Communications in Mathematics and Applications
Vol. 16, No. 2, pp. 405–418, 2025
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v16i2.3006

http://www.rgnpublications.com

Research Article

Rough Δ^m-Statistical Convergence of order α of Generalized Difference Sequences in Intuitionistic Fuzzy Normed Spaces

Gursimran Kaur*[1] and Meenakshi Chawla[2]

[1] *Department of Mathematics, Chandigarh University, Gharuan 140413, Punjab, India*
[2] *Department of Applied Sciences, Chandigarh Engineering College, Chandigarh Group of Colleges, Jhanjeri 140307, Punjab, India*
***Corresponding author:** gsimran.kaur94@gmail.com

Received: January 22, 2025 **Revised:** August 14, 2025 **Accepted:** October 20, 2025

Abstract. The prime direction of this research article is to explicate the perception of rough Δ^m-statistical convergence of order α of generalized difference sequences in intuitionistic fuzzy normed spaces. We showed certain rough convergence features, which give some new functional tools in the face of uncertainty, such as intuitionistic fuzzy normed spaces. We demonstrated some basic properties and examples which generates the results that this perception is more generic. Further, we added the set of Δ^m-statistical limit points and set of Δ^m-statistical cluster points and their relationship of rough Δ^m-statistically convergence of generalized difference sequences in these spaces.

Keywords. Intuitionistic normed spaces, Statistical convergence, Rough statistical convergence, Difference sequence, Generalized difference sequence

Mathematics Subject Classification (2020). 46S40, 11B39, 03E72, 40G15

Copyright © 2025 Gursimran Kaur and Meenakshi Chawla. *This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.*

1. Introduction

The most interesting generalization of the concept of classical convergence of sequences was coined by Zygmund [32] and stated as statistical convergence in 1930. Steinhaus [30] and Fast [8] also presented the notion of statistical convergence simultaneously in the same year 1951. A sequence $x = \{x_k\}$ converges to L statistically, if the natural density of the set K is zero, where $K = \{k \le n : |x_k - l| \ge \epsilon\}$ to each $\epsilon > 0$. The natural density (symptomatic as $\delta(K)$) of the set K, which is a subset of $\mathbb{N}$, is defined as: $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \le n\}|$, where the bars on the vertical denotes the order of enclosed set. If $\{x_{k_i}\}$ is a sub-sequence of the sequence $x = \{x_k\}$ of $\mathbb{R}$ and $A = \{k_i : i \in \mathbb{N}\}$, then we denote it by $\{x_A\}$. In the event, $\delta(A) = 0$, $\{x_A\}$ is claimed as a sub-sequence of natural density zero or a thin sub-sequence of the sequence x. However, if $\{x_A\}$ does not occupy natural density zero or fails to have natural density then it is claimed as non-thin sub-sequence of x. The notion was introduced to deal with the theory of series summation and has been studied by various researchers in different spaces such as intuitionistic fuzzy normed spaces (Mohiuddine and Lohani [15]), random 2-normed spaces (Mursaleen [17]), probabilistic normed spaces (Karakus and Demırcı [10]). It has also been studied for different sequences such as ordinary sequences (Šalát [27]), double sequences (Mursaleen and Edely [18]), triple sequences (Şahiner *et al*. [26]) and multiple sequences (Móricz [16]) (also see, Chawla *et al*. [3], and Reena *et al*. [24]).

Zadeh [31] introduced the theory of fuzzy sets in 1965 for the first time. Later, it was studied and identified to be applicable in several branches of science by many researchers. The study of fuzzy sets is applicable in computer programming (Giles [9]), operation research (Prade [22]), decision-making (Lootsma [13]), statistics (Nguyen and [19]), engineering (Ross [25]). Atanassov [1] founded the proposition of intutionistic fuzzy sets. The perception of intutionistic fuzzy metric space was proposed by Park [20] in 2004. Saadadti and Park [28] the ordinary normed linear space established *Intuitionistic Fuzzy Normed Linear Space* (*IFNS*). *IFNS* plays important role in mathematical modeling in day to day life situations.

The perception of rough convergence which is an abstraction of usual convergence of sequences, was inaugurated by Phu [21] for normed linear spaces with the perceptions of rough limits, degree of roughness, rough continuity of linear operators and rough Cauchy sequences. The perception of rough convergence was broadened to rough statistical convergence by Aytar [2] and studied the characteristics of convexity, closeness of the set of rough statistical limit points and rough statistical cluster points of a sequence.

Following this line, Maity [14] extended the concept of rough statistical convergence to rough statistical convergence of order α ($\alpha \in (0,1]$) in normed linear spaces. Recently, Demir and Gümüş [5] defined the rough convergence for difference sequences in finite normed linear spaces. More generalizations and applications using the rough convergence, statistical convergence and generalized convergence in different aspects can be studied and explored.

In 1981, Kizmaz [12] introduced the difference sequence spaces $X(\Delta)$ for $X = l_\infty, c, c_0$, where $X(\Delta)$ is a Banach Space. Further, Et and Çolak [6] generalized the notion of difference sequences. In the present paper, we are introducing the notion rough Δ^m-statistical convergence of order α for the generalized difference sequences in the *Intuitionistic Fuzzy Normed Space* (*IFNS*).

Definition 1.1 ([29])**.** A binary operation $\star : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if it has the following axioms:

(i) $\star$ is associative, commutative and continuous,

(ii) $p \star 1 = p$, for all $p \in [0,1]$,

(iii) $p \star q \le r \star s$ if $p \le r$ and $q \le s$, for each $p,q,r,s \in [0,1]$.

Definition 1.2 ([29])**.** A binary operation $\cdot : [0,1]\times[0,1]$ is continuous t-conorm if it has the following axioms:

(i) $\cdot$ is associative, commutative and continuous,

(ii) $p\cdot 0 = p$, for all $p \in [0,1]$,

(iii) $p\cdot q \le r\star s$ if $p \le r$ and $q \le s$, for each $p,q,r,s \in [0,1]$.

