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Research Article

Some Properties of Pythagorean Fuzzy Normed Ideals

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Abstract. Pythagorean fuzzy sets are an advanced extension of fuzzy sets, building upon the framework of intuitionistic fuzzy sets and offering a more comprehensive solution to the limitations inherent in intuitionistic fuzzy theory. This study introduces the concept of *Pythagorean Fuzzy Normed Ideals* (PFNIs). It explores the intrinsic product between two PFNIs, presenting key results related to this operation. Additionally, the study introduces the concept of characteristic functions in the context of PFNIs and discusses several significant properties of these functions. Furthermore, important findings concerning the epimorphism of Pythagorean fuzzy normed ideals are also presented.

Keywords. Intuitionistic fuzzy set, Pythagorean fuzzy set, Pythagorean fuzzy normed ideal

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [15], who assigned in unit interval $[0,1]$ to each element in a set, reflecting the degree of its membership, denoted by ϱ. *Fuzzy Sets* (FSs) represent a specific area within set theory, allowing elements to have degrees of membership that range from complete inclusion to total exclusion. The membership value lies between 0 and 1, where 0 indicates non-membership, 1 indicates full membership, and intermediate values represent varying degrees of partial membership.

However, fuzzy set theory has limitations as it does not account for non-membership functions nor does it consider the potential impact of hesitation. To address these shortcomings, Atanassov [4] developed the framework of *Intuitionistic Fuzzy Sets* (IFSs). IFSs incorporate three components: membership grade ϱ, non-membership grade ϖ, hesitation margin π, which represents the degree of uncertainty where an element is neither fully a member nor a non-member. This leads to the conditions $\varrho+\varpi \le 1$ and $\varrho+\varpi+\pi = 1$, offering a more nuanced representation of uncertainty.

In contrast to the conditions observed in *Intuitionistic Fuzzy Sets* (IFSs), there are scenarios where the inequality $\varrho+\varpi \ge 1$ holds. To address this, *Pythagorean Fuzzy Sets* (PFSs) were introduced within the IFS framework. Proposed by Yager [13], and Yager and Abbasov [14], PFSs provide a distinct approach to handling the ambiguity associated with membership grade ϱ and non-membership grade ϖ. These grades satisfy the conditions $0 \le \varrho \le 1$ and $0 \le \varpi \le 1$, and they are constrained by the relation $\varrho^2+\varpi^2+\pi^2 \le 1$, where π represents the degree of hesitation. Pythagorean fuzzy sets offer a more comprehensive and accurate representation of uncertain data compared to intuitionistic fuzzy sets, and they perform better in simulating real-world scenarios involving uncertainty and confusion.

Algebraic structures are fundamental in mathematics and have wide-ranging applications in fields such as control engineering, computer science, theoretical physics, and information science. Alhaleem and Ahmad [5] explored the concept of IF-normed sub-rings. Emniyet and Şahin [9] studied fuzzy normed prime ideals and maximal ideals, this extends these concepts to apply the framework of IFSs to prime and maximal normed ideals. Additionally, Adak and Salokolaei [2] introduced the concept of rough PFSs, and Adak and Kumar [1] examined the characteristics of *Pythagorean Fuzzy Ideals* (PFI) in relation to Γ-near rings.

The article is organized into the following sections: Section 2 offers an overview and definitions of key concepts, including normed spaces, normed rings, FSs, IFSs, and Pythagorean fuzzy sets. In Section 3, we examine image and inverse image of *Pythagorean Fuzzy Normed Ideals* (PFNIs) and *Pythagorean Fuzzy Normed Sub-Rings* (PFNSR), highlighting some of their key properties based on epimorphism. Finally, Section 4 presents the conclusion.

2. Preliminaries

Initially, we present several definitions that are essential for the rest of the paper:

Definition 2.1 ([11]). A linear space L defined as a normed space if a real number $\|\xi\|$ that satisfies the following conditions:

- $\|\xi\| \ge 0$, for every $\xi \in L$ where $\xi = 0$ then $\|\xi\| = 0$;
- $\|a \cdot \xi\| = |a| \cdot \|\xi\|$;
- $\|\xi+\nu\| \le \|\xi\| + \|\nu\|$, for all $\xi,\nu \in L$.

