \subseteq S$. Since $\cap P_i \neq \emptyset$, there exists at least one element $x \in \cap P_i$. Therefore, $x \in P_i$, for all i. So, $x \in P$. Thus, $x \in R$ or $x \in S$. Suppose $x \in R$, since $R \cap S = \emptyset$, we have $x \notin R$. Therefore, $R \not\subset S$. Thus, $P \subseteq R$. This contradicts $P = R \cup S$. Hence, P is FPF-β connected. □

Example 3.24. Let $X = \{x_1, x_2, x_3\}$ and define FPF-β connected subsets A and B of X are

$$A = \{\langle x_1, 0.85, 0.1, 0.05\rangle, \langle x_2, 0.8, 0.15, 0.05\rangle\},$$

$$B = \{\langle x_2, 0.8, 0.15, 0.05\rangle, \langle x_3, 0.75, 0.2, 0.05\rangle\},$$

$$A \cup B = \{\langle x_1, 0.85, 0.1, 0.05\rangle, \langle x_2, 0.8, 0.15, 0.05\rangle, \langle x_3, 0.75, 0.2, 0.05\rangle\}.$$

Since $A \cap B \neq \emptyset$ (i.e., A and B share x_2), their union $A \cup B$ is also FPF-β connected.

4. Conclusion

In this paper, we introduced the concept of Fermatean picture fuzzy continuous mapping within the framework of Fermatean picture fuzzy topological spaces and explored its fundamental properties. By extending the existing notions of continuity in fuzzy topologies to include the Fermatean and Picture fuzzy paradigms, we provided a robust framework for addressing uncertainty in topological structures. Additionally, we investigated a key topological concept FPF-β connectedness in Fermatean picture fuzzy topological space, revealing their compatibility and applicability within this extended fuzzy environment.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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