Definition 1.3 ([28])**.** Consider $\mathcal{X}$ be a vector space, $\star$ be continuous t-norm, $\cdot$ be continuous t-conorm, η,ψ be the fuzzy sets and a five tuple $(\mathcal{X},\eta,\psi,\star,\cdot)$ is intuitionistic fuzzy norm space *IFNS* on $\mathcal{X}\times(0,\infty)$ if for every $x,y \in \mathcal{X}$ and $s,t>0$,

(i) $\eta(x,t)+\psi(y,t) \le 1$,

(ii) $\eta(x,t) > 0$,

(iii) $\eta(x,t) = 1 \iff x = 0$,

(iv) $\eta(cx,t) = \eta\left(x,\frac{t}{|c|}\right)$, for $c \ne 0$,

(v) $\eta(x,t)\star\psi(y,s) \le \eta(x+y,t+s)$,

(vi) $\eta(x,\cdot):(0,\infty)\to[0,1]$ is continuous,

(vii) $\lim_{t\to\infty}\eta(x,t) = 1$ and $\lim_{t\to 0}\eta(x,t) = 0$,

(viii) $\psi(x,t) < 1$,

(ix) $\psi(x,t) = 0 \iff x = 0$,

(x) $\psi(cx,t) = \psi\left(x,\frac{t}{|c|}\right)$, for $c \ne 0$,

(xi) $\psi(x,t)\cdot\psi(y,s) \ge \psi(x+y,t+s)$,

(xii) $\psi(x,\cdot):(0,\infty)\to[0,1]$ is continuous,

(xiii) $\lim_{t\to\infty}\psi(x,t) = 0$ and $\lim_{t\to 0}\psi(x,t) = 1$.

Then (η,ψ) is said to be intuitionistic fuzzy norm.

Definition 1.4 ([28])**.** Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS*. A sequence $x=\{x_k\}$ in $\mathcal{X}$ is convergent to $L\in\mathcal{X}$ with respect to the intuitionistic fuzzy norm (η,ψ) if for every $\epsilon>0$ and $t>0$, $\exists\ k_0 \in \mathbb{N}$ such that

$$\eta(x_k-L,t) > 1-\epsilon \ \text{ and } \ \psi(x_k-L,t) < \epsilon, \quad \text{for all } k \ge k_0.$$

Symptomatic as (η,ψ)-$\lim x = L$.

Definition 1.5 ([28])**.** Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS*. A sequence $x=\{x_k\}$ in $\mathcal{X}$ is Cauchy sequence with respect to the intuitionistic fuzzy norm (η,ψ) if for every $\epsilon>0$ and $t>0$, $\exists\ k_0\in\mathbb{N}$ such that

$$\eta(x_k-x_s,t) > 1-\epsilon \ \text{ and } \ \psi(x_k-x_s,t) < \epsilon, \quad \text{for all } r,s \ge k_0.$$

Definition 1.6 ([11])**.** Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS*. A sequence $x=\{x_k\}$ in $\mathcal{X}$ is statistical convergent to $L\in\mathcal{X}$ with respect to the intuitionistic fuzzy norm (η,ψ) if for every $\epsilon>0$ and $t>0$, $\exists\ k_0\in\mathbb{N}$ such that

$$\delta(\{k\le n : \eta(x_k-L,t) \le 1-\epsilon \ \text{ or } \ \psi(x_k-L,t) \ge \epsilon\}) = 0$$

or

$$\delta(\{k\le n : \eta(x_k-L,t) > 1-\epsilon \ \text{ and } \ \psi(x_k-L,t) < \epsilon\}) = 1.$$

Symptomatic as $St(\eta,\psi)$-$\lim x = L$.

Definition 1.7 ([23]). Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS* and $\alpha\in(0,1]$. A sequence $x=\{x_k\}$ in $\mathcal{X}$ is statistical convergent of order α to $L\in\mathcal{X}$ with respect to the intuitionistic fuzzy norm (η,ψ) if for every $\epsilon>0$ and $t>0$, $\exists\ k_0\in\mathbb{N}$ such that

$$\lim_{n\to\infty}\frac{1}{n^\alpha}(\{k\le n:\eta(x_k-L,t)\le 1-\epsilon \text{ or } \psi(x_k-L,t)\ge\epsilon\})=0$$

or

$$\lim_{n\to\infty}\frac{1}{n^\alpha}(\{k\le n:\eta(x_k-L,t)> 1-\epsilon \text{ or } \psi(x_k-L,t)<\epsilon\})=0.$$

Definition 1.8 ([6]). Consider m be a non-negative integer, then the generalized difference operator $\Delta^m x_k$ is determined as

$$\Delta^m x_k=\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1},$$

where $\Delta^0 x_k=x_k$, for all $k\in\mathbb{N}$.

Definition 1.9 ([7]). Consider m be a fixed positive integer. A sequence $x=\{x_k\}$ is named as Δ^m-statistically convergent to ξ if to each $\epsilon>0$, we have

$$\delta\left(\{k\le n:|\Delta^m x_k-\xi|\ge\epsilon\}\right)=\lim_{n\to\infty}\frac{1}{n}|\{k\le n:|\Delta^m x_k-\xi|\ge\epsilon\}|=0.$$

It is symptomatic as St-$\lim\Delta^m x_k=\xi$.

Definition 1.10 ([4]). Consider m as a fixed positive integer and $\alpha\in(0,1]$ be given. A sequence $x=\{x_k\}$ is named as Δ^m-statistically convergent of order α to ξ if to each $\epsilon>0$, we have

$$\delta_\alpha(\{k\le n:|\Delta^m x_k-\xi|\ge\epsilon\})=\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\le n:|\Delta^m x_k-\xi|\ge\epsilon\}|=0.$$

It is symptomatic as St^α-$\lim\Delta^m x_k=\xi$.

Definition 1.11 ([2]). Consider X be a normed linear space and r be some non-negative number. Then $x=\{x_k\}$ be a sequence in X is named as rough statistically convergent to $\xi\in X$ if to each $\epsilon>0$ and, we have

$$\delta\ (\{k\le n:|x_k-\xi|\ge r+\epsilon\})=0.$$

It is symptomatic as r-St-$\lim x_k=\xi$.

2. Main Results

In this part, we initiate with the definition of rough Δ^m-statistical convergence in *IFNS*, moving with the progression of the perception, we elaborate the results for generalized difference sequences:

Definition 2.1. Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS* and r be a non-negative real number. Then a sequence $x=\{x_k\}$ in X is named as rough convergent to $\xi\in X$ with respect to intutionistic fuzzy norm (η,ψ) as for every $\epsilon>0$ and $\lambda\in(0,1)$ there exists $k_0\in\mathbb{N}$ in such a manner that

$$\eta(\Delta^m x_k-\xi;r+\epsilon)<1-\lambda \text{ and } \psi(\Delta^m x_k-\xi;r+\epsilon)>\lambda, \quad \text{for all } k\ge k_0.$$

Symptomatic as $r_{(\eta,\psi)}$-$\lim_{k\to\infty}(\Delta^m x_k)=\xi$.