Definition 2.2 ([11]). A ring R is defined as a normed ring (NR) if it has a norm $\|\ \|$, where $\|\ \| : NR \to \mathbb{R}$ with

(i) $\|\xi\| = 0 \Leftrightarrow \xi = 0$,

(ii) $\|\xi+\nu\| \leq \|\xi\|+\|\nu\|$, for all $\xi,\nu \in L$,

(iii) $\|\xi\| = \|-\xi\|$, (and hence $\|1_A\| = 1 = \|-1\|$ if identity exists), and

(iv) $\xi\nu \leq \|\xi\|\,\|\nu\|$,

for any elements $\xi,\nu \in R$.

Definition 2.3 ([9])**.** Operation $*:[0,1]\times[0,1]\to[0,1]$ to be considered as t-norm, if it is meet commutativity, associativity, monotonicity, and the presence of a neutral element 1.
We will briefly utilize the t-norm and denote it as $\xi * \nu$ in place of $*(\xi,\nu)$.

Definition 2.4 ([9])**.** Operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is considered as s-norm if it satisfies the conditions of commutativity, associativity, monotonicity, and the existence of a neutral element 0.
We will briefly utilize s-norm and denote it as $\xi\diamond\nu$ in place of $\diamond(\xi,\nu)$.

Definition 2.5 ([15])**.** A fuzzy set F in a universal set X is

$$F = \{\langle x,\varrho_F(x)\rangle : x \in X\},$$

where $\varrho_F : X \to [0,1]$ is membership function.
Complement of ϱ is defined by $\bar{\varrho}(x) = 1-\varrho(x)$ for all $x \in X$ and denoted by $\bar{\varrho}$.

Definition 2.6 ([4])**.** An *Intuitionistic Fuzzy Set* (IFS) A in X is defined as

$$A = \{\langle x,\varrho_A(x),\varpi_A(x)\rangle : x \in X\},$$

where $\varrho_A(x)$ and $\varpi_A(x)$ are represents belonging and non-belonging grade of $x \in X$ respectively, with

$$0 \leq \varrho_A(x)+\varpi_A(x) \leq 1,$$

for all $x \in X$ and $\varrho_A : X \to [0,1]$, $\varpi_A : X \to [0,1]$.
The indeterminacy $h_A(x) = 1-\varrho_A(x)-\varpi_A(x)$.

Definition 2.7 ([14])**.** A Pythagorean fuzzy set P in X is given by

$$P = \{\langle x,\varrho_P(x),\varpi_P(x)\rangle \mid x \in X\},$$

where $\varrho_P(x) : X \to [0,1]$ and $\varpi_P(x) : X \to [0,1]$, with condition that

$$0 \leq (\varrho_P(x))^2+(\varpi_P(x))^2 \leq 1,$$

for all $x \in X$.
The indeterminacy $h_P(x) = \sqrt{1-(\varrho_P(x))^2-(\varpi_P(x))^2}$.

Definition 2.8 ([13])**.** Let $A = (\varrho_A,\varpi_A)$ and $B = (\varrho_B,\varpi_B)$, be two PFSs, then

(1) $A^c = \{\langle \xi,\varpi_A(\xi),\varrho_A(\xi)\rangle : \xi \in R\}$,

(2) $A\cup B = \{\langle \xi,\max(\varrho_A(\xi),\varrho_B(\xi)),\min(\varpi_A(\xi),\varpi_B(\xi))\rangle : \xi \in R\}$,

(3) $A\cap B = \{\langle \xi,\min(\varrho_A(\xi),\varrho_B(\xi)),\max(\varpi_A(\xi),\varpi_B(\xi))\rangle : \xi \in R\}$.

