Communications in Mathematics and Applications
Vol. 16, No. 3, pp. 845–859, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i3.3015

http://www.rgnpublications.com

Research Article

Certain Applications via (α,β)-$\mathcal{Z}$-Contraction Type Fixed Points in Bipolar Parametric Metric Space

K. Jyothirmayi Rani*[1] and V. Nagaraju[2]

Department of Mathematics, University College of Science (Osmania University), Telangana, India
*Corresponding author: jyothirmai.rani2013@gmail.com

Received: January 29, 2025 **Revised:** April 17, 2025 **Accepted:** May 4, 2025

Abstract. This work defines (α,β)-$\mathcal{Z}$ contraction mappings to establish fixed point results in a bipolar parametric metric space. Our findings generalize and enhance several well-known results from the fixed point (FP) theory literature. A good example is also provided to confirm the veracity of the acquired results. In addition, we provide applications for homotopy and integral equations, as well as an explanation of the significance of the obtained results.

Keywords. Bipolar parametric metric space, Fixed point, Covariant map, Contravariant map, (α,β)-$\mathcal{Z}$ Contraction maps.

Mathematics Subject Classification (2020). 47H10, 54H25

1. Introduction

In order to demonstrate the existence and ultimately the uniqueness of solutions to a variety of mathematical models such as, variational inequalities, approximation theory, optimization problems, and initial and boundary value problems in ordinary and partial differential equations, fixed point theorems are crucial tools. Over the past few years various writers have built various forms of generalized metric spaces using various approaches. The existence of FPs in contraction maps in BPMS (*Bipolar Metric Spaces*) which are generalizations of the Banach contraction principle, is one among the most latest developments in FP theory. The principles of BPMS was created in 2016 by Mutlu and Gürdal [14]. They, then looked into some fundamental fixed point (FP) and common fixed point (CFP) results for covariant and contravariant maps under contractive conditions (Mutlu *et al*. [15]). A significant amount of substantial work has been done on BPMSs (see, Gürdal *et al*. [4], Gaba *et al*. [5], Kishore *et al*. [9–12], Mutlu *et al*. [16], Naresh *et al*. [17], Rao *et al*. [19], Reddy *et al*. [20]).

Recently, Hussain *et al*. [6,7] introduced and explored the idea that parametric metric spaces are a natural generalization of metric spaces. As a generalization of parametric metric space, Kumar *et al*. [13] established fixed point theorems and introduced the idea of binary operation at the place of non-negative parameter $\mathfrak{t}$. The concept of BPPMS (*Bipolar Parametric Metric Space*) was introduced and some FP theorems were proved on this space by Pasha *et al*. [18].

The concept of (α,β)-admissible mappings was recently presented by Chandok [1] and derived some fixed point findings. Afterwards, many authors then obtained a generalization of the result, as an example, refer to Cho [2], and Dewangan *et al*. [3]. Although, Khojasteh *et al*. [8] presented a brand-new category of functions known as simulation functions. In [8], they demonstrated the validity of many fixed point theorems and demonstrated that numerous results found in the literature are only the outcome of their own findings.

Within the context of bipolar parametric metric space, we provide several fixed point theorems in this paper by using (α,β)-$\mathcal{Z}$-type contraction mappings via simulation function. Additionally, we could be able to provide appropriate and pertinent examples related to homotopy and integral equations.

2. Preliminaries

In this section, first we recall some basic results.

Definition 2.1 ([18]). Suppose $\mathfrak{d}_c : \mathcal{S}\times\Im\times(0,\infty)\to\mathbb{R}^+$ is a function defined on two non-empty sets $\mathcal{S}$ and $\Im$ such that

(a) $\mathfrak{d}_c(\mathfrak{x},\mathfrak{y},c)=0$, for all $c>0$ then $\mathfrak{x}=\mathfrak{y}$, for all $(\mathfrak{x},\mathfrak{y})\in\mathcal{S}\times\Im$.

(b) $\mathfrak{x}=\mathfrak{y}$, then $\mathfrak{d}_c(\mathfrak{x},\mathfrak{y},c)=0$, for all $c>0$ and $(\mathfrak{x},\mathfrak{y})\in\mathcal{S}\times\Im$

(c) $\mathfrak{d}_c(\mathfrak{x},\mathfrak{y},c)=\mathfrak{d}_c(\mathfrak{y},\mathfrak{x},c)$, for all $c>0$ and $\mathfrak{x},\mathfrak{y}\in\mathcal{S}\cap\Im$

(d) $\mathfrak{d}_c(\mathfrak{x},\mathfrak{y},c)\le\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c)+\mathfrak{d}_c(\Delta,\mathfrak{z},c)+\mathfrak{d}_c(\Delta,\mathfrak{y},c)$, for all $c>0$, $\mathfrak{x},\Delta\in\mathcal{S}$ and $\mathfrak{y},\mathfrak{z}\in\Im$.

The triplet $(\mathcal{S},\Im,\mathfrak{d}_c)$ is called a BPPMS.

Example 2.1 ([18]). For all $\mathfrak{x}\in\mathcal{S}$, $\mathfrak{y}\in\Im$ and $c>0$, let $\mathcal{S}=[-1,0]$ and $\Im=[0,1]$ be equipped with $\mathfrak{d}_c(\mathfrak{x},\mathfrak{y},c)=c|\mathfrak{x}-\mathfrak{y}|$. Consequently, $(\mathcal{S},\Im,\mathfrak{d}_c)$ is a complete BPPMS.

Definition 2.2 ([18]). Let $(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})$ and $(\mathcal{S}_2,\Im_2,\mathfrak{d}_{c2})$ be BPPMSs and $\Omega:\mathcal{S}_1\cup\Im_1\to\mathcal{S}_2\cup\Im_2$ be a function. If $\Omega(\mathcal{S}_1)\subseteq\mathcal{S}_2$ and $\Omega(\Im_1)\subseteq\Im_2$, then Ω is known as a covariant map or a map from $(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})$ to $(\mathcal{S}_2,\Im_2,\mathfrak{d}_{c2})$ and is written as $\Omega:(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})\rightrightarrows(\mathcal{S}_2,\Im_2,\mathfrak{d}_{c2})$. If $\Omega:(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})\rightrightarrows(\Im_2,\mathcal{S}_2,\bar{\mathfrak{d}}_{c_2})$ is a map, then Ω is known as a contravariant map from $(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})$ to $(\mathcal{S}_2,\Im_2,\mathfrak{d}_{c2})$ and this is denoted as $\Omega:(\mathcal{S}_1,\Im_1,\mathfrak{d}_{c1})\rightleftarrows(\mathcal{S}_2,\Im_2,\mathfrak{d}_{c2})$.

Definition 2.3 ([18]). Let $(\mathcal{S},\Im,\mathfrak{d}_c)$ be a BPPMS. Then

(z_1) The points of the sets $\mathcal{S}$, $\Im$, and $\mathcal{S}\cap\Im$ are referred to as left, right, and central points, respectively. A sequence on $(\mathcal{S},\Im,\mathfrak{d}_c)$ that consists solely of left, right, or central points is termed a left, right, or central sequence.

(z_2) If $\eth_c(\mathfrak{x}_a,\mathfrak{y},c) < \wp$ for all $a \geq a_0$ and $c > 0$ means that a left sequence $\{\mathfrak{x}_a\}$ converges to a right point $\mathfrak{y}$ if and only if to each $\wp > 0$ there exists an $a_0 \in \mathbb{N}$.

Similar to this, a right sequence $\{\mathfrak{y}_a\}$ converges to a left point $\mathfrak{x}$ if and only if we can locate a $a_0 \in \mathbb{N}$ satisfying, whenever $a \geq a_0$, $c > 0$, $\eth_c(\mathfrak{x},\mathfrak{y}_a,c) < \wp$.

Definition 2.4 ([18]). Let $(\mathcal{S},\Im,\eth_c)$ be a BPPMS.

(i) A sequence $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\})$ on the set $\mathcal{S}\times\Im$ is termed as a bisequence on $(\mathcal{S},\Im,\eth_c)$.

(ii) The bisequence $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\})$ is said to be convergent if sequences $\{\mathfrak{x}_a\}$ and $\{\mathfrak{y}_a\}$ are convergent. This bisequence is termed to be biconvergent if $\{\mathfrak{x}_a\}$ and $\{\mathfrak{y}_a\}$ converge to the same point $u \in \mathcal{S}\cap\Im$.

(iii) $(\{\mathfrak{x}_a\},\{\mathfrak{y}_a\})$ is a bisequence for $(\mathcal{S},\Im,\eth_c)$ is known as a Cauchy bisequence if, to each $\wp > 0$, we can locate a number $a_0 \in \mathbb{N}$, which, for every positive integer $a,b \geq a_0$, $c > 0$, $\eth_c(\mathfrak{x}_a, \mathfrak{y}_b,c) < \wp$, is called a Cauchy bisequence.

Definition 2.5 ([18]). Let $(\mathcal{S}_1,\Im_1,\eth_{c1})$ and $(\mathcal{S}_2,\Im_2,\eth_{c2})$ be BPPMSs.

(i) A map $\Omega : (\mathcal{S}_1,\Im_1,\eth_{c1}) \rightrightarrows (\mathcal{S}_2,\Im_2,\eth_{c2})$ is termed to be left continuous at a point $\sigma_0 \in \mathcal{S}_1$, if to each $\wp > 0$, we could find a $\delta > 0$ satisfying $\eth_{c1}(\sigma_0,\eta,c) < \delta$, $\eth_{c2}(\Omega(\sigma_0), \Omega(\eta),c) < \wp$ whenever $\eta \in \Im_1$, $c > 0$. It is right continuous at a point $\eta_0 \in \Im_1$ if to each $\wp > 0$, we could find a $\delta > 0$ satisfying $\eth_{c1}(\sigma,\eta_0,c) < \mathcal{S}_1$, $\eth_{c2}(\Omega(\sigma),\Omega(\eta_0),c) < \wp$ whenever $\sigma \in \mathcal{S}_1$, $c > 0$. If Ω is continuous at each point $\sigma \in \mathcal{S}_1$ and $\eta \in \Im_1$, then it is called continuous.