Definition 2.2. Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS* and r be a non-negative real number. Then a sequence $x=\{x_k\}$ in X is named as rough-statistically convergent to $\xi\in X$ with respect to intutionistic fuzzy norm (η,ψ) as for every $\epsilon>0$ and $\lambda\in(0,1)$ in such a manner that

$$\delta(\{k\le n:\eta(\Delta^m x_k-\xi;r+\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;r+\epsilon)\ge\lambda\})=0$$

or

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n:\eta(\Delta^m x_k-\xi;r+\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;r+\epsilon)\ge\lambda\}|=0.$$

Symptomatic as r-$St_{(\eta,\psi)}$-$\lim_{k\to\infty}(\Delta^m x_k)=\xi$.

Remark 2.1. In the event if $r=0$, (where r is named as roughness degree) then the perception of rough statistical convergence equates with the perception of statistical convergence in an *IFNS*$(\mathcal{X},\eta,\psi,\star,\cdot)$.

Definition 2.3. Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS*, r be a non-negative real number and $\alpha\in(0,1]$. Then a sequence $x=\{x_k\}$ in X is named as rough-statistically convergent of order α to $\xi\in X$ with respect to intutionistic fuzzy norm (η,ψ) as for every $\epsilon>0$ and $\lambda\in(0,1)$ in such a manner that

$$\delta_\alpha(\{k\le n:\eta(\Delta^m x_k-\xi;r+\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;r+\epsilon)\ge\lambda\})=0$$

or

$$\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k\le n:\eta(\Delta^m x_k-\xi;r+\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;r+\epsilon)\ge\lambda\}|=0.$$

Symptomatic as r-$St^\alpha_{(\eta,\psi)}$-$\lim_{k\to\infty}(\Delta^m x_k)=\xi$.

Remark 2.2. The limit of rough statistical convergence in *IFNS* is not unique.

Example 2.1. Let $(\mathbb{R},|\cdot|)$ be a real normed space, where $|\cdot|$ is the usual norm for the set of real numbers. Let $a\star b=ab$ and $a\cdot b=\min\{1,a+b\}$, also, $\eta(\Delta^m x_k,t)=\frac{t}{t+|\Delta^m x_k|}$ and $\psi(\Delta^m x_k,t)=\frac{|\Delta^m x_k|}{t+|\Delta^m x_k|}$, for $x\in\mathcal{X}$ and $t>0$. Then, $(\mathcal{X},\eta,\psi,\star,\cdot)$ is an *IFNS*. Define a sequence $x=\{x_k\}$ as

$$\Delta^m x_k=\begin{cases}(-1^k), & \text{if } k\ne n^2,\\ 0, & \text{otherwise.}\end{cases}$$

Take $\alpha=1$, then, we have

$$St^\alpha_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k}=\begin{cases}\phi, & \text{if } r<1,\\ [1-r,r-1], & \text{otherwise}\end{cases}$$

and $St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k}=\phi$, for all $k\ge 0$. Thus, this sequence is not convergent as it is unbounded. Also, this sequence is not rough convergent in *IFNS*$(\mathcal{X},\eta,\psi,\star,\cdot)$, for any r.

Definition 2.4. Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS* and r be a non-negative real number. Then a sequence $x=\{x_k\}$ in X is named as rough Δ^m-statistically bounded as for the intutionistic fuzzy norm (η,ψ) to each $\epsilon>0$, $\lambda\in(0,1)$, there exists a real number $M>0$ in such a manner,

$$\delta(\{k\le n:\eta(\Delta^m x_k;M)\le 1-\lambda \text{ or } \psi(\Delta^m x_k;M)\ge\lambda\})=0.$$

Definition 2.5. Consider $(\mathcal{X},\eta,\psi,\star,\cdot)$ be an *IFNS*, r be a non-negative real number and $\alpha\in(0,1]$ be given. Then a sequence $x=\{x_k\}$ in X is named as rough Δ^m-statistically bounded of order α as for the intutionistic fuzzy norm (η,ψ) to each $\epsilon>0$, $\lambda\in(0,1)$, there exists a real number

$M > 0$ in such a manner,

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k; M) \le 1 - \lambda \text{ or } \psi(\Delta^m x_k; M) \ge \lambda\}) = 0.$$

Definition 2.6. Consider $(\mathcal{X}, \eta, \psi, \star, \cdot)$ be a *IFNS*, r be a non-negative real number and $\alpha \in (0,1]$ be given. Then a sequence $x = \{x_k\}$ in X and $C \in X$ is named as rough Δ^m-statistical cluster point of order α as for the intutionistic fuzzy norm (η, ψ) if, to each $\epsilon > 0$, $\lambda \in (0,1)$ in such a manner,

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k - C; r + \epsilon) < 1 - \lambda \text{ and } \psi(\Delta^m x_k - C; r + \epsilon) > \lambda\}) > 0$$

or

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k - C; r + \epsilon) < 1 - \lambda \text{ and } \psi(\Delta^m x_k - C; r + \epsilon) > \lambda\}) \ne 0,$$

where C is known as r-$St^\alpha_{(\eta,\psi)}$-Δ^m-cluster point of order α of a sequence $x = \{x_k\}$.
Let $\Gamma^r_{St^\alpha_{(\eta,\psi)}}(\Delta^m x)$ denotes the set of all rough Δ^m-statistical cluster points of order α of a sequence $x = \{x_k\}$ in *IFNS*$(\mathcal{X}, \eta, \psi, \star, \cdot)$.