Definition 2.9. A PFS $A = \{(\xi,\varrho_A(\xi),\varpi_A(\xi)) : \xi \in NR\}$ is a *Pythagorean Fuzzy Normed Sub-Ring* (PFNSR) of NR if:

(i) $\varrho_A(\xi-\nu) \geq \varrho_A(\xi) * \varrho_A(\nu)$,

(ii) $\varrho_A(\xi\nu) \geq \varrho_A(\xi) * \varrho_A(\nu)$,

(iii) $\varpi_A(\xi-\nu) \leq \varpi_A(\xi) \diamond \varpi_A(\nu)$,

(iv) $\varpi_A(\xi\nu) \leq \varpi_A(\xi) \diamond \varpi_A(\nu)$.

Definition 2.10. A PFS $A = \{(\xi, \varrho_A(\xi), \varpi_A(\xi)) : \xi \in NR\}$ is a *Pythagorean Fuzzy Normed Ideal* (PFNI) of NR if:

(i) $\varrho_A(\xi-\nu) \geq \varrho_A(\xi) * \varrho_A(\nu)$,

(ii) $\varrho_A(\xi\nu) \geq \varrho_A(\xi) \diamond \varrho_A(\nu)$,

(iii) $\varpi_A(\xi-\nu) \leq \varpi_A(\xi) \diamond \varpi_A(\nu)$,

(iv) $\varpi_A(\xi\nu) \leq \varpi_A(\xi) * \varpi_A(\nu)$.

3. Main Results

This section delineates various results of PFNIs, leading to the derivation of fundamental results.

Definition 3.1. Consider two Pythagorean fuzzy subsets, denoted as A and B of NR. The intrinsic operations $A \circ B = (\varrho_{A \odot B}, \varpi_{A \otimes B}) = (\varrho_A \odot \varrho_B, \varpi_A \otimes \varpi_B)$ specified as follows:

$$\varrho_{A\odot B}(\xi) = \begin{cases} \diamond_{\xi=\nu z} (\varrho_A(\nu) * \varrho_B(z)), & \text{if } \xi = \nu z, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varpi_{A\otimes B}(\xi) = \begin{cases} \diamond_{\xi=\nu z} (\varpi_A(\nu) \diamond \varpi_B(z)), & \text{if } \xi = \nu z, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.1. *Let A and B be PFNRI and PFNLI of NR, respectively. It follows that $A \circ B \subseteq A \cap B$.*

Proof. Let $A \cap B$ be a PFNI of NR.

Consider A is a PFNRI and B is a PFNLI.

Let $\varrho_{A\odot B}(\xi) = \diamond_{\xi=\nu z} (\varrho_A(\nu) * \varrho_B(z))$ and let $\varpi_{A\otimes B}(\xi) = \diamond_{\xi=\nu z} (\varpi_A(\nu) \diamond \varpi_B(z))$.

Since, A is a PFNRI and B is PFNLI, then

$$\varrho_A(\nu) \leq \varrho_A(\nu z) = \varrho_A(\xi) \quad \text{and} \quad \varrho_B(z) \leq \varrho_B(\nu z) = \varrho_B(\xi)$$

and

$$\varpi_A(\xi) = \varpi_A(\nu z) \geq \varpi_A(\nu) \quad \text{and} \quad \varpi_B(z) = \varpi_B(\nu z) \geq \varpi_B(z).$$

Thus,

$$\begin{aligned} \varrho_{A\odot B}(\xi) &= \diamond_{\xi=\nu z} (\varrho_A(\nu) * \varrho_B(z)) \\ &= \min((\varrho_A(\nu), \varrho_B(z))) \\ &\leq \min(\varrho_A(\xi), \varrho_B(\xi)) \\ &\leq \varrho_{A\cap B}(\xi) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\varpi_{A\otimes B}(\xi) &= \underset{\xi=\nu z}{*}(\varpi_A(\nu)\diamond\varpi_B(z))\\ &= \max((\varpi_A(\nu),\varpi_B(z)))\\ &\geq \max(\varpi_A(\xi),\varpi_B(\xi))\\ &\geq \varpi_{A\cap B}(\xi).\end{aligned} \tag{3.2}$$