(ii) a contravariant map $\Omega : (\mathcal{S}_1,\Im_1,\eth_{c1}) \leftrightarrows (\mathcal{S}_2,\Im_2,\eth_{c2})$ is continuous if it has a covariant map $\Omega : (\mathcal{S}_1,\Im_1,\eth_{c1}) \rightrightarrows (\mathcal{S}_2,\Im_2,\bar{\eth}_{c_2})$.

It follows from this definition that a covariant or contravariant Ω from $(\mathcal{S}_1,\Im_1,\eth_{c1})$ to $(\mathcal{S}_2,\Im_2,\eth_{c2})$ is continuous, iff left sequence $\{\pi_a\}$ converges to a right point ς on $(\mathcal{S}_1,\Im_1,\eth_{c1})$ implies $\{\Omega(\pi_a)\} \to \Omega(\varsigma)$ on $(\mathcal{S}_2,\Im_2,\eth_{c2})$ and right sequence $\{\varsigma_a\}$ converges to a left point π on $(\mathcal{S}_1,\Im_1,\eth_{c1})$ implies $\{\Omega(\varsigma_a)\} \to \Omega(\pi)$ on $(\mathcal{S}_2,\Im_2,\eth_{c2})$.

Definition 2.6 ([8]). If $\mathfrak{F}$ meets the following criteria, it is referred to as a simulation function: $\mathfrak{F} : [0,\infty)\times[0,\infty) \to \mathbb{R}$

(i) $\mathfrak{F}(0,0) = 0$;

(ii) $\mathfrak{F}(\mathfrak{x},\mathfrak{w}) < \mathfrak{w} - \mathfrak{x}$, for all $\mathfrak{x},\mathfrak{w} > 0$;

(iii) if $\{\eta_a\},\{\sigma_a\} \subseteq (0,\infty)$ such that $\lim\limits_{a\to\infty} \eta_a = \lim\limits_{a\to\infty} \sigma_a = \ell \in (0,\infty)$, then $\lim\limits_{a\to\infty} \sup \mathfrak{F}(\eta_a,\sigma_a) < 0$

Example 2.2 ([8]). Let $\mathfrak{F} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ be defined as

(i)
$$\mathfrak{F}(\sigma,\varsigma) = \begin{cases} 1, & \text{if } (\sigma,\varsigma) = (0,0), \\ \iota\varsigma - \sigma, & \text{if otherwise,} \end{cases}$$
where $\iota \in (0;1)$. Then, $\mathfrak{F}$ is a simulation function.

(ii) $\mathfrak{F}(\sigma,\varsigma) = \tau\varsigma - \sigma$, for all $\sigma,\varsigma \in [0,\infty)$ and $\tau \in [0,1)$, then $\mathfrak{F}$ is a simulation function.

(iii) $\mathfrak{F}(\sigma,\varsigma) = \varsigma - \varphi(\varsigma) - \sigma$, for all $\sigma,\varsigma \in [0,\infty)$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semi continuous function such that $\varphi(\sigma) = 0$ iff $\sigma = 0$. Then, $\mathfrak{F}$ is a simulation function.

3. Main Results

We prove FP theorems on BPPMS in this section.

Definition 3.1. Suppose $\Omega : \mathcal{S} \cup \Im \to \mathcal{S} \cup \Im$ be a covariant map, $\alpha, \beta : \mathcal{S} \times \Im \to \mathbb{R}^+$ and $\mathcal{S}, \Im$ be two non-empty sets then Ω is a (α,β)-admissible mapping if

$$\alpha(\mathfrak{x},\mathfrak{y}) \geq 1 \text{ and } \beta(\mathfrak{x},\mathfrak{y}) \geq 1 \text{ implies that } \alpha(\Omega\mathfrak{x},\Omega\mathfrak{y}) \geq 1 \text{ and } \beta(\Omega\mathfrak{x},\Omega\mathfrak{y}) \geq 1, \quad \text{for all } (\mathfrak{x},\mathfrak{y}) \in (\mathcal{S},\Im).$$

Definition 3.2. Consider the BPPMS $(\mathcal{S},\Im,\mathfrak{d}_c)$. Assume a covariant map $\Omega : \mathcal{S} \cup \Im \to \mathcal{S} \cup \Im$ and $\alpha, \beta : \mathcal{S} \times \Im \to \mathbb{R}^+$, such Ω is called to be an (α,β)-$\mathcal{Z}$-contraction with respect to $\mathfrak{F}$ if

$$\mathfrak{F}(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c),\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c)) \geq 0, \tag{3.1}$$

for all $\mathfrak{x} \in \mathcal{S}$, $\mathfrak{z} \in \Im$ and $c > 0$ where $\mathfrak{F}$ is a simulation function.

Theorem 3.1. *Let $(\mathcal{S},\Im,\mathfrak{d}_c)$ be a complete BPPMS, and $\Omega : (\mathcal{S},\Im,\mathfrak{d}_c) \rightrightarrows (\mathcal{S},\Im,\mathfrak{d}_c)$ is a (α,β)-$\mathcal{Z}$-contraction covariant mapping with regard to a simulation function $\mathfrak{F}$ if there exist $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\eta(p) < p$ such that*

$$\mathfrak{F}(\eta(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c)),\eta(\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c))) \geq 0, \quad \textit{for all } (\mathfrak{x},\mathfrak{z}) \in (\mathcal{S},\Im). \tag{3.2}$$

Assume that

(i) *Ω is (α,β)-admissible;*

(ii) *there exist elements $\sigma_0 \in \mathcal{S}$ and $\varsigma_0 \in \Im$ such that $\alpha(\sigma_0,\Omega\varsigma_0) \geq 1$ and $\beta(\sigma_0,\Omega\varsigma_0) \geq 1$ and $\alpha(\Omega\sigma_0,\varsigma_0) \geq 1$ and $\beta(\Omega\sigma_0,\varsigma_0) \geq 1$;*

(iii) *Ω is $\mathfrak{d}_c$-continuous.*

The function $\Omega : \mathcal{S} \cup \Im \to \mathcal{S} \cup \Im$ contains a UFP (unique fixed point).

Proof. By condition (ii) of this theorem there exist elements $\sigma_0 \in \mathcal{S}$ and $\varsigma_0 \in \Im$ such that $\alpha(\sigma_0,\Omega\varsigma_0) \geq 1$, $\beta(\sigma_0,\Omega\varsigma_0) \geq 1$ and $\alpha(\Omega\sigma_0,\varsigma_0) \geq 1$, $\beta(\Omega\sigma_0,\varsigma_0) \geq 1$. Define the bisequence $(\{\sigma_a\},\{\varsigma_a\}) \subseteq (\mathcal{S},\Im)$ such that $\Omega(\sigma_a) = \sigma_{a+1}$ and $\Omega(\varsigma_a) = \varsigma_{a+1}$, for all $a \in \mathbb{N} \cup \{0\}$. If $\sigma_a = \sigma_{a+1} = \varsigma_a = \varsigma_{a+1}$ for some a, then $\sigma_a \in \mathcal{S} \cap \Im$ is FP of Ω and this completes the proof. Now assume that $\sigma_a \neq \sigma_{a+1}$ and $\varsigma_a \neq \varsigma_{a+1}$, for all $a \in \mathbb{N} \cup \{0\}$. Since from (i), Ω is an (α,β)-admissible, we derive

$$\alpha(\sigma_0,\Omega\varsigma_0) = \alpha(\sigma_0,\varsigma_1) \geq 1 \Rightarrow \alpha(\Omega\sigma_0,\Omega\varsigma_1) = \alpha(\sigma_1,\varsigma_2) \geq 1.$$

Continuing this process, we get

$$\alpha(\sigma_a,\varsigma_{a+1}) \geq 1, \quad \text{for all } a \geq 0. \tag{3.3}$$

Similarly,

$$\beta(\sigma_a,\varsigma_{a+1}) \geq 1, \quad \text{for all } a \geq 0. \tag{3.4}$$

From (3.2), (3.3) and (3.4), we have

$$\begin{aligned} 0 &\leq \mathfrak{F}(\eta(\alpha(\sigma_a,\varsigma_{a-1})\beta(\sigma_a,\varsigma_{a-1})\mathfrak{d}_c(\Omega\sigma_a,\Omega\varsigma_{a-1},c)),\eta(\mathfrak{d}_c(\sigma_a,\varsigma_{a-1},c))) \\ &= \mathfrak{F}(\eta(\alpha(\sigma_a,\varsigma_{a-1})\beta(\sigma_a,\varsigma_{a-1})\mathfrak{d}_c(\sigma_{a+1},\varsigma_a,c)),\eta(\mathfrak{d}_c(\sigma_a,\varsigma_{a-1},c))) \\ &< \eta(\mathfrak{d}_c(\sigma_a,\varsigma_{a-1},c)) - \eta\left(\alpha(\sigma_a,\varsigma_{a-1})\beta(\sigma_a,\varsigma_{a-1})\mathfrak{d}_c(\sigma_{a+1},\varsigma_a,c)\right). \end{aligned}$$

Therefore, for all $a = 0,1,2,3,\ldots$,

$$\eta(\alpha(\sigma_a,\varsigma_{a-1})\beta(\sigma_a,\varsigma_{a-1})\mathfrak{d}_c(\sigma_{a+1},\varsigma_a,c)) < \eta(\mathfrak{d}_c(\sigma_a,\varsigma_{a-1},c)).$$