With the perspective of the above definitions, we obtain the progression and results of the perception:

Theorem 2.1. *Consider $x = \{x_k\}$ be a sequence in IFNS$(\mathcal{X}, \eta, \psi, \star, \cdot)$. Then $x = \{x_k\}$ is Δ^m-statistically bounded of order α iff $\exists$ a non-negative real number $r > 0$ in such a manner, $St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k} \ne \phi$.*

Proof. Necessary Part: Consider a sequence $x = \{x_k\}$ be Δ^m-statistically bounded of order α in *IFNS*$(\mathcal{X}, \eta, \psi, \star, \cdot)$. By the definition of Δ^m-statistically bounded, to each $\epsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$, $\exists$ $M > 0$ in such a manner,

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k; M) \le 1 - \lambda \text{ or } \psi(\Delta^m x_k; M) \ge \lambda\}) = 0.$$

Let $K = \{k \le n : \eta(\Delta^m x_k; M) \le 1 - \lambda \text{ or } \psi(\Delta^m x_k; M) \ge \lambda\} = 0$. For $k \in K^c$, we have $\eta(\Delta^m x_k; M) > 1 - \lambda$ and $\psi(\Delta^m x_k; M) < \lambda$.
Also,

$$\begin{aligned}\eta(\Delta^m x_k; r + M) &\ge \min\{\eta(0; r), \eta(\Delta^m x_k; M)\} \\ &= \min\{1, \eta(\Delta^m x_k; M)\} > 1 - \lambda\end{aligned}$$

and

$$\begin{aligned}\psi(\Delta^m x_k; r + M) &\le \max\{\psi(0; r), \psi(\Delta^m x_k; M)\} \\ &= \max\{0, \psi(\Delta^m x_k; M)\} < \lambda\end{aligned}$$

thus

$$0 \in St^\alpha_{(\eta,\psi)}\text{-}\lim{}^r_{\Delta^m x_k}.$$

that is, the set $St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k}$ contains the origin of X.
Hence $St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k} \ne \phi$.
Sufficient Part: Consider $St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k} \ne \phi$ for some $r > 0$. Then, exists some $\xi \in X$ in such a manner, $\xi \in St^\alpha_{(\eta,\psi)}$-$\lim^r_{\Delta^m x_k}$, then, to each $\epsilon > 0$ and $\lambda \in (0,1)$, we have

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k - \xi; M) \le 1 - \lambda \text{ or } \psi(\Delta^m x_k; M) \ge \lambda\}) = 0.$$

Therefore, almost all $\Delta^m x_k$'s are contained in some ball with center ξ.
Thus, $x = \{x_k\}$ is Δ^m-statistically bounded of order α in $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$. □

Next, we elaborate the algebraic characterization of rough Δ^m-statistically convergent sequences of order α in $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$.

Theorem 2.2. *Consider $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences in $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$. Then, for some non-negative number r and $\alpha \in (0,1]$, the following statements holds:*

(i) *If $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k} = \xi_1$ and $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m y_k} = \xi_2$, then $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m (x_k+y_k)} = \xi_1+\xi_2$.*

(ii) *If $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k} = \xi$ and $c \in \mathbb{R}$, then $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m cx_k} = c\xi$.*

Proof. (i) Consider $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k} = \xi_1$ and $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m y_k} = \xi_2$. For $\epsilon \in (0,1)$, take $\lambda \in (0,1)$ in such a manner, $(1-\lambda)*(1-\lambda) > 1-\epsilon$. For some non-negative real number r, define

$$\mathcal{K}_1 = \left\{k \le n : \eta\left(\Delta^m x_k - \xi; \frac{r+\epsilon}{2}\right) \le 1-\lambda \text{ or } \psi\left(\Delta^m x_k - \xi; \frac{r+\epsilon}{2}\right) \ge \lambda\right\},$$

$$\mathcal{K}_2 = \left\{k \le n : \eta\left(\Delta^m y_k - \xi; \frac{r+\epsilon}{2}\right) \le 1-\lambda \text{ or } \psi\left(\Delta^m y_k - \xi; \frac{r+\epsilon}{2}\right) \ge \lambda\right\}.$$

Since $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k} = \xi_1$, therefore α-density of the set $\mathcal{K}_1 = 0$.
Also, for $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m y_k} = \xi_2$, we get the α-density of the set $\mathcal{K}_2 = 0$.
To each $\epsilon > 0$, let $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$.
Then, $\delta_\alpha(\mathcal{K})$ is zero, which implies $\delta_\alpha(\mathbb{N}-\mathcal{K}) = 1$.
Let $k \in \mathbb{N}-\mathcal{K}$. Then

$$\begin{aligned}\eta(\Delta^m(x_k+y_k)-(\xi_1+\xi_2); r+\epsilon) &\ge \eta\left(\Delta^m x_k - \xi_1; \frac{r+\epsilon}{2}\right) * \eta\left(\Delta^m y_k - \xi_2; \frac{r+\epsilon}{2}\right)\\ &> (1-\lambda) * 1-\lambda)\\ &> 1-\epsilon\end{aligned}$$

and

$$\begin{aligned}\psi(\Delta^m(x_k+y_k)-(\xi_1+\xi_2); r+\epsilon) &\le \psi\left(\Delta^m x_k - \xi_1; \frac{r+\epsilon}{2}\right) * \psi\left(\Delta^m y_k - \xi_2; \frac{r+\epsilon}{2}\right)\\ &< \lambda * \lambda\\ &< \epsilon.\end{aligned}$$

This shows that

$$\delta_\alpha\{k \le n : \eta(\Delta^m x_k - \xi_1 + \Delta^m y_k - \xi_2; \epsilon + r) \le 1-\lambda \text{ or } \psi(\Delta^m x_k - \xi_1 + \Delta^m y_k - \xi_2; \epsilon + r) \ge \lambda\} = 0.$$

Hence $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m (x_k+y_k)} = \xi_1+\xi_2$.