By (3.1) and (3.2) the proof is concluded. □

Theorem 3.2. *Let $A=(\varrho_A,\varpi_A)$ be a PFNI of NR, then we have for all $\xi\in NR$:*

(i) $\varrho_A(0)\geq\varrho_A(\xi)$ *and* $\varpi_A(0)\leq\varpi_A(\xi)$,

(ii) $\varrho_A(-\xi)=\varrho_A(\xi)$ *and* $\varpi_A(-\xi)=\varpi_A(\xi)$,

(iii) *if* $\varrho_A(\xi-\nu)=\varrho(0)$ *then* $\varrho_A(\xi)=\varrho_A(\nu)$,

(iv) *if* $\varpi_A(\xi-\nu)=\varpi_A(0)$, *then* $\varpi_A(\xi)=\varpi_A(\nu)$.

Proof. (i) As A is a PFNI, then

$$\varrho_A(0)=\varrho_A(\xi-\nu)\geq\varrho_A(\xi)*\varrho_A(\xi)=\varrho_A(\xi)$$

and

$$\varpi_A(0)=\varpi_A(\xi-\xi)\leq\varpi_A(\xi)\diamond\varpi_A(\xi)=\varpi_A(\xi).$$

(ii) $\varrho_A(-\xi)=\varrho_A(0-\xi)\geq\varrho_A(0)*\varrho_A(\xi)=\varrho_A(\xi)$

and

$$\varrho_A(\xi)=\varrho_A(0-(-\xi))\geq\varrho_A(0)*\varrho_A(-\xi)=\varrho_A(-\xi).$$

Therefore,

$$\varrho_A(-\xi)=\varrho_A(\xi).$$

Also,

$$\varpi_A(-\xi)=\varpi_A(0-\xi)\leq\varpi_A(0)\diamond\varpi_A(\xi)$$

and

$$\varpi_A(\xi)=\varpi_A(0-(-\xi))\leq\varpi_A(0)\diamond\varpi_A(-\xi)=\varpi_A(-\xi).$$

Therefore,

$$\varpi_A(-\xi)=\varpi_A(\xi).$$

(iii) Since $\varrho_A(\xi-\nu)=\varrho_A(0)$, then

$$\varrho_A(\nu)=\varrho_A(\xi-(\xi-\nu))\geq\varrho_A(\xi)*\varrho_A(\xi-\nu)=\varrho_A(\xi)*\varrho_A(0)\geq\varrho_A(\xi).$$

Similarly,

$$\varrho_A(\xi)=\varrho_A((\xi-\nu)-(-\nu))\geq\varrho_A(\xi-\nu)*\varrho_A(-\nu)=\varrho_A(0)*\varrho_A(\nu)\geq\varrho_A(\nu).$$

Consequently, $\varrho_A(\xi)=\varrho_A(\nu)$.

(iv) It can be prove using (iii). □

Theorem 3.3. *Let A be a PFNI of NR, then $\Delta A = (\varrho_A, \varrho^c_A)$, is a PFNI of NR.*

Proof. Let $\xi, \nu \in NR$,

$$\begin{aligned}\varrho^c_A(\xi-\nu) &= 1-\varrho_A(\xi-\nu)\\ &\le 1-\min\{\varrho_A(\xi),\varrho_A(\nu)\}\\ &= \max\{1-\varrho_A(\xi),1-\varrho_A(\nu)\}\\ &= \min\{\varrho^c_A(\xi),\varrho^c_A(\nu)\}.\end{aligned}$$

Then

$$\varrho^c_A(\xi-\nu) \le \varrho^c_A(\xi)\diamond\varrho^c_A,$$

$$\begin{aligned}\varrho^c_A(\xi\nu) &= 1-\varrho_A(\xi\nu)\\ &\le 1-\max\{\varrho_A(\xi),1-\varrho_A(\nu)\}\\ &= \min\{\varrho^c_A(\xi),\varrho^c_A(\nu)\}.\end{aligned}$$