By using the property of η, we get

$$\alpha(\sigma_a,\varsigma_{a-1})\beta(\sigma_a,\varsigma_{a-1})\mathfrak{d}_c(\sigma_{a+1},\varsigma_a,c) < \mathfrak{d}_c(\sigma_a,\varsigma_{a-1},c) \tag{3.5}$$

and also we have

$$\alpha(\Omega\sigma_0,\varsigma_0) = \alpha(\sigma_1,\varsigma_0) \geq 1 \Rightarrow \alpha(\Omega\sigma_1,\Omega\varsigma_0) = \alpha(\sigma_2,\varsigma_1) \geq 1.$$

Continuing this process, we get

$$\alpha(\sigma_{a+1},\varsigma_a) \geq 1, \quad \text{for all } a \geq 0. \tag{3.6}$$

Similarly,

$$\beta(\sigma_{a+1},\varsigma_a) \geq 1, \quad \text{for all } a \geq 0. \tag{3.7}$$

From (3.2), (3.6) and (3.7), we have

$$\begin{aligned}0 &\leq \mathfrak{F}(\eta(\alpha(\sigma_{a-1},\varsigma_a)\beta(\sigma_{a-1},\varsigma_a)\mathfrak{d}_c(\Omega\sigma_{a-1},\Omega\varsigma_a,c)),\eta(\mathfrak{d}_c(\sigma_{a-1},\varsigma_a,c)))\\ &= \mathfrak{F}(\eta(\alpha(\sigma_{a-1},\varsigma_a)\beta(\sigma_{a-1},\varsigma_a)\mathfrak{d}_c(\sigma_a,\varsigma_{a+1},c)),\eta(\mathfrak{d}_c(\sigma_{a-1},\varsigma_a,c)))\\ &< \eta(\mathfrak{d}_c(\sigma_{a-1},\varsigma_a,c)) - \eta(\alpha(\sigma_{a-1},\varsigma_a)\beta(\sigma_{a-1},\varsigma_a)\mathfrak{d}_c(\sigma_a,\varsigma_{a+1},c)).\end{aligned}$$

Therefore, for all $a = 0,1,2,3,\ldots$,

$$\eta(\alpha(\sigma_{a-1},\varsigma_a)\beta(\sigma_{a-1},\varsigma_a)\mathfrak{d}_c(\sigma_a,\varsigma_{a+1},c)) < \eta(\mathfrak{d}_c(\sigma_{a-1},\varsigma_a,c)).$$

By using the property of η, we get

$$\alpha(\sigma_{a-1},\varsigma_a)\beta(\sigma_{a-1},\varsigma_a)\mathfrak{d}_c(\sigma_a,\varsigma_{a+1},c) < \mathfrak{d}_c(\sigma_{a-1},\varsigma_a,c). \tag{3.8}$$

Moreover

$$\alpha(\sigma_0,\varsigma_0) \geq 1 \Rightarrow \alpha(\Omega\sigma_0,\Omega\varsigma_0) = \alpha(\sigma_1,\varsigma_1) \geq 1.$$

Continuing this process, we get

$$\alpha(\sigma_a,\varsigma_a) \geq 1, \quad \text{for all } a \geq 0. \tag{3.9}$$

Similarly,

$$\beta(\sigma_a,\varsigma_a) \geq 1, \quad \text{for all } a \geq 0. \tag{3.10}$$

From (3.2), (3.9) and (3.10), we have

$$\begin{aligned}0 &\leq \mathfrak{F}(\eta(\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\Omega\sigma_a,\Omega\varsigma_a,c)),\eta(\mathfrak{d}_c(\sigma_a,\varsigma_a,c)))\\ &= \mathfrak{F}(\eta(\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\sigma_{a+1},\varsigma_{a+1},c)),\eta(\mathfrak{d}_c(\sigma_a,\varsigma_a,c)))\\ &< \eta(\mathfrak{d}_c(\sigma_a,\varsigma_a,c)) - \eta(\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\sigma_{a+1},\varsigma_{a+1},c)).\end{aligned}$$

Therefore, for all $a = 0,1,2,3,\ldots$,

$$\eta(\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\sigma_{a+1},\varsigma_{a+1},c)) < \eta(\mathfrak{d}_c(\sigma_a,\varsigma_a,c)).$$

By using the property of η, we get

$$\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\sigma_{a+1},\varsigma_{a+1},c) < \mathfrak{d}_c(\sigma_a,\varsigma_a,c). \tag{3.11}$$

According to eqs. (3.5), (3.8) and (3.11), the bisequences $\{\mathfrak{d}_c(\sigma_a,\varsigma_a,c)\}$ are nonincreasing bisequences of non-negative real numbers. For $\theta \geq 0$, they should biconverge.

Assume that $\theta > 0$. Using the properties of η, eqs. (3.5), (3.8) and (3.11), and the condition (iii) in Definition 2.6, we obtain

$$0 \leq \limsup_{a\to\infty} \mathfrak{F}(\eta(\alpha(\sigma_a,\varsigma_a)\beta(\sigma_a,\varsigma_a)\mathfrak{d}_c(\sigma_{a+1},\varsigma_{a+1},c)),\eta(\mathfrak{d}_c(\sigma_a,\varsigma_a,c))) < 0$$

which is a contraction. Therefore, $\theta = 0$, this implies

$$\lim_{a\to\infty} \eth_c(\sigma_a,\varsigma_a,c)=0,\ \lim_{a\to\infty} \eth_c(\sigma_{a+1},\varsigma_a,c)=0 \quad\text{and}\quad \lim_{a\to\infty} \eth_c(\sigma_a,\varsigma_{a+1},c)=0. \tag{3.12}$$

We shall demonstrate that the bisequence $(\{\sigma_a\},\{\varsigma_a\})$ is a Cauchy bisequence sequence. Assume $(\{\sigma_a\},\{\varsigma_a\})$ does not form a Cauchy bisequence. Then, there exists $\wp > 0$ for which we can assume subsequences $(\{\sigma_{b_z}\},\{\varsigma_{b_z}\})$ and $(\{\sigma_{a_z}\},\{\varsigma_{a_z}\})$ of $(\{\sigma_a\},\{\varsigma_a\})$ with $b(z) > a(z) > z$ such that for every z and $c > 0$,

$$\eth_c(\sigma_{a_z},\varsigma_{b_z},c) \ge \wp, \quad \eth_c(\sigma_{b_z},\varsigma_{a_z},c) \ge \wp \tag{3.13}$$

and

$$\eth_c(\sigma_{a_z},\varsigma_{b_{z-1}},c) < \wp, \quad \eth_c(\sigma_{b_z},\varsigma_{a_{z-1}},c) < \wp. \tag{3.14}$$

By the condition (iv) in Definition 2.1 and using eqs. (3.13) and (3.14), we get

$$\begin{aligned}\wp &\le \eth_c(\sigma_{a_z},\varsigma_{b_z},c)\\ &\le \eth_c(\sigma_{a_z},\varsigma_{b_{z-1}},c)+\eth_c(\sigma_{b_z},\varsigma_{b_{z-1}},c)+\eth_c(\sigma_{b_z},\varsigma_{b_z},c)\\ &< \wp+\eth_c(\sigma_{b_z},\varsigma_{b_{z-1}},c)+\eth_c(\sigma_{b_z},\varsigma_{b_z},c).\end{aligned}$$

Letting $z \to \infty$ in the above inequalities and by using eqs. 3.12and (3.13), we get

$$\lim_{z\to\infty} \eth_c(\sigma_{a_z},\varsigma_{b_z},c)=\wp. \tag{3.15}$$

Also, from the condition (iv) in Definition 2.1, we have

$$\begin{aligned}&\eth_c(\sigma_{a_z},\varsigma_{b_z},c) \le \eth_c(\sigma_{a_z},\varsigma_{b_{z+1}},c)+\eth_c(\sigma_{b_{z+1}},\varsigma_{b_{z+1}},c)+\eth_c(\sigma_{b_{z+1}},\varsigma_{b_z},c),\\ &|\eth_c(\sigma_{a_z},\varsigma_{b_{z+1}},c)-\eth_c(\sigma_{a_z},\varsigma_{b_z},c)| \le \eth_c(\sigma_{b_{z+1}},\varsigma_{b_{z+1}},c)+\eth_c(\sigma_{b_{z+1}},\varsigma_{b_z},c).\end{aligned}$$

On taking limit as $z \to \infty$ on both sides of above inequality and using eqs. (3.12) and (3.15), we get

$$\lim_{z\to\infty} \eth_c(\sigma_{a_z},\varsigma_{b_{z+1}},c)=\wp. \tag{3.16}$$

Similarly, it is easy to demonstrate that

$$\lim_{z\to\infty} \eth_c(\sigma_{a_{z+1}},\varsigma_{b_z},c)=\lim_{z\to\infty} \eth_c(\sigma_{b_{z+1}},\varsigma_{a_z},c)=\lim_{z\to\infty} \eth_c(\sigma_{b_z},\varsigma_{a_z},c)=\lim_{z\to\infty} \eth_c(\sigma_{b_z},\varsigma_{a_{z+1}},c)=\wp. \tag{3.17}$$

Furthermore, Ω is a (α,β)-admissible, thus we have

$$\begin{aligned}&\alpha(\sigma_{a_z},\varsigma_{b_z},c)\ge 1, \quad \beta(\sigma_{a_z},\varsigma_{b_z},c)\ge 1,\\ &\alpha(\sigma_{b_z},\varsigma_{a_z},c)\ge 1, \quad \beta(\sigma_{b_z},\varsigma_{a_z},c)\ge 1.\end{aligned} \tag{3.18}$$