(ii) Let $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m x_k} = \xi$ and $c \in \mathbb{R}$. Then, for given $\epsilon > 0$ and $\lambda > 0$, we have

$$\mathcal{K}_{(\eta,\psi)} = \{k \in \mathbb{N} : \eta(\Delta^m x_k - \xi; \epsilon + r) \le 1-\lambda \text{ or } \psi(\Delta^m x_k - \xi; \epsilon + r) \ge \lambda\}.$$

We have

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\mathcal{K}_{(\eta,\psi)}| = 0.$$

Now, two cases arise:

Case (i): If $c \neq 0$,

$$\implies \quad \delta_\alpha\{\mathcal{K}_{(\eta,\psi)}\} = 0$$

$\implies \quad \delta_\alpha[\mathcal{K}_{(\eta,\psi)}]^c = 1$

Let $k \in \delta_\alpha[\mathcal{K}_{(\eta,\psi)}]^c$, then

$$\begin{aligned}\eta(c\Delta^m x_k - c\xi, \epsilon + r) &= \eta(c(\Delta^m x_k - \xi), \epsilon + r)\\ &\geq \eta\left(\Delta^m x_k - \xi, \frac{\epsilon + r}{|c|}\right)\\ &\geq \eta(\Delta^m x_k - \xi, \epsilon + r) \star \eta\left(0, \frac{\epsilon + r}{|c|} - \epsilon + r\right)\\ &= \eta(\Delta^m x_k - \xi, \epsilon + r) \star 1 > 1 - \lambda\end{aligned}$$

and

$$\begin{aligned}\psi(c\Delta^m x_k - c\xi, \epsilon + r) &= \psi(c(\Delta^m x_k - \xi), \epsilon + r)\\ &\leq \psi\left(\Delta^m x_k - \xi, \frac{\epsilon + r}{|c|}\right)\\ &\leq \psi(\Delta^m x_r - \xi, \epsilon + r) \star \psi\left(0, \frac{\epsilon + r}{|c|} - \epsilon + r\right)\\ &= \psi(\Delta^m x_k - \xi, \epsilon + r) \star 0 < 1 - \lambda.\end{aligned}$$

$\implies \quad \delta_\alpha[\mathcal{K}_{(\eta,\psi)}]^c = 1$

$$\implies \quad \lim_{n\to\infty} \frac{1}{n^\alpha}\{k \in \mathbb{N} : \eta(c\Delta^m x_k - c\xi; \epsilon + r) \geq 1 - \lambda \text{ and } \psi(c\Delta^m x_k - c\xi; \epsilon + r) \leq \lambda\} = 1$$

Therefore, $St^\alpha_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m c x_k} = c\xi$.

Case (ii): If $c = 0$, then

$\eta(0\Delta^m x_k, \epsilon + r) = \eta(0, \epsilon + r) = 1 > 1 - \lambda$ and

$\psi(0\Delta^m x_k, \epsilon + r) = \psi(0, \epsilon + r) = 0 < \lambda$

$\implies \quad St^\alpha_{(\eta,\psi)}\text{-}\lim^r_{\Delta^m c x_k} = c\xi$. □

Theorem 2.3. *Consider* $0 < \beta \leq \alpha \leq 1$ *then* $r\text{-}St^\alpha_{(\eta,\psi)} \subseteq r\text{-}St^\beta_{(\eta,\psi)}$ *where* $r\text{-}St^\alpha_{(\eta,\psi)}$ *and* $r\text{-}St^\beta_{(\eta,\psi)}$ *represent the sets of all rough statistically convergent of order* α *and* β*, respectively.*

Proof. Let $x = \{x_k\}$ be a sequence in $IFNS(\mathcal{X}, \eta, \psi, \star, \cdot)$. If $0 < \beta \leq \alpha \leq 1$ then as for $\epsilon > 0$ and some $r > 0$ with limit point ξ, we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^\beta}\{k \in \mathbb{N} : \eta(\Delta^m x_k - \xi; r + \epsilon) \geq 1 - \lambda \text{ and } \psi(\Delta^m x_k - \xi; r + \epsilon) \leq \lambda\}\\ &\quad \leq \lim_{n\to\infty} \frac{1}{n^\alpha}\{k \in \mathbb{N} : \eta(\Delta^m x_k - \xi; r + \epsilon) \geq 1 - \lambda \text{ and } \psi(\Delta^m x_k - \xi; r + \epsilon) \leq \lambda\}.\end{aligned}$$

Therefore, $r\text{-}St^\alpha_{(\eta,\psi)} \subseteq r\text{-}St^\beta_{(\eta,\psi)}$. □

Example 2.2. Let $(\mathbb{R}, |\cdot|)$ be a real normed space, where $|\cdot|$ is the usual norm for the set of real numbers, define $\eta(\Delta^m x_k, t) = \frac{t}{t + |\Delta^m x_k|}$ and $\psi(\Delta^m x_k, t) = \frac{|\Delta^m x_k|}{t + |\Delta^m x_k|}$ for $x \in \mathcal{X}$ and $t > 0$. Then, $(\mathcal{X}, \eta, \psi, \star, \cdot)$ is an *IFNS*. Define a sequence $x = \{x_k\}$ as

$$\Delta^m x_k = \begin{cases} k, & \text{if } k \neq n^2, \\ 0, & \text{otherwise} \end{cases}$$

gives $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} = [-r,r]$ and its subsequence gives $\Delta^m x' = \{1,4,9,\dots\}$. Thus, $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} = \phi$, which is not exact for the non-thin subsequences of rough statistical convergent sequences of order α in an $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$ which is elaborated in the following result:

Theorem 2.4. *Consider $\Delta^m x' = \{\Delta^m x_{k_i}\}$ be a non-thin sub-sequence of a sequence $\Delta^m x = \{\Delta^m x_k\}$ in $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$, $\alpha \in (0,1]$ then $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} \subset St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x'_k}$.*

Proof. The proof of this theorem is trivial, so we are omitting it. □

Theorem 2.5. *For $\alpha \in (0,1]$ and $r > 0$, the set $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$ of the sequence $x = \{x_k\}$ in $IFNS(\mathcal{X},\eta,\psi,\star,\cdot)$ is a closed set.*

Proof. If $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} = \phi$, result is trivial.
If $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} \neq \phi$ for some $r > 0$. Consider $y = \{y_k\}$ be a sequence in $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$ which converges as for the intuitionistic fuzzy norm (η,ψ) to $y_0 \in X$.
Then, for each $\epsilon > 0$ and $\lambda \in (0,1)$ $\exists$ $k_0 \in \mathbb{N}$ in such a manner,

$$\eta\left(\Delta^m x_k - y_0; \frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \psi\left(\Delta^m x_k - y_0; \frac{\epsilon}{2}\right) < \lambda, \quad \text{for all } k \geq k_0.$$

Let us select, $y_n \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$ with $n > k_0$ in such a manner,

$$\delta_\alpha\left(\left\{k \leq n : \eta\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \psi\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) \geq \lambda\right\}\right) = 0 \tag{2.1}$$

for $j \in \delta_\alpha\left(\left\{k \leq n : \eta\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \psi\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) < \lambda\right\}\right)$, we have

$$\eta\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) > 1-\lambda \text{ or } \psi\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) < \lambda.$$

Then,

$$\eta(\Delta^m x_j - y_0; r+\epsilon) \geq \min\left\{\eta\left(\Delta^m x_j - y_n, \frac{\epsilon}{2}\right), \eta\left(y_n - y_0, \frac{\epsilon}{2}\right)\right\} > 1-\lambda$$

and

$$\psi(\Delta^m x_j - y_0; r+\epsilon) \leq \max\left\{\eta\left(\Delta^m x_j - y_n, \frac{\epsilon}{2}\right), \psi\left(y_n - y_0, \frac{\epsilon}{2}\right)\right\} < \lambda.$$

Now, for $j \in \{k \leq n : \eta(\Delta^m x_k - y_0; r+\epsilon) > 1-\lambda \text{ and } \psi(\Delta^m x_k - y_0; r+\epsilon) < \lambda\}$.