Then

$$\varrho^c_A(\xi\nu) \le \varrho^c_A(\xi) * \varrho^c_A(\nu).$$

Accordingly, $\Delta A = (\varrho_A, \varrho^c_A)$ is a PFNI of NR. □

Theorem 3.4. *If A is a PFNI of NR, then $\diamond A = (\varpi^c_A, \varpi_A)$ is a PFNI of NR.*

Proof. Let $\xi, \nu \in NR$,

$$\begin{aligned}\varpi^c_A(\xi-\nu) &= 1-\varpi_A(\xi-\nu)\\ &\ge 1-\max\{\varpi_A(\xi),\varpi_A(\nu)\}\\ &= \min\{1-\varpi_A(\xi),1-\varpi_A(\nu)\}\\ &= \min\{\varpi^c_A(\xi),\varpi^c_A(\nu)\}.\end{aligned}$$

Then

$$\varpi^c_A(\xi-\nu) \ge \varpi^c_A(\xi) * \varpi^c_A(\nu),$$

$$\begin{aligned}\varpi^c_A(\xi\nu) &= 1-\varpi_A(\xi\nu)\\ &\ge 1-\min\{\varpi_A(\xi),\varpi_A(\nu)\}\\ &= \max\{1-\varrho_A(\xi),1-\varpi_A(\nu)\}\\ &= \max\{\varpi^c_A(\xi),\varpi^c_A(\nu)\}.\end{aligned}$$

Then $\varpi^c_A(\xi\nu) \ge \varpi^c_A(\xi)\diamond\varpi^c_A(\nu)$.
Therefore, $\diamond A = (\varpi^c_A, \varpi_A)$ is a PFNI of NR. □

Theorem 3.5. *PFS $A = (\varrho_A, \varpi_A)$ is a PFNI of NR if fuzzy subsets ϱ_A and ϖ^c_A are PFNIs of NR.*

Proof. Let $\xi, \nu \in NR$,

$$\begin{aligned}1-\varpi_A(\xi-nu) &= \varpi^c_A(\xi-\nu)\\ &\ge \min\{\varpi^c_A(\xi),\varpi^c_A(\nu)\}\\ &= \min\{(1-\varpi_A(\xi)),(1-\varpi_A(\nu))\}\\ &= 1-\max\{\varpi_A(\xi),\varpi_A(\nu)\}.\end{aligned}$$

Then

$$\begin{aligned}\varpi_A(\xi-\nu) &\le \varpi_A(\xi)\diamond\varpi_A(\nu),\\ 1-\varpi_A(\xi\nu) &= \varpi_A^c(\xi\nu)\\ &\ge \max\{\varpi_A^c(\xi),\varpi_A^c(\nu)\}\\ &= \max\{(1-\varpi_A(\xi)),(1-\varpi_A(\nu))\}\\ &= 1-\min\{\varpi_A(\xi),\varpi_A(\nu)\}.\end{aligned}$$

Then, $\varpi_A^c(\xi\nu) \le \varpi_A(\xi) * \varpi_A(\nu)$.
Consequently, $A = (\varrho_A, \varpi_A)$ is a PFNI of NR. □

Definition 3.2. Let $A(\neq \phi) \subseteq NR$, then Pythagorean characteristic function of A is $\lambda_A = (\varrho_{\lambda_A}, \varpi_{\lambda_A})$, where

$$\varrho_{\lambda_A}(\xi) = \begin{cases}1, & \text{if } \xi\in A,\\ 0, & \text{if } \xi\notin A\end{cases} \quad\text{and}\quad \varpi_{\lambda_A}(\xi) = \begin{cases}0, & \text{if } \xi\in A,\\ 1, & \text{if } \xi\notin A.\end{cases}$$

Theorem 3.6. *For subset A $(\neq \phi)$ of NR, A is a subring of NR if and only if $\lambda_A = (\varrho_{\lambda_A}, \varpi_{\lambda_A})$ is a PFNSR of NR.*