From the fact that Ω is an (α,β)-$\mathcal{Z}$-contraction with respect to $\mathfrak{F}$, along with eqs. (3.15), (3.18) and the condition (iii) in Definition 2.6, we get

$$0 \le \lim_{z\to\infty}\sup \mathfrak{F}(\eta(\alpha(\sigma_{a_z},\varsigma_{b_z})\beta(\sigma_{a_z},\varsigma_{b_z})\eth_c(\sigma_{a_{z+1}},\varsigma_{b_{z+1}},c)),\eta(\eth_c(\sigma_{a_z},\varsigma_{b_z},c))) < 0.$$

This is a contradiction, so $(\{\sigma_a\},\{\varsigma_a\})$ represents a Cauchy bisequence. Because $(\mathcal{S},\mathfrak{I},\eth_c)$ is complete,$(\{\sigma_a\},\{\varsigma_a\})$ converges to the point $\pi \in \mathcal{S}\cap\mathfrak{I}$ and

$$\{\Omega(\varsigma_a)\}=\{\varsigma_{a+1}\} \to \pi \in \mathcal{S}\cap\mathfrak{I}$$

ensures that $\{\Omega(\varsigma_a)\}$ has unique limit. Since Ω is continuous, $\Omega(\varsigma_a) \to \Omega(\pi)$, implying that $\Omega(\pi)=\pi$. Hence, π is an FP of Ω. If ς is any FP of Ω, then $\Omega(\varsigma)=\varsigma$ implies that $\varsigma \in \mathcal{S}\cap\mathfrak{I}$ and

$\varsigma \neq \pi$, then from eq. (3.2) and the condition (ii) in Definition 2.6, we have

$$0 \le \mathfrak{F}(\eta(\alpha(\pi,\varsigma)\beta(\pi,\varsigma)\mathfrak{d}_c(\Omega\pi,\Omega\varsigma,c)),\eta(\mathfrak{d}_c(\pi,\varsigma,c)))$$
$$< \eta(\mathfrak{d}_c(\pi,\varsigma,c)) - \eta(\alpha(\pi,\varsigma)\beta(\pi,\varsigma)\mathfrak{d}_c(\Omega\pi,\Omega\varsigma,c)).$$

By using the property of η, we have

$$0 < \mathfrak{d}_c(\pi,\varsigma,c) - \alpha(\pi,\varsigma)\beta(\pi,\varsigma)\mathfrak{d}_c(\Omega\pi,\Omega\varsigma,c) \le 0,$$

which is a contradiction, so $\pi = \varsigma$. □

Theorem 3.2. *Let $(\mathcal{S},\mathfrak{I},\mathfrak{d}_c)$ be a complete BPPMS and $\Omega : (\mathcal{S},\mathfrak{I},\mathfrak{d}_c) \rightrightarrows (\mathcal{S},\mathfrak{I},\mathfrak{d}_c)$ be a (α,β)-$\mathcal{Z}$-contraction covariant mapping with respect to a simulation function $\mathfrak{F}$ if there exist $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\eta(p) < p$ such that*

$$\mathfrak{F}(\eta(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c)),\eta(\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c))) \ge 0, \quad \textit{for all } (\mathfrak{x},\mathfrak{z}) \in (\mathcal{S},\mathfrak{I}). \tag{3.19}$$

Assume that

(i) *Ω is (α,β)-admissible;*

(ii) *there exist elements $\sigma_0 \in \mathcal{S}$ and $\varsigma_0 \in \mathfrak{I}$ such that $\alpha(\sigma_0,\Omega\varsigma_0) \ge 1$ and $\beta(\sigma_0,\Omega\varsigma_0) \ge 1$ and $\alpha(\Omega\sigma_0,\varsigma_0) \ge 1$ and $\beta(\Omega\sigma_0,\varsigma_0) \ge 1$;*

(iii) *if $(\{\sigma_a\},\{\varsigma_a\})$ is a bisequence in $(\mathcal{S},\mathfrak{I})$ such that $\alpha(\sigma_a,\varsigma_{a+1}) \ge 1$, $\beta(\sigma_a,\varsigma_{a+1}) \ge 1$, $\alpha(\sigma_{a+1},\varsigma_a) \ge 1$, $\beta(\sigma_{a+1},\varsigma_a) \ge 1$, for all a then there exists sub-bisequences $(\{\sigma_{a_z}\},\{\varsigma_{a_z}\})$ of $(\{\sigma_a\},\{\varsigma_a\})$ such that $\alpha(\sigma_{a_z},\varsigma_{a_{z+1}}) \ge 1$, $\beta(\sigma_{a_z},\varsigma_{a_{z+1}}) \ge 1$, $\alpha(\sigma_{a_{z+1}},\varsigma_{a_z}) \ge 1$, $\beta(\sigma_{a_{z+1}},\varsigma_{a_z}) \ge 1$, for all $z \in \mathbb{N}$ and $\alpha(\sigma,\Omega\varsigma) \ge 1$, $\beta(\sigma,\Omega\varsigma) \ge 1$ also $\alpha(\Omega\sigma,\varsigma) \ge 1$, $\beta(\Omega\sigma,\varsigma) \ge 1$.*

Then, the function $\Omega : \mathcal{S} \cup \mathfrak{I} \to \mathcal{S} \cup \mathfrak{I}$ has a UFP (Unique Fixed Point).

Proof. After demonstrating Theorem 3.1, we create a bisequence. $(\{\sigma_a\},\{\varsigma_a\}) \subseteq (\mathcal{S},\mathfrak{I})$ such that $\Omega(\sigma_a) = \sigma_{a+1}$ and $\Omega(\varsigma_a) = \varsigma_{a+1}$, for all $a \in \mathbb{N} \cup \{0\}$ which converges to some $\pi \in \mathfrak{I}$ and $\varsigma \in \mathcal{S}$ respectively. From definition of (α,β)-admissible mapping and condition (iii) of hypothesis, there exists a sub-bisequence $(\{\sigma_{a_z}\},\{\varsigma_{a_z}\})$ of $(\{\sigma_a\},\{\varsigma_a\})$ such that $\alpha(\sigma_{a_z},\varsigma_{a_{z+1}}) \ge 1$, $\beta(\sigma_{a_z},\varsigma_{a_{z+1}}) \ge 1$, $\alpha(\sigma_{a_{z+1}},\varsigma_{a_z}) \ge 1$, $\beta(\sigma_{a_{z+1}},\varsigma_{a_z}) \ge 1$, for all $z \in \mathbb{N}$ and $\alpha(\varsigma,\Omega\varsigma_{a_z}) \ge 1$, $\beta(\varsigma,\Omega\varsigma_{a_z}) \ge 1$ also $\alpha(\Omega\sigma_{a_z},\pi) \ge 1$, $\beta(\Omega\sigma_{a_z},\pi) \ge 1$. Thus applying eq. (3.19) for all z, we have

$$0 \le \mathfrak{F}(\eta(\alpha(\sigma_{a_z},\pi)\beta(\sigma_{a_z},\pi)\mathfrak{d}_c(\Omega\sigma_{a_z},\Omega\pi,c)),\eta(\mathfrak{d}_c(\sigma_{a_z},\pi,c)))$$
$$= \mathfrak{F}(\eta(\alpha(\sigma_{a_z},\pi)\beta(\sigma_{a_z},\pi)\mathfrak{d}_c(\sigma_{a_{z+1}},\Omega\pi,c)),\eta(\mathfrak{d}_c(\sigma_{a_z},\pi,c)))$$
$$< \eta(\mathfrak{d}_c(\sigma_{a_z},\pi,c)) - \eta(\alpha(\sigma_{a_z},\pi)\beta(\sigma_{a_z},\pi)\mathfrak{d}_c(\sigma_{a_{z+1}},\Omega\pi,c)).$$

By suing the property η, we have

$$0 < \mathfrak{d}_c(\sigma_{a_z},\pi,c) - \alpha(\sigma_{a_z},\pi)\beta(\sigma_{a_z},\pi)\mathfrak{d}_c(\sigma_{a_{z+1}},\Omega\pi,c)$$

which is equivalent to

$$\mathfrak{d}_c(\sigma_{a_{z+1}},\Omega\pi,c) = \mathfrak{d}_c(\Omega\sigma_{a_z},\Omega\pi,c) \le \alpha(\sigma_{a_z},\pi)\beta(\sigma_{a_z},\pi)\mathfrak{d}_c(\Omega\sigma_{a_z},\Omega\pi,c)$$
$$< \mathfrak{d}_c(\sigma_{a_z},\pi,c).$$

Letting $z \to \infty$ in the above, we have $\mathfrak{d}_c(\pi,\Omega\pi,c) = 0$. Moreover, from eq. (3.19), we have

$$0 \le \mathfrak{F}(\eta(\alpha(\varsigma,\varsigma_{a_z})\beta(\varsigma,\varsigma_{a_z})\mathfrak{d}_c(\Omega\varsigma,\Omega\varsigma_{a_z},c)),\eta(\mathfrak{d}_c(\varsigma,\varsigma_{a_z},c)))$$
$$< \eta(\mathfrak{d}_c(\varsigma,\varsigma_{a_z},c)) - \eta(\alpha(\varsigma,\varsigma_{a_z})\beta(\varsigma,\varsigma_{a_z})\mathfrak{d}_c(\Omega\varsigma,\varsigma_{a_{z+1}},c)).$$

By suing the property η, we have

$$0 < \eth_c(\varsigma,\varsigma_{a_z},c) - \alpha(\varsigma,\varsigma_{a_z})\beta(\varsigma,\varsigma_{a_z})\eth_c(\Omega\varsigma,\varsigma_{a_{z+1}},c)$$

which is equivalent to

$$\eth_c(\Omega\varsigma,\varsigma_{a_{z+1}},c) = \eth_c(\Omega\varsigma,\Omega\varsigma_{a_z},c) \le \alpha(\varsigma,\varsigma_{a_z})\beta(\varsigma,\varsigma_{a_z})\eth_c(\Omega\varsigma,\varsigma_{a_{z+1}},c) < \eth_c(\varsigma,\varsigma_{a_z},c).$$