We have the following inclusion

$$\begin{aligned}&\left\{k \leq n : \eta\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) > 1-\lambda \text{ and } \psi\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) < \lambda\right\}\\ &\quad \subseteq \{k \leq n : \eta(\Delta^m x_k - y_0; r+\epsilon) > 1-\lambda \text{ and } \psi(\Delta^m x_k - y_0; r+\epsilon) < \lambda\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\delta_\alpha(\{k \leq n : \eta(\Delta^m x_k - y_0; r+\epsilon) \leq 1-\lambda \text{ or } \psi(\Delta^m x_k - y_0; r+\epsilon) \geq \lambda\})\\ &\quad \leq \delta_\alpha\left(\left\{k \leq n : \eta\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \psi\left(\Delta^m x_k - y_n; r+\frac{\epsilon}{2}\right) \geq \lambda\right\}\right).\end{aligned}$$

Using eq. (2.1), we get

$$\delta_\alpha(\{k \leq n : \eta(\Delta^m x_k - y_0; r+\epsilon) \leq 1-\lambda \text{ or } \psi(\Delta^m x_k - y_0; r+\epsilon) \geq \lambda\}) = 0$$

$$\implies \quad y_0 \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}.$$ □

In the next theorem, we introduce the convexity of the set $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$.

Theorem 2.6. *For $\alpha \in (0,1]$ and $r > 0$, the rough statistical limit set of order α $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$ of the sequence $x = \{x_k\}$ in IFNS$(\mathcal{X},\eta,\psi,\star,\cdot)$ for the intuitionistic fuzzy norm (η,ψ) is a convex set.*

Proof. Let us select $\xi_1,\xi_2 \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$. For the convexity of the set $St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$, we need to show that $[(1-\beta)\xi_1 + \beta\xi_2] \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$, for some $\beta \in (0,1)$. Then, for each $\epsilon > 0$ select $\lambda \in (0,1)$, we define

$$\mathcal{K}_1 = \left\{k \le n : \eta\left(\Delta^m x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \le 1-\lambda \text{ or } \psi\left(\Delta^m x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \ge \lambda\right\},$$
$$\mathcal{K}_2 = \left\{k \le n : \psi\left(\Delta^m x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \le 1-\lambda \text{ or } \psi\left(\Delta^m x_k - \xi_2; \frac{r+\epsilon}{2\beta} \ge \lambda\right)\right\}.$$

Since $\xi_1,\xi_2 \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$, we have

$$\delta_\alpha(\mathcal{K}_1) = \delta_\alpha(\mathcal{K}_2) = 0.$$

For every $k \in {\mathcal{K}_1}^c \cap {\mathcal{K}_2}^c$, we have

$$\begin{aligned}
\eta(\Delta^m x_k - [(1-\beta)\xi_1 + \beta\xi_2]) &= \eta((1-\beta)(\Delta^m x_k - \xi_1) + \beta(\Delta^m x_k - \xi_2); r+\epsilon)\\
&\ge \min\left\{\eta\left((1-\beta)(\Delta^m x_k - \xi_1); \frac{r+\epsilon}{2}\right), \eta\left(\beta(\Delta^m x_k - \xi_2); \frac{r+\epsilon}{2}\right)\right\}\\
&= \min\left\{\eta\left((\Delta^m x_k - \xi_1); \frac{r+\epsilon}{2(1-\beta)}\right), \eta\left((\Delta^m x_k - \xi_2); \frac{r+\epsilon}{2\beta}\right)\right\}\\
&> (1-\lambda),\\
\psi(\Delta^m x_k - [(1-\beta)\xi_1 + \beta\xi_2]) &= \psi((1-\beta)(\Delta^m x_k - \xi_1) + \beta(\Delta^m x_k - \xi_2); r+\epsilon)\\
&\le \max\left\{\psi\left((1-\beta)(\Delta^m x_k - \xi_1); \frac{r+\epsilon}{2}\right), \psi\left(\beta(\Delta^m x_k - \xi_2); \frac{r+\epsilon}{2}\right)\right\}\\
&= \max\left\{\psi\left((\Delta^m x_k - \xi_1); \frac{r+\epsilon}{2(1-\beta)}\right), \psi\left((\Delta^m x_k - \xi_2); \frac{r+\epsilon}{2\beta}\right)\right\}\\
&< \lambda.
\end{aligned}$$

Thus,

$$\delta_\alpha(\{k \le n : \eta(\Delta^m x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) \le 1-\lambda \text{ or } \psi(\Delta^m x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) \ge 1-\lambda\}) = 0.$$

Hence, $[(1-\beta)\xi_1 + \beta\xi_2] \in St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k} \implies St^{\alpha}_{(\eta,\psi)}\text{-}\lim^{r}_{\Delta^m x_k}$ is a convex set. □

Theorem 2.7. *For $\alpha \in (0,1]$ and $r > 0$, a sequence $x = \{x_k\}$ is rough Δ^m-statistically convergent of order α to ξ in IFNS$(\mathcal{X},\eta,\psi,\star,\cdot)$ for the intuitionistic fuzzy norm (η,ψ) if $\exists$ a sequence $y = \{y_k\}$ in X which is Δ^m-statistically convergent of order α to $\xi \in X$ as for the intutionistic fuzzy norm (η,ψ) and for every $\lambda \in (0,1)$ satisfies $\eta(\Delta^m x_k - \Delta^m y_k; r) > 1-\lambda$ and $\psi(\Delta^m x_k - \Delta^m y_k; r) < \lambda$ $\forall$ $k \le n$.*