Proof. Let A be a subring of NR, $\xi,\nu \in NR$.
If $\xi,\nu\in A$, then by Pythagorean characteristic function properties $\varrho_{\lambda_A}(\xi) = 1 = \varrho_{\lambda_A}(\nu)$, $\varpi_{\lambda_A}(\xi) = 0 = \varpi_{\lambda_A}(\nu)$.
Since, A is a subring, so

$$\xi-\nu,\ \xi\nu\in A.$$

Thus,

$$\varrho_{\lambda_A}(\xi-\nu)1 = 1*1 = \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu) \ \text{ and }\ \varrho_{\lambda_A}(\xi\nu) = 1 = 1*1 = \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu).$$

Also,

$$\varpi_{\lambda_A}(\xi-\nu)0 = 0\diamond 0 = \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu) \ \text{ and }\ \varpi_{\lambda_A}(\xi\nu) = 0 = 0\diamond 0 = \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu),$$

this implies,

$$\begin{aligned}&\varrho_{\lambda_A}(\xi-\nu) \ge \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu) \ \text{ and }\ \varrho_{\lambda_A}(\xi\nu) \ge \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu),\\ &\varpi_{\lambda_A}(\xi-\nu) \le \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu) \text{ and } \varpi_{\lambda_A}(\xi\nu) \le \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu).\end{aligned}$$

Similar manner for $\xi,\nu\notin A$.
Hence, $\lambda_A(\varrho_{\lambda_A}, \varpi_{\lambda_A})$ is a PFNSR of NR.
Conversely, for Pythagorean characteristic function $\lambda_A = (\varrho_{\lambda_A}, \varpi_{\lambda_A})$ is a PFNSR of NR. Let $\xi,\nu\in A$, then $\varrho_{\lambda_A}(\xi) = 1 = \varrho_{\lambda_A}(\nu)$ and $\varpi_{\lambda_A}(\xi) = 0 = \varpi_{\lambda_A}(\nu)$. So,

$$\begin{aligned}&\varrho_{\lambda_A}(\xi-\nu) \ge \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu) \ge 1*1 \ge 1, && \text{also } \varrho_{\lambda_A}(\xi-\nu)\le 1,\\ &\varrho_{\lambda_A}(\xi\nu) \ge \varrho_{\lambda_A}(\xi)*\varrho_{\lambda_A}(\nu) \ge 1*1 \ge 1, && \text{also } \varrho_{\lambda_A}(\xi\nu)\le 1,\\ &\varpi_{\lambda_A}(\xi-\nu) \le \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu) \le 0\diamond 0 \le 0, && \text{also } \varpi_{\lambda_A}(\xi-\nu)\ge 0,\\ &\varpi_{\lambda_A}(\xi\nu) \le \varpi_{\lambda_A}(\xi)\diamond\varpi_{\lambda_A}(\nu) \le 0\diamond 0 \le 0, && \text{also } \varpi_{\lambda_A}(\xi\nu)\ge 0,\end{aligned}$$

then

$$\varrho_{\lambda_A}(\xi-\nu) = 1,\ \varrho_{\lambda_A}(\xi\nu) = 1 \ \text{ and }\ \varpi_{\lambda_A}(\xi-\nu) = 0,\ \varpi_{\lambda_A}(\xi\nu) = 0,$$

therefore, $\xi - \nu$, $\xi\nu \in A$.
Hence, A is a subring of NR. □

Theorem 3.7. *Let I $(\neq \phi)$ subset of NR, then I is an ideal of NR if and only if $\lambda_I = (\varrho_{\lambda_I}, \varpi_{\lambda_I})$ is a PFNI of NR.*

Proof. Let I be an ideal of NR, $\xi, \nu \in NR$.