Letting $z \to \infty$ in the above, we have $\eth_c(\Omega\varsigma,\varsigma,c) = 0$. Now, we show that $\pi = \varsigma \in \mathcal{S} \cap \Im$, then from (3.19), we have

$$0 \le \mathfrak{F}(\eta(\alpha(\pi,\varsigma)\beta(\pi,\varsigma)\eth_c(\Omega\pi,\Omega\varsigma,c)),\eta(\eth_c(\pi,\varsigma,c))) < \eta(\eth_c(\pi,\varsigma,c)) - \eta(\alpha(\pi,\varsigma)\beta(\pi,\varsigma)\eth_c(\Omega\pi,\Omega\varsigma,c)).$$

By using the property of η, we have

$$0 < \eth_c(\pi,\varsigma,c) - \alpha(\pi,\varsigma)\beta(\pi,\varsigma)\eth_c(\Omega\pi,\Omega\varsigma,c) \le 0,$$

which is a contradictory $\pi = \varsigma$. It is possible to demonstrate that π is a fixed point of Ω using the same reasoning as above, Similar arguments as provided in the proof of Theorem 3.1 are used to obtain the uniqueness of the fixed point of Ω. □

Corollary 3.1. *Suppose $(\mathcal{S},\Im,\eth_c)$ be a complete BPPMS and $\Omega : (\mathcal{S},\Im,\eth_c) \rightrightarrows (\mathcal{S},\Im,\eth_c)$ be a covariant mapping satisfying*

$$\eta\left(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\eth_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c)\right) \le \kappa\eta(\eth_c(\mathfrak{x},\mathfrak{z},c)), \quad \text{for all } (\mathfrak{x},\mathfrak{z}) \in (\mathcal{S},\Im), \tag{3.20}$$

where $\kappa \in [0,1)$ and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\eta(p) \le p$ and $\eta(0) = 0$. Also, assume that

(i) *Ω is (α,β)-admissible;*

(ii) *there exist elements $\sigma_0 \in \mathcal{S}$ and $\varsigma_0 \in \Im$ such that $\alpha(\sigma_0,\Omega\varsigma_0) \ge 1$ and $\beta(\sigma_0,\Omega\varsigma_0) \ge 1$ and $\alpha(\Omega\sigma_0,\varsigma_0) \ge 1$ and $\beta(\Omega\sigma_0,\varsigma_0) \ge 1$;*

(iii) *Ω is $\eth_c$-continuous.*

Then, the function $\Omega : \mathcal{S} \cup \Im \to \mathcal{S} \cup \Im$ has a UFP.

Proof. Following Theorem 3.1, by taking as a $\mathfrak{F}$-simulation function, $\mathfrak{F}(\sigma,\varsigma) = \kappa\varsigma - \sigma$. □

Example 3.3. Let $\mathcal{S} = [-1,0]$ and $\Im = [0,1]$ with $\eth_c(\sigma,\eta,c) = c|\sigma - \eta|$, for each $\sigma \in \mathcal{S}$, $\eta \in \Im$ and $c > 0$. Then, $(\mathcal{S},\Im,\eth_c)$ is a complete BPPMS. Define $\Omega : \mathcal{S} \cup \Im \rightrightarrows \mathcal{S} \cup \Im$ given by

$$\Omega(\sigma) = \begin{cases} \frac{\sigma}{6}, & \text{if } \sigma \in [-1,0], \\ 0, & \text{if } \sigma \in (0,1], \end{cases}$$

and let $\mathfrak{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ as $\mathfrak{F}(\sigma,\varsigma) = \kappa\varsigma - \sigma$ where $\kappa = \frac{1}{2} \in [0,1)$. We define two mappings $\alpha,\beta : \mathcal{S} \times \Im \to \mathbb{R}^+$ as

$$\alpha(\sigma,\varsigma) = \begin{cases} \frac{3}{2}, & \text{if } \sigma,\varsigma \in [-1,0], \\ 0, & \text{if } \sigma,\varsigma \in (0,1] \end{cases} \text{ and } \beta(\sigma,\varsigma) = \begin{cases} \frac{5}{3}, & \text{if } \sigma,\varsigma \in [-1,0], \\ 0, & \text{if } \sigma,\varsigma \in (0,1]. \end{cases}$$

Also, let $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\eta(t) = t$, for all $t \ge 0$. Let $\sigma \in \mathcal{S}$ and $\varsigma \in \Im$, such that $\alpha(\sigma,\varsigma) \ge 1$ and $\beta(\sigma,\varsigma) \ge 1$. Since $\sigma,\varsigma \in [-1,0]$ and so $\Omega\sigma \in [-1,0]$, $\Omega\varsigma \in [-1,0]$ and $\alpha(\Omega\sigma,\Omega\varsigma) \ge 1$ and $\beta(\Omega\sigma,\Omega\varsigma) \ge 1$.

Thus, Ω is (α,β)-admissible. Condition (ii) is satisfied with $\sigma_0 = 0$ and $\varsigma_0 = -1$ and eq. (3.2) is also satisfied in Theorem 3.1 with $(\sigma_a,\varsigma_a) = \left(-\frac{1}{n},\frac{1}{n}\right)$, for all $n \in \mathbb{N}$. If $\sigma,\varsigma \in [-1,0]$ then $\alpha(\sigma,\varsigma) = \frac{3}{2}$ and $\beta(\sigma,\varsigma) = \frac{5}{3}$. Now

$$\begin{aligned}\eta(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c)),\eta(\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c)) &= \alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c),\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c)\\ &= \left(\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)c\left|\frac{\mathfrak{x}}{6}-\frac{\mathfrak{z}}{6}\right|,c|\mathfrak{x}-\mathfrak{z}|\right)\end{aligned}$$

implies that

$$\begin{aligned}\mathfrak{F}(\eta(\alpha(\mathfrak{x},\mathfrak{z})\beta(\mathfrak{x},\mathfrak{z})\mathfrak{d}_c(\Omega\mathfrak{x},\Omega\mathfrak{z},c)),\eta(\mathfrak{d}_c(\mathfrak{x},\mathfrak{z},c))) &= \mathfrak{F}\left(\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)c\left|\frac{\mathfrak{x}}{6}-\frac{\mathfrak{z}}{6}\right|,c|\mathfrak{x}-\mathfrak{z}|\right)\\ &= \kappa c|\mathfrak{x}-\mathfrak{z}| - \left(\frac{3}{2}\right)\left(\frac{5}{3}\right)c\left|\frac{\mathfrak{x}}{6}-\frac{\mathfrak{z}}{6}\right|\\ &= \kappa c|\mathfrak{x}-\mathfrak{z}| - \frac{5}{12}c|\mathfrak{x}-\mathfrak{z}|\\ &= \left(\frac{1}{2}-\frac{5}{12}\right)c|\mathfrak{x}-\mathfrak{z}|\\ &= \frac{1}{6}c|\mathfrak{x}-\mathfrak{z}| \geq 0.\end{aligned}$$

As a result, Theorem 3.1's requirements are all met. Given that $\beta(\sigma,\varsigma) = \alpha(\sigma,\varsigma) = 0$. Due to the satisfaction of all the conditions in Corollary 3.1, Ω has a UFP with $\sigma = 0$.

4. Applications

4.1 Application to the Existence of Solutions of Integral Equations

As an application of Theorem 3.1, we examine the existence and unique solution of integral equations in this section.

Theorem 4.1. *Let us consider the integral equation*

$$\sigma(\vartheta) = \mathfrak{b}(\vartheta) + \int_{\Upsilon_1\cup\Upsilon_2} \mathcal{G}(\vartheta,\eta,\sigma(\eta))d\eta, \quad \vartheta \in \Upsilon_1\cup\Upsilon_2,$$

where $\Upsilon_1\cup\Upsilon_2$ is a Lebesgue measurable set. Suppose

(T1) $\mathcal{G} : (\Upsilon_1^2\cup\Upsilon_2^2)\times[0,\infty) \to [0,\infty)$ *and* $\mathfrak{b} \in L^\infty(\Upsilon_1)\cup L^\infty(\Upsilon_2)$,

(T2) *there is a continuous function* $\theta : \Upsilon_1^2\cup\Upsilon_2^2 \to [0,\infty)$ *and* $\Im = \frac{5}{6} \in (0,1)$ *s.t.*

$$|\mathcal{G}(\vartheta,\eta,\sigma(\eta)) - \mathcal{G}(\vartheta,\eta,\varsigma(\eta))| \leq \Im\theta(\vartheta,\eta)(|\sigma(\eta)-\varsigma(\eta)|,$$

for $(\vartheta,\eta) \in \Upsilon_1^2\cup\Upsilon_2^2$;

(T3) *let* $\phi,\varphi : L^\infty(\Upsilon_1)^2\cup L^\infty(\Upsilon_2)^2 \to [0,\infty)$ *such that for each* $\eta \in \Upsilon_1\cup\Upsilon_2$ *and* $(\sigma,\varsigma) \in L^\infty(\Upsilon_1)^2\cup L^\infty(\Upsilon_2)^2$, $\phi(\sigma(\eta),\varsigma(\eta)) > 0 \Rightarrow \phi(\Omega\sigma(\eta),\Omega\varsigma(\eta)) > 0$ *and* $\varphi(\sigma(\eta),\varsigma(\eta)) > 0 \Rightarrow \varphi(\Omega\sigma(\eta),\Omega\varsigma(\eta)) > 0$;

(T4) *for each* $\eta \in \Upsilon_1\cup\Upsilon_2$ *and* $(\{\sigma_a\},\{\varsigma_a\}) \subseteq (L^\infty(\Upsilon_1),L^\infty(\Upsilon_2))$ *be a bisequence such that* $\sigma_a \to \sigma$ *and* $\varsigma_a \to \varsigma$ *in* $\varsigma \in L^\infty(\Upsilon_1),\sigma \in L^\infty(\Upsilon_2)$ *and* $\phi(\sigma_a(\eta),\varsigma_{a+1}(\eta)) > 0$, $\varphi(\sigma_a(\eta),\varsigma_{a+1}(\eta)) > 0$ *for all* $a \in \mathbb{N}$ *then* $\phi(\sigma_a(\eta),\sigma(\eta)) > 0$, $\varphi(\sigma_a(\eta),\sigma(\eta)) > 0$ *and* $\phi(\varsigma(\eta),\varsigma_a(\eta)) > 0$, $\varphi(\varsigma(\eta),\varsigma_a(\eta)) > 0$, $a \in \mathbb{N}$;

(T5) $\| \int_{\Upsilon_1\cup\Upsilon_2} \theta(\vartheta,\eta)d\eta\| \leq 1$, *i.e.*, $\sup_{\vartheta\in\Upsilon_1\cup\Upsilon_2} \int_{\Upsilon_1\cup\Upsilon_2} \phi(\vartheta,\eta)\psi(\vartheta,\eta)d\eta$.