Proof. Consider $\epsilon > 0$ and let $\Delta^m y_k \xrightarrow{r\text{-}St^{\alpha}_{(\eta,\psi)}} \xi$ and $\eta(\Delta^m x_k - \Delta^m y_k; r) > 1-\lambda$ and $\psi\Delta^m x_k - \Delta^m y_k; r) < \lambda$, for all $k \in \mathbb{N}$. For given $\lambda \in (0,1)$ define

$$A_1 = \lim_{n\to\infty} \{k \le n : \eta(\Delta^m y_k - \xi; \epsilon) \le 1-\lambda \text{ or } \psi(\Delta^m y_k - \xi; \epsilon) \ge \lambda\},$$
$$A_2 = \lim_{n\to\infty} \{k \le n : \eta(\Delta^m x_k - \Delta^m y_k; r) \le 1-\lambda \text{ or } \psi(\Delta^m x_k - \Delta^m y_k; r) \ge \lambda\}.$$

Clearly, $\delta_\alpha(A_1) = 0$ and $\delta_\alpha(A_2) = 0$. For $k \in {A_1}^c \cap {A_2}^c$, we have

$$\eta(\Delta^m x_k - \xi; r+\epsilon) \ge \min\{\eta(\Delta^m x_k - \Delta^m y_k; r), \eta(\Delta^m y_k - \xi; \epsilon)\} > 1-\lambda$$

and

$$\psi(\Delta^m x_k-\xi;r+\epsilon)\le\max\{\psi(\Delta^m x_k-\Delta^m y_k;r),\psi(\Delta^m y_k-\xi;\epsilon)\}<\lambda.$$

Then

$$\eta(\Delta^m x_k-\xi;r+\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-\xi;r+\epsilon)<\lambda,\quad \text{for all } k\in {A_1}^c\cap {A_2}^c,$$

$$\implies \lim_{n\to\infty}\{k\le n:\eta(\Delta^m x_k-\xi;\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;\epsilon)\ge\lambda\}\subseteq A_1\cup A_2.$$

Then,

$$\delta_\alpha\{k\le n:\eta(\Delta^m x_k-\xi;\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;\epsilon)\ge\lambda\}\le\delta_\alpha(A_1)+\delta_\alpha(A_2).$$

Then, we have

$$\delta_\alpha\{k\le n:\eta(\Delta^m x_k-\xi;\epsilon)\le 1-\lambda \text{ or } \psi(\Delta^m x_k-\xi;\epsilon)\ge\lambda\}=0.$$

Hence, the sequence $\Delta^m x_k \xrightarrow{r\text{-}St^\alpha_{(\eta,\psi)}} \xi$. □

Theorem 2.8. *Consider the set of all r-$St^\alpha_{(\eta,\psi)}$-Δ^m-cluster points of order α be $\Gamma_{St^\alpha_{(\eta,\psi)}}(\Delta^m x)$ of a sequence $x=\{x_k\}$ in IFNS$(\mathfrak{X},\eta,\psi,\star,\cdot)$, for $\alpha\in(0,1]$ and r be some non-negative real number. Then, for arbitrary $C\in\Gamma_{St^\alpha_{(\eta,\psi)}}(\Delta^m x)$ and $\lambda\in(0,1)$ in such a manner, $\eta(\xi-C;r)>1-\lambda$ and $\psi(\xi-C;r)<\lambda$ for all $\xi\in\Gamma^r_{St_{(\eta,\psi))^\alpha}}(\Delta^m x)$.*

Proof. For $C\in\Gamma_{St^\alpha_{(\eta,\psi)}}(\Delta^m x)$ then to each $\epsilon>0$ and $\lambda\in(0,1)$, we have

$$\delta_\alpha\{k\le n:\eta(\Delta^m x_k-C;\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;\epsilon)<\lambda\}=0. \tag{2.2}$$

Now, its sufficient to show that if $\xi\in\mathfrak{X}$ which gives $\eta(\xi-C;\epsilon)>1-\lambda$ and $\psi(\xi-C;\epsilon)<\lambda$, then $\xi\in\Gamma_{St^\alpha_{(\eta,\psi)}}(\Delta^m x)$.

Let $j\in\{k\le n:\eta(\Delta^m x_k-C;\epsilon)>1-\lambda$ and $\psi(\Delta^m x_k-C;\epsilon)<\lambda\}$ then $\eta(\Delta^m x_j-C;\epsilon)>1-\lambda$ and $\psi(\Delta^m x_j-C;\epsilon)<\lambda$.

Now,

$$\eta(\Delta^m x_j-C;r+\epsilon)\ge\min\{\eta(\Delta^m x_j-C;\epsilon),\eta(\xi-C;r)\}>1-\lambda$$

and

$$\psi(\Delta^m x_j-C;r+\epsilon)\ge\max\{\psi(\Delta^m x_j-C;\epsilon),\psi(\xi-C;r)\}<\lambda$$

$$\implies \eta(\xi-C;r+\epsilon)>1-\lambda \text{ and } \psi(\xi-C;r+\epsilon)<\lambda.$$

Hence

$$j\in\{k\le n:\eta(\Delta^m x_k-C;r+\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;r+\epsilon)<\lambda\}.$$

Now, the next inclusion holds,

$$\begin{aligned}&\{k\le n:\eta(\Delta^m x_k-C;\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;\epsilon)<\lambda\}\\&\quad\subseteq\{k\le n:\eta(\Delta^m x_k-C;r+\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;r+\epsilon)<\lambda\}.\end{aligned}$$

Then

$$\begin{aligned}&\delta_\alpha\{k\le n:\eta(\Delta^m x_k-C;\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;\epsilon)<\lambda\}\\&\quad\le\delta_\alpha\{k\le n:\eta(\Delta^m x_k-C;r+\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-C;r+\epsilon)<\lambda\}.\end{aligned}$$

By eq. (2.2), we now have

$$\delta_\alpha(\{k\le n:\eta(\Delta^m x_k-\xi;r+\epsilon)>1-\lambda \text{ and } \psi(\Delta^m x_k-\xi;r+\epsilon)<\lambda\})>0.$$

Therefore,

$$\xi \in \Gamma^r_{St_{(\eta,\psi)})^\alpha}(\Delta^m x). \qquad \square$$

Conclusion

We have presented the perception of rough Δ^m-statistical convergence of order α for the sequences in intuitionistic fuzzy normed spaces in this paper which is more generalized than the corresponding results of Δ^m-statistical convergence of the sequences in intutionistic fuzzy normed spaces. We proved some important results for these perceptions and explored some examples that show that this approach is more broad than the findings of normed spaces.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems* **20**(1) (1986), 87 – 96, DOI: 10.1016/S0165-0114(86)80034-3.