Case I. If $\xi, \nu \in I$, then $\xi\nu \in I$,

$$\varrho_{\lambda_I}(\xi) = 1, \quad \varrho_{\lambda_I}(\nu) = 1,$$
$$\varpi_{\lambda_I}(\xi) = 0, \quad \varpi_{\lambda_I}(\nu) = 0.$$

Thus,

$$\varrho_{\lambda_I}(\xi\nu) = 1 \text{ and } \varpi_{\lambda_I}(\xi\nu) = 0.$$

Again,

$$\varrho_{\lambda_I}(\xi\nu) = 1 = \varrho_{\lambda_I}(\xi) \diamond \varrho_{\lambda_I}(\nu) \text{ and } \varpi_{\lambda_I}(\xi\nu) = 0 = \varpi_{\lambda_I}(\xi) * \varpi_{\lambda_I}(\nu).$$

Case II. If $\xi \notin I$ or $\nu \notin I$, thus $\xi, \nu \notin I$, then

$$\varrho_{\lambda_I}(\xi) = 0 \text{ or } \varrho_{\lambda_I}(\nu) = 0$$

and

$$\varpi_{\lambda_I}(\xi) = 1 \text{ or } \varpi_{\lambda_I}(\nu) = 1.$$

Thus,

$$\varrho_{\lambda_I}(\xi\nu) = 1 \geq \varrho_{\lambda_I}(\xi) \diamond \varrho_{\lambda_I}(\nu) \text{ and } \varpi_{\lambda_I}(\xi\nu) = 0 \leq \varpi_{\lambda_I}(\xi) * \varpi_{\lambda_I}(\nu).$$

Hence, $\lambda_I = (\varrho_{\lambda_I}, \varpi_{\lambda_I})$ is a PFNI of NR.
On the other hand, we suppose $\lambda_I = (\varrho_{\lambda_I}, \varpi_{\lambda_I})$ is a PFNI of NR. □

Theorem 3.8. *Let A and B be two PFNLI (or, PFNRI) of NR, then $A_* \cap B_* \subseteq (A \cap B)_*$.*

Proof. Let $\xi \in A_* \cap B_*$, then

$$\varrho_A(\xi) = \varrho_A(0), \varrho_B(\xi) = \varrho_B(0) \text{ and } \varpi_A(\xi) = \varpi_A(0), \varpi_B(\xi) = \varpi_B(0),$$

$$\begin{aligned}\varrho_{A\cap B}(\xi) &= \min\{\varrho_A(\xi), \varrho_B(\xi)\} \\ &= \min\{\varrho_A(0), \varrho_B(0)\} \\ &= \varrho_{A\cap B}(0)\end{aligned}$$

and

$$\begin{aligned}\varpi_{A\cap B}(\xi) &= \min\{\varpi_A(\xi), \varpi_B(\xi)\} \\ &= \min\{\varpi_A(0), \varpi_B(0)\} \\ &= \varpi_{A\cap B}(0).\end{aligned}$$

Thus,

$$\xi \in (A \cap B)_*.$$

Hence, $A_* \cap B_* \subseteq (A \cap B)_*$. □

Theorem 3.9. *If A is a PFNI of NR, then $f(A)$ is also a PFNI of NR', where $f: NR \to NR'$ be an epimorphism.*

Proof. Suppose $A = \{(\xi, \varrho_A(\xi), \varpi_A(\xi)) : \xi \in NR\}$,

$$f(A) = \{(\nu, \underset{f(\xi)=\nu}{\diamond} \varrho_A(\xi) \underset{f(\xi)=\nu}{*} \varpi_A(\xi) : \xi \in NR, \nu \in NR')\}.$$

Let $\nu_1, \nu_2 \in NR'$, then there exists $\xi_1, \xi_2 \in NR$ such that $f(\xi_1) = \nu_1$ and $f(\xi_2) = \nu_2$,

(i)
$$\begin{aligned} \varrho_{f(A)}(\nu_1 - \nu_2) &= \underset{f(\xi_1-\xi_2)=\nu_1-\nu_2}{\diamond} \varrho_A(\xi_1 - \xi_2) \\ &\geq \underset{f(\xi_1)=\nu_1, f(\xi_2)=\nu_2}{\diamond} (\varrho_A(\xi_1) * \varrho_A(\xi_2)) \\ &\geq \left(\underset{f(\xi_1)=\nu_1}{\diamond} \varrho_A(\xi_1)\right) * \left(\underset{f(\xi_2)=\nu_2}{\diamond} \varrho_A(\xi_2)\right) \\ &\geq \varrho_{f(A)}(\nu_1) * \varrho_{f_A}(\nu_2). \end{aligned}$$