Then, the integral equation has a unique solution in $L^\infty(\Upsilon_1)\cup L^\infty(\Upsilon_2)$.

Proof. Suppose $\mathcal{S} = L^{\infty}(\Upsilon_1)$ and $\Im = L^{\infty}(\Upsilon_2)$ be two normed linear spaces, where Υ_1,Υ_2 are Lebesgue measurable sets and $m(\Upsilon_1 \cup \Upsilon_2) < \infty$.
Consider $\mathfrak{d}_c : \mathcal{S} \times \Im \to \mathbb{R}^+$ to be defined by $\mathfrak{d}_c(\sigma,\eta) = c\|\sigma-\eta\|_{\infty}$ to each $(\sigma,\eta) \in \mathcal{S}\times\Im$ and $c > 0$. Then $(\mathcal{S},\Im,\mathfrak{d}_c)$ is a complete BPPMS. Define the covariant mapping $\Omega : L^{\infty}(\Upsilon_1)\cup L^{\infty}(\Upsilon_2) \to L^{\infty}(\Upsilon_1)\cup L^{\infty}(\Upsilon_2)$ by

$$\Omega(\sigma(\vartheta)) = \mathfrak{b}(\vartheta) + \int_{\Upsilon_1\cup\Upsilon_2} \mathcal{G}(\vartheta,\eta,\sigma(\eta))d\eta, \quad \vartheta \in \Upsilon_1\cup\Upsilon_2.$$

Now, for all $(\sigma,\varsigma) \in L^{\infty}(\Upsilon_1)^2 \cup L^{\infty}(\Upsilon_2)^2$, $\phi(\sigma(\eta),\varsigma(\eta)) > 0$ and $\varphi(\sigma(\eta),\varsigma(\eta)) > 0$ $\forall$ $\eta \in \Upsilon_1\cup\Upsilon_2$, we have

$$\begin{aligned}
\mathfrak{d}_c(\Omega\sigma(\vartheta),\Omega\varsigma(\vartheta),c) &= c\|\Omega\sigma(\vartheta)-\Omega\varsigma(\vartheta)\| \\
&= c\left|\mathfrak{b}(\vartheta) + \int_{\Upsilon_1\cup\Upsilon_2} \mathcal{G}(\vartheta,\eta,\sigma(\eta))d\eta - \left(\mathfrak{b}(\vartheta) + \int_{\Upsilon_1\cup\Upsilon_2} \mathcal{G}(\vartheta,\eta,\varsigma(\eta))d\eta\right)\right| \\
&\le c\int_{\Upsilon_1\cup\Upsilon_2} |\mathcal{G}(\vartheta,\eta,\sigma(\eta)) - \mathcal{G}(\vartheta,\eta,\varsigma(\eta))|d\eta \\
&\le c\int_{\Upsilon_1\cup\Upsilon_2} \Im\theta(\vartheta,\eta)(|\sigma(\eta)-\varsigma(\eta)|)d\eta \\
&\le c\Im(\|\sigma(\eta)-\varsigma(\eta)\|_{\infty})\int_{\Upsilon_1\cup\Upsilon_2} \theta(\vartheta,\eta)d\eta \\
&\le c\Im(\|\sigma(\eta)-\varsigma(\eta)\|_{\infty}) \sup_{\vartheta\in\Upsilon_1\cup\Upsilon_2}\int_{\Upsilon_1\cup\Upsilon_2} \theta(\vartheta,\eta)d\eta \\
&\le \Im(c\|\sigma(\eta)-\varsigma(\eta)\|_{\infty}) \\
&\le \Im\mathfrak{d}_c(\sigma,\varsigma,c). \qquad (4.1)
\end{aligned}$$

Let $\mathfrak{F} : [0,\infty)\times[0,\infty) \to \mathbb{R}$ as $\mathfrak{F}(\sigma,\varsigma) = \frac{5}{6}\varsigma - \sigma$ for all $\sigma,\varsigma \in [0,\infty)$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\psi(t) = t$ to each $t \ge 0$. For $\eta \in \Upsilon_1\cup\Upsilon_2$ the following is defined: $\alpha,\beta : \mathcal{S}\times\Im \to \mathbb{R}^+$ as $\alpha(\sigma,\varsigma) = \begin{cases} 1, & \text{if } \phi(\sigma(\eta),\varsigma(\eta)) > 0, \\ 0, & \text{otherwise,} \end{cases}$ and $\beta(\sigma,\varsigma) = \begin{cases} 1, & \text{if } \varphi(\sigma(\eta),\varsigma(\eta)) > 0, \\ 0, & \text{otherwise.} \end{cases}$
Now, using (4.1), we get

$$\frac{5}{6}\psi(\mathfrak{d}_c(\sigma,\varsigma,c)) - \psi\left(\alpha(\sigma,\varsigma)\beta(\sigma,\varsigma)\mathfrak{d}_c(\Omega\sigma(\vartheta),\Omega\varsigma(\vartheta),c)\right) = \frac{5}{6}\mathfrak{d}_c(\sigma,\varsigma,c) - \mathfrak{d}_c(\Omega\sigma(\vartheta),\Omega\varsigma(\vartheta),c) \ge 0.$$

Hence

$$\mathfrak{F}\left(\psi\left(\alpha(\sigma,\varsigma)\beta(\sigma,\varsigma)\mathfrak{d}_c(\Omega\sigma(\vartheta),\Omega\varsigma(\vartheta),c)\right), \psi(\mathfrak{d}_c(\sigma,\varsigma,c))\right) \ge 0.$$

Therefore, the mapping Ω is (α,β)-$\mathcal{Z}$-type contraction. Now using (T_3), we get

$$\begin{aligned}
\alpha(\sigma,\varsigma) \ge 1 &\Rightarrow \phi(\sigma(\eta),\varsigma(\eta)) > 0 \\
&\Rightarrow \phi(\Omega\sigma(\eta),\Omega\varsigma(\eta)) > 0 \\
&\Rightarrow \alpha(\Omega\sigma,\Omega\varsigma) \ge 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta(\sigma,\varsigma) \ge 1 &\Rightarrow \varphi(\sigma(\eta),\varsigma(\eta)) > 0 \\
&\Rightarrow \varphi(\Omega\sigma(\eta),\Omega\varsigma(\eta)) > 0 \\
&\Rightarrow \beta(\Omega\sigma,\Omega\varsigma) \ge 1.
\end{aligned}$$

Consequently, (α,β)-admissible mapping Ω exists. As Theorem 3.1's entire hypothesis is confirmed, and as a result, there is only one solution to the integral equation. □

4.2 Application to the Existence of Solutions of Homotopy

In this section, we investigate the possibility of a single solution to homotopy theory.

Theorem 4.2. *Suppose $(\mathcal{S},\mathfrak{I},d_c)$ be a complete BPPMS, $(\mathfrak{E},\mathfrak{G})$ and $(\bar{\mathfrak{E}},\bar{\mathfrak{G}})$ be an open and closed subset of $(\mathcal{S},\mathfrak{I})$ such that $(\mathfrak{E},\mathfrak{G})\subseteq(\bar{\mathfrak{E}},\bar{\mathfrak{G}})$. Suppose $\mathfrak{H}_c:(\bar{\mathfrak{E}}\cup\bar{\mathfrak{G}})\times[0,1]\to\mathcal{S}\cup\mathfrak{I}$ be an operator with following conditions are satisfying,*

(i) $\wp\neq\mathfrak{H}_c(\wp,s)$ *for each* $\wp\in\partial\mathfrak{E}\cup\partial\mathfrak{G}$ *and* $s\in[0,1]$ (*here* $\partial\mathfrak{E}\cup\partial\mathfrak{G}$ *is boundary of* $\mathfrak{E}\cup\mathfrak{G}$ *in* $\mathcal{S}\cup\mathfrak{I}$);

(ii) *for all* $\wp\in\bar{\mathfrak{E}}$, $\imath\in\bar{\mathfrak{G}}$, $s\in[0,1]$ *and* $\eta:\mathbb{R}^+\to\mathbb{R}^+$ *with* $\eta(a+b)\leq\eta(a)+\eta(b)$, $\eta(p)\leq p$ *and* $\eta(0)=0$ *also* $c>0$, $\kappa\in[0,1)$ *such that*

$$\eta(d_c(\mathfrak{H}_c(\wp,s),\mathfrak{H}_c(\imath,s),c))\leq\kappa\eta(d_c(\wp,\imath,c)).$$

(iii) $\exists\, M\geq0\ni d_c(\mathfrak{H}_c(\wp,s),\mathfrak{H}_c(\imath,t),c)\leq Mc|s-t|$, *for every* $\wp\in\bar{\mathfrak{E}}$, $\imath\in\bar{\mathfrak{G}}$ *and* $s,t\in[0,1]$.