[2] S. Aytar, Rough statistical convergence, *Numerical Functional Analysis and Optimization* **29**(3-4) (2008), 291 – 303, DOI: 10.1080/01630560802001064.

[3] M. Chawla, M. S. Saroa and V. Kumar, On Λ-statistical convergence of order α in random 2-normed space, *Miskolc Mathematical Notes* **16**(2) (2015), 1003 – 1015, DOI: 10.18514/MMN.2015.821.

[4] M. Çınar and M. Et, Δ^m-statistical convergence of order $\tilde{\alpha}$ for double sequences of functions, *Facta Universitatis, Series: Mathematics and Informatics* **35**(2) (2020), 393 – 404, DOI: 10.22190/FUMI2002393C.

[5] N. Demir and H. Gümüş, Rough convergence for difference sequences, *New Trends in Mathematical Sciences* **8**(2) (2020), 22 – 28, DOI: 10.20852/ntmsci.2020.402.

[6] M. Et and R. Colak, On some generalized difference sequence spaces, *Soochow Journal of Mathematics* **21** (1995), 377 – 386.

[7] M. Et and F. Nuray, Δ^m-Statistical convergence, *Indian Journal of Pure and Applied Mathematics* **32**(6) (2001), 961 – 969.

[8] H. Fast, Sur la convergence statistique, *Colloquium Mathematicae* **2**(3-4) (1951), 241 – 244, https://eudml.org/doc/209960.

[9] R. Giles, A computer program for fuzzy reasoning, *Fuzzy Sets and Systems* **4**(3) (1980), 221 – 234, DOI: 10.1016/0165-0114(80)90012-3.

[10] S. Karakus and K. Demırcı, Statistical convergence of double sequences on probabilistic normed spaces, *International Journal of Mathematics and Mathematical Sciences* **2007**(1), 014737, DOI: 10.1155/2007/14737.

[11] S. Karakus, K. Demırcı and O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, *Chaos, Solitons & Fractals* **35**(4) (2008), 763 – 769, DOI: 10.1016/j.chaos.2006.05.046.

[12] H. Kizmaz, On certain sequence spaces, *Canadian Mathematical Bulletin* **24**(2) (1981), 169 – 176, DOI: 10.4153/CMB-1981-027-5.

[13] F. A. Lootsma, *Fuzzy Logic for Planning and Decision Making*, 1st edition, Springer, New York, NY, x + 198 pages (1997), DOI: 10.1007/978-1-4757-2618-3.

[14] M. Maity, A note on rough statistical convergence of order α, *arXiv:1603.00183*, 7 pages (2016), DOI: 10.48550/arXiv.1603.00183.

[15] S. A. Mohiuddine and Q. M. D. Lohani, On generalized statistical convergence in intuitionistic fuzzy normed space, **42**(3) (2009), 1731 – 1737, DOI: 10.1016/j.chaos.2009.03.086.

[16] F. Móricz, Statistical convergence of multiple sequences, *Archiv der Mathematik* **81** (2003), 82 – 89, DOI: 10.1007/s00013-003-0506-9.

[17] M. Mursaleen, On statistical convergence in random 2-normed spaces, *Acta Scientiarum Mathematicarum* **76** (2010), 101 – 109, DOI: 10.1007/BF03549823.

[18] Mursaleen and O. H. H. Edely, Statistical convergence of double sequences, *Journal of Mathematical Analysis and Applications* **288**(1) (2003), 223 – 231, DOI: 10.1016/j.jmaa.2003.08.004.

[19] H. T. Nguyen and B. Wu, *Fundamentals of Statistics with Fuzzy Data*, Studies in Fuzziness and Soft Computing series, Springer, Berlin — Heidelberg, x + 196 pages (2006), DOI: 10.1007/11353492.

[20] J. H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* **22**(5) (2004), 1039 – 1046, DOI: 10.1016/j.chaos.2004.02.051.

[21] H. X. Phu, Rough convergence in normed linear spaces, *Numerical Functional Analysis and Optimization* **22**(1-2) (2001), 199 – 222, DOI: 10.1081/NFA-100103794.

[22] H. M. Prade, Operations research with fuzzy data in fuzzy sets, in: *Fuzzy Sets*, P. P. Wang and S. K. Chang (editors), Springer, Boston, MA, DOI: 10.1007/978-1-4684-3848-2_14.

[23] Reena, Meenakshi and T. Bansal, Statistical Λ-convergence of order α in intuitionistic fuzzy normed spaces, *International Journal on Emerging Technologies* **10**(2b)(2019), 43 – 47.

[24] Reena, Meenakshi and V. Kumar, Generalized statistical convergence of α in random n-normed space, *Advances and Applications in Mathematical Sciences* **18**(9) (2019), 715 – 729.

[25] T. J. Ross, *Fuzzy Logic with Engineering Applications*, John Wiley & Sons, Ltd., xxi + 585 pages (2010), DOI: 10.1002/9781119994374.

[26] A. Şahiner, M. Gürdal and F. K. Düden, Triple sequences and their statistical convergences, *Selçuk Journal of Applied Mathematics* **8**(2) (2007), 49 – 55.

[27] T. Šalát, On statistically convergent sequences of real numbers, *Mathematica Slovaca* **30**(2) (1980), 139 – 150, URL: https://eudml.org/doc/34081.

[28] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological space, *Chaos, Solitons & Fractals* **27**(2) (2006), 331 – 344, DOI: 10.1016/j.chaos.2005.03.019.

[29] B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific Journal of Mathematics* **10**(1) (1960) 313 – 334.

[30] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloquium Mathematicum* **2** (1951), 73 – 74.

[31] L. A. Zadeh, Fuzzy sets, *Information and Control*, **8**(3) (1965), 338 – 353, DOI: 10.1016/S0019-9958(65)90241-X.

[32] A. Zygmund, On the convergence of lacunary trigonometric series, *Fundamenta Mathematicae* **16**(1) (1930), 90 – 107, URL: http://eudml.org/doc/212510.