Then $\mathfrak{H}_c(\cdot,0)$ *has a fixed point* $\iff$ $\mathfrak{H}_c(\cdot,1)$ *has a fixed point.*

Proof. Suppose the sets

$$\Theta=\{s\in[0,1]:\mathfrak{H}_c(\wp,s)=\wp,\ \text{for some } \wp\in\mathcal{P}\},$$

$$\Upsilon=\{t\in[0,1]:\mathfrak{H}_c(\jmath,t)=\jmath,\ \text{for some } \jmath\in\mathfrak{G}\}.$$

Assuming that there is a FP for $\mathfrak{H}_c(\cdot,0)$ in $\mathfrak{E}\cup\mathfrak{G}$, we obtain that $0\in\Theta\cap\Upsilon$. In light of this, $\Theta\cap\Upsilon\neq\phi$.

We now demonstrate that, given the connectedness $\Theta=\Upsilon=[0,1]$, $\Theta\cap\Upsilon$ is both closed and open in $[0,1]$. Consequently, there is a fixed point for $\mathfrak{H}_c(\cdot,0)$ in $\Theta\cap\Upsilon$. We first demonstrate the closure of $\Theta\cap\Upsilon$ in $[0,1]$. To see this, let $(\{\sigma_a\}_{a=1}^{\infty},\{\varsigma_a\}_{a=1}^{\infty})\subseteq(\Theta,\Upsilon)$ with $(\sigma_a,\varsigma_a)\to(\alpha,\alpha)\in[0,1]$ as $a\to\infty$. We have to show that $\alpha\in\Theta\cap\Upsilon$.

Since $(\sigma_a,\varsigma_a)\in(\Theta,\Upsilon)$, for $a=0,1,2,3,\cdots$, there exists sequences $(\{\wp_a\},\{\jmath_a\})$ with $\wp_{a+1}=\mathfrak{H}_c(\wp_a,\sigma_a)$, $\jmath_{a+1}=\mathfrak{H}_c(\jmath_a,\varsigma_a)$.

Consider

$$\begin{aligned}\eta(d_c(\wp_a,\jmath_{a+1},c))&=\eta(d_c(\mathfrak{H}_c(\wp_{a-1},\sigma_{a-1}),\mathfrak{H}_c(\jmath_a,\varsigma_a),c))\\&\leq\eta(d_c(\mathfrak{H}_c(\wp_{a-1},\sigma_{a-1}),\mathfrak{H}_c(\jmath_a,\sigma_{a-1}),c))+\eta(d_c(\mathfrak{H}_c(\wp_a,\sigma_a),\mathfrak{H}_c(\jmath_a,\sigma_{a-1}),c))\\&\quad+\eta(d_c(\mathfrak{H}_c(\wp_a,\sigma_a),\mathfrak{H}_c(\jmath_a,\varsigma_a),c))\\&\leq\kappa\eta(d_c(\wp_{a-1},\jmath_a,c))+Mc|\sigma_a-\sigma_{a-1}|+Mc|\sigma_a-\varsigma_a|.\end{aligned}$$

Letting $a\to\infty$ and using property of η, we have

$$\begin{aligned}\lim_{a\to\infty}d_c(\wp_a,\jmath_{a+1},c)&\leq\lim_{a\to\infty}\kappa d_c(\wp_{a-1},\jmath_a,c)\\&\leq\lim_{a\to\infty}\kappa^2d_c(\wp_{a-2},\jmath_{a-1},c)\\&\ \ \vdots\\&\leq\lim_{a\to\infty}\kappa^a d_c(\wp_0,\jmath_1,c)=0.\end{aligned}$$

Similar lines, we can prove that

$$\begin{aligned}\lim_{a\to\infty} d_c(\wp_{a+1}, \jmath_a, c) &\le \lim_{a\to\infty} \kappa d_c(\wp_a, \jmath_{a-1}, c)\\ &\le \lim_{a\to\infty} \kappa^2 d_c(\wp_{a-1}, \jmath_{a-2}, c)\\ &\vdots\\ &\le \lim_{a\to\infty} \kappa^a d_c(\wp_1, \jmath_0, c) = 0.\end{aligned}$$

Moreover,

$$\begin{aligned}\lim_{a\to\infty} d_c(\wp_a, \jmath_p, c) &\le \lim_{a\to\infty} \kappa d_c(\wp_{a-1}, \jmath_{a-1}, c)\\ &\le \lim_{a\to\infty} \kappa^2 d_c(\wp_{a-2}, \jmath_{a-2}, c)\\ &\vdots\\ &\le \lim_{a\to\infty} \kappa^a d_c(\wp_0, \jmath_0, c) = 0.\end{aligned}$$

We now prove $(\{\wp_a\},\{\jmath_a\})$ is a Cauchy bisequence. Suppose there are $\epsilon > 0$ and $\{b_k\},\{a_k\}$ so that for $a_k > b_k > k$,

$$d_c(\wp_{a_k}, \jmath_{b_k}, c) \ge \epsilon, \quad d_c(\wp_{a_{k-1}}, \jmath_{b_k}, c) < \epsilon \tag{4.2}$$

and

$$d_c(\wp_{b_k}, \jmath_{a_k}, c) \ge \epsilon, \quad d_c(\wp_{b_k}, \jmath_{a_{k-1}}, c) < \epsilon. \tag{4.3}$$

With respect to (4.2) and triangle inequality, we obtain

$$\begin{aligned}\epsilon &\le d_c(\wp_{a_k}, \jmath_{b_k}, c)\\ &\le d_c(\wp_{a_k}, \jmath_{a_{k-1}}, c) + d_c(\wp_{a_{k-1}}, \jmath_{a_{k-1}}, c) + d_c(\wp_{a_{k-1}}, \jmath_{b_k}, c)\\ &< d_c(\wp_{a_k}, \jmath_{a_{k-1}}, c) + d_c(\mathfrak{H}_c(\wp_{a-2}, \sigma_{a-2}), \mathfrak{H}_c(\jmath_{a-2}, \varsigma_{a-2}), c) + \epsilon\\ &< d_c(\wp_{a_k}, \jmath_{a_{k-1}}, c) + Mc|\sigma_{a-2} - \varsigma_{a-2}| + \epsilon.\end{aligned}$$

Letting $a \to \infty$ in the above inequality, we get

$$\lim_{a\to\infty} d_c(\wp_{a_k}, \jmath_{b_k}, c) = \epsilon. \tag{4.4}$$

Using (4.3), one can prove

$$\lim_{a\to\infty} d_c(\wp_{b_k}, \jmath_{p_k}, c) = \epsilon. \tag{4.5}$$

For all $k \in \mathbb{N}$, by (ii) we have

$$\eta(d_c(\wp_{a_{k+1}}, \jmath_{b_{k+1}}, c)) \le \kappa\eta(d_c(\wp_{a_k}, \jmath_{b_k}, c))$$

and

$$\eta(d_c(\wp_{b_{k+1}}, \jmath_{a_{k+1}}), c) \le \kappa\eta(d_c(\wp_{b_k}, \jmath_{p_k}, c)).$$

We get at the limit, $\eta(\epsilon) \le \kappa\eta(\epsilon)$, by applying (4.4) and (4.5) which is contradictory. Therefore, $(\{\wp_a\},\{\jmath_a\})$ is Cauchy bi-sequence in $(\mathfrak{E},\mathfrak{G})$. To be thorough, $\tau \in \mathfrak{E} \cap \mathfrak{G}$ exists with

$$\lim_{a\to\infty} \wp_{a+1} = \tau = \lim_{a\to\infty} \jmath_{a+1}. \tag{4.6}$$

We have

$$\begin{aligned}\eta(d_c(\mathfrak{H}_c(\tau,\alpha), \jmath_{a+1}), c) &= \eta(d_c(\mathfrak{H}_c(\tau,\alpha), \mathfrak{H}_c(\jmath_a, \varsigma_a), c))\\ &\le \kappa\eta(d_c(\tau, \jmath_a, c)).\end{aligned}$$

By taking the limsup on both sides and property of η, we have $d_c(\mathfrak{H}_c(\tau,\alpha),\tau,c)=0$ implies that $\mathfrak{H}_c(\tau,\alpha)=\tau$. Therefore, $\alpha\in\Theta\cap\Upsilon$. Clearly, $\Theta\cap\Upsilon$ closed in $[0,1]$. Let $(\sigma_0,\varsigma_0)\in\Theta\times\Upsilon$, there exists bi-sequences $(\wp_0,\jmath_0)$ with $\wp_0=\mathfrak{H}_c(\wp_0,\sigma_0)$, $\jmath_0=\mathfrak{H}_c(\jmath_0,\varsigma_0)$. Since $\mathfrak{E}\cup\mathfrak{G}$ is open, then there exist $\delta>0$ such that $B_{d_c}(\wp_0,\delta)\subseteq\mathfrak{E}\cup\mathfrak{G}$ and $B_{d_c}(\delta,\jmath_0)\subseteq\mathfrak{E}\cup\mathfrak{G}$. Choose $\sigma\in(\sigma_0-\epsilon,\sigma_0+\epsilon)$, $\varsigma\in(\varsigma_0-\epsilon,\varsigma_0+\epsilon)$ such that $|\sigma-\varsigma_0|\le\frac{1}{M^a}<\frac{\epsilon}{2}$, $|\varsigma-\sigma_0|\le\frac{1}{M^a}<\frac{\epsilon}{2}$, $|\sigma_0-\sigma|\le\frac{1}{M^a}<\frac{\epsilon}{2}$ and $|\sigma_0-\varsigma_0|\le\frac{1}{M^a}<\frac{\epsilon}{2}$.
Then for, $\jmath\in\bar{B}_{\mathfrak{E}\cup\mathfrak{G}}(\wp_0,\delta)=\{\jmath,\jmath_0,\in\mathfrak{G},c>0/d_c(\wp_0,\jmath,c)\le d_c(\wp_0,\jmath_0,c)+\delta\}$,
$\wp\in\bar{B}_{\mathfrak{E}\cup\mathfrak{G}}(\delta,\jmath_0)=\{\wp,\wp_0\in\mathfrak{E},c>0/d_c(\wp,\jmath_0,c)\le d_c(\wp_0,\jmath_0,c)+\delta\}$,

$$\begin{aligned}d_c(\mathfrak{H}_c(\wp,\sigma),\jmath_0,c)&=d_c(\mathfrak{H}_c(\wp,\sigma),\mathfrak{H}_c(\jmath_0,\varsigma_0),c)\\&\le d_c(\mathfrak{H}_c(\wp,\sigma),\mathfrak{H}_c(\jmath_0,\sigma),c)+d_c(\mathfrak{H}_c(\wp_0,\sigma_0),\mathfrak{H}_c(\jmath_0,\sigma),c)\\&\quad+d_c(\mathfrak{H}_c(\wp_0,\sigma_0),\mathfrak{H}_c(\jmath_0,\varsigma_0),c)\\&\le d_c(\mathfrak{H}_c(\wp,\sigma),\mathfrak{H}_c(\jmath_0,\sigma),c)+Mc|\sigma_0-\sigma|++Mc|\sigma_0-\varsigma_0|\\&\le d_c(\mathfrak{H}_c(\wp,\sigma),\mathfrak{H}_c(\jmath_0,\sigma),c)+\frac{2c}{M^{a-1}}.\end{aligned}$$

Letting $a\to\infty$ and using η property, then we have

$$\begin{aligned}\eta(d_c(\mathfrak{H}_c(\wp,\sigma),\jmath_0,c))&\le\eta(d_c(\mathfrak{H}_c(\wp,\sigma),\mathfrak{H}_c(\jmath_0,\sigma),c))\\&\le\kappa\eta(d_c(\wp_0,\jmath,c))\\&<\eta(d_c(\wp_0,\jmath,c))\end{aligned}$$

implies that

$$\begin{aligned}d_c(\mathfrak{H}_c(\wp,\sigma),\jmath_0,c)&<d_c(\wp_0,\jmath,c)\\&\le d_c(\wp_0,\jmath_0,c)+\delta.\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}d_c(\wp_0,\mathfrak{H}_c(\jmath,\varsigma),c))&<d_c(\wp,\jmath_0,c)\\&\le d_c(\wp_0,\jmath_0,c)+\delta\end{aligned}$$

On the other hand,

$$d_c(\wp_0,\jmath_0,c)=d_c(\mathfrak{H}_c(\wp_0,\sigma_0),\mathfrak{H}_c(\jmath_0,\varsigma_0,),c)\le Mc|\sigma_0-\varsigma_0|<\frac{c}{M^{a-1}}\to 0 \text{ as } a\to\infty.$$

So $\wp_0=\jmath_0$ and hence $\sigma=\varsigma$. Thus, for each fixed $\sigma\in(\sigma_0-\epsilon,\sigma_0+\epsilon)$, $\mathfrak{H}_c(\cdot,\sigma):\bar{B}_{\Theta\cup\Upsilon}(\wp_0,\delta)\to\bar{B}_{\Theta\cup\Upsilon}(\wp_0,\delta)$. Consequently, we deduce that there is a fixed point for $\mathfrak{H}_c(\cdot,\sigma)$ in $\bar{\mathfrak{E}}\cap\bar{\mathfrak{G}}$. But which has to be in $\mathfrak{E}\cup\mathfrak{G}$. Consequently, for $\sigma\in(\sigma_0-\epsilon,\sigma_0+\epsilon)$, $\sigma\in\Theta\cap\Upsilon$. For this reason, $(\sigma_0-\epsilon,\sigma_0+\epsilon)\subseteq\Theta\cap\Upsilon$. It is obvious that in $[0,1]$, $\Theta\cap\Upsilon$ is open. A similar procedure can be used to demonstrate the opposite. □

5. Conclusion

This paper uses contractive mappings of the (α,β)-$\mathcal{Z}$-type via simulation functions to demonstrate certain FPT in the context of complete BPPMS, along with appropriate examples that highlight the main findings. Applications for integral equations and homotopy are also given.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Chandok, Some fixed points theorems for (α,β)-admissible Geraghty type contractive mappings and related results, *Mathematical Sciences* **9**(3) (2015), 127 – 135, DOI: 10.1007/s40096-015-0159-4.

[2] S.-H. Cho, Fixed point theorem for (α,β)-Z-contractions in metric spaces, *International Journal of Mathematical Analysis* **13**(4) (2019), 161 – 174, DOI: 10.12988/ijma.2019.9318.

[3] A. Dewangan, A. K. Dubey, M. D. Pandey and R. P. Dubey, Fixed points for (α,β)-admissible mapping via simulation functions, *Communications in Mathematics and Applications* **12**(4) (2021), 1101 – 1111, DOI: 10.26713/cma.v12i4.1615.

[4] U. Gürdal, A. Mutlu and K. Özkan, Fixed point results for $\alpha-\mathcal{S}$ contractive mappings in bipolar metric spaces, *Journal of Inequalities, Special Functions* **11**(1) (2020), 64 – 75.

[5] Y. U. Gaba, M. Aphane and H. Aydi, (α,BK)-Contractions in bipolar metric spaces, *Journal of Mathematics* **2021** (2021), Article ID 5562651, DOI: 10.1155/2021/5562651.

[6] N. Hussain, P. Salimi and V. Parvaneh, Fixed point results for various contractions in parametric and fuzzy b-metric spaces, *Journal of Nonlinear Sciences and Applications* **8**(5) (2015), 719 – 739, DOI: 10.22436/jnsa.008.05.24.

[7] N. Hussain, S. Khaleghizadeh, P. Salimi and A. A. N. Abdou, A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces, *Abstract and Applied Analysis* **2014** (2014), Article ID 690139, DOI: 10.1155/2014/690139.

[8] F. Khojasteh, S. Shukla and S. Radenović, A new approach to the study of fixed point theory for simulation functions, *Filomat* **29**(6) (2015), 1189 – 1194, DOI: 10.2298/FIL1506189K.

[9] G. N. V. Kishore, D. R. Prasad, B. S. Rao and V. S. Baghavan, Some applications via common coupled fixed point theorems in bipolar metric spaces, *Journal of Critical Reviews* **7**(2) (2019), 601 – 607.

[10] G. N. V. Kishore, K. P. R. Rao, A. Sombabu and R. V. N. S. Rao, Related results to hybrid pair of mappings and applications in bipolar metric spaces, *Journal of Mathematics* **2019** (2019), Article ID 8485412, DOI: 10.1155/2019/8485412.

[11] G. N. V. Kishore, K. P. R. Rao, H. IsIk, B. S. Rao and A. Sombabu, Covariant mappings and coupled fixed point results in bipolar metric spaces, *International Journal of Nonlinear Analysis and Applications* **12**(1) (2021), 1 – 15, DOI: 10.22075/ijnaa.2021.4650.

[12] G. N. V. Kishore, R. P. Agarwal, B. S. Rao and R. V. N. S. Rao, Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications, *Fixed Point Theory and Applications* **21**(1) (2018), Article number: 21, DOI: 10.1186/s13663-018-0646-z.

[13] M. Kumar, O. Ege, V. Mor, P. Kumar and M. De la Sen, Boyd-Wong type contractions in generalized parametric bipolar metric space, *Heliyon* **10**(2) (2024), e23998, DOI: 10.1016/j.heliyon.2024.e23998.

[14] A. Mutlu and U. Gürdal, Bipolar metric spaces and some fixed point theorems, *Journal of Nonlinear Sciences and Applications* **9**(9) (2016), 5362 – 5373, DOI: 10.22436/jnsa.009.09.05.

[15] A. Mutlu, K. Özkan and U. Gürdal, Coupled fixed point theorems on bipolar metric spaces, *European Journal of Pure and Applied Mathematics* **10**(4) (2017), 655 – 667, https://www.ejpam.com/index.php/ejpam/article/view/3019.

[16] A. Mutlu, K. Özkan and U. Gürdal, Locally and weakly contractive principle in bipolar metric spaces, *TWMS Journal of Applied and Engineering Mathematics* **10**(2) (2020), 379 – 388, URL: https://jaem.isikun.edu.tr/web/images/articles/vol.10.no.2/11.pdf.

[17] P. Naresh, G. U. Reddy and B. S. Rao, Some quadruple fixed points of integral type contraction mappings in bipolar metric spaces with applications, *Indian Journal of Science and Technology* **16**(25) (2023), 1843 – 1856, DOI: 10.17485/IJST/v16i25.2058.

[18] M. I. Pasha, K. R. K. Rao, G. Mani, A. J. Gnanaprakasam and S. Kumar, Solving a fractional differential equation via the bipolar parametric metric space, *Journal of Mathematics* **2024** (2024), Article ID 5533347, DOI: 10.1155/2024/5533347.

[19] B. S. Rao, G. N. V. Kishore and G. K. Kumar, Geraghty type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy, *International Journal of Mathematics Trends and Technology* **63**(1) (2018), 25 – 34.

[20] G. U. Reddy, C. Ushabhavani and B. S. Rao, Existence fixed point solutions for C-class functions in bipolar metric spaces with applications, *International Journal of Analysis and Applications* **21** (2023), Article ID 7, DOI: 10.28924/2291-8639-21-2023-7.