(ii)
$$\begin{aligned} \varrho_{f(A)}(\nu_1\nu_2) &= \underset{f(\xi_1\xi_2)=\nu_1\nu_2}{\diamond} \varrho_A(\xi_1\xi_2) \\ &\geq \underset{f(\xi_2)=\nu_2}{\diamond} \varrho_A(\xi_2) \\ &\geq \varrho_{f(A)}(\nu_2). \end{aligned}$$

(iii)
$$\begin{aligned} \varpi_{f_A}(\nu_1 - \nu_2) &= \underset{f(\xi_1-\xi_2)=\nu_1-\nu_2}{*} \varpi_A(\xi_1 - \xi_2) \\ &\leq \underset{f(\xi_1)=\nu_1, f(\xi_2)=\nu_2}{*} (\varpi_A(\xi_1) \diamond \varpi_A(\xi_2)) \\ &\leq \left(\underset{f(\xi_1)=\nu_1}{*} \varpi_A(\xi_1)\right) \diamond \left(\underset{f(\xi_2)=\nu_2}{*} \varpi_A(\xi_2)\right). \end{aligned}$$

(iv)
$$\begin{aligned} \varpi_{f(A)}(\nu_1\nu_2) &= \underset{f(\xi_1\xi_2)=\nu_1\nu_2}{*} \varpi_A(\xi_1\xi_2) \\ &\leq \underset{f(\xi_2)=\nu_2}{*} \varpi_A(\xi_2) \\ &\leq \varpi_{f(A)}(\nu_2). \end{aligned}$$

Hence, $f(A)$ is a PFNLI.
Similarly, $f(A)$ is a PFNRI.
Then, $f(A)$ is a PFNI of NR. □

Theorem 3.10. *If B is a PFNI of NR', then $f^{-1}(B)$ is also PFNI of NR, where $f: NR \to NR'$ be an epimorphism.*

Proof. Suppose $B = \{(\nu, \varrho_B(\nu), \varpi_B(\nu)) : \nu \in NR'\}$,

$$f^{-1}(B) = \{(\xi, \varrho_{f^{-1}(B)}(\xi), \varpi_{f^{-1}B}(\xi) : \xi \in NR\},$$

where $\varrho_{f^{-1}B}(r) = \varrho_B(f(\xi))$ and $\varpi_{f^{-1}B}(\xi) = \varpi_B(f(\xi))$ for every $\xi \in NR$.
Let $\xi_1, \xi_2 \in NR$, then

(i)
$$\begin{aligned} \varrho_{f^{-1}(B)}(\xi_1 - \xi_2) &= \varrho_B(f(\xi_1 - \xi_2)) \\ &= \varrho_B(f(\xi_1) - f(\xi_2)) \\ &\geq \varrho_B(f(\xi_1) * \varrho_B f(\xi_2)) \\ &\geq \varrho_{f^{-1}(B)}(\xi_1) * \varrho_{f^{-1}(B)}(\xi_2). \end{aligned}$$

(ii)
$$\begin{aligned}\varrho_{f^{-1}(B)}(\xi_1\xi_2) &= \varrho_B(f(\xi_1\xi_2))\\ &= \varrho_B(f(\xi_1)f(\xi_2))\\ &\geq \varrho_B(f(\xi_2))\\ &\geq \varrho_{f^{-1}(B)}(\xi_2).\end{aligned}$$

(iii)
$$\begin{aligned}\varpi_{f^{-1}(B)}(\xi_1-\xi_2) &= \varpi_B(f(\xi_1-\xi_2))\\ &= \varpi_B(f(\xi_1)-f(\xi_2))\\ &\leq \varpi_B(f(\xi_1)\diamond\varpi_B f(\xi_2))\\ &\leq \varpi_{f^{-1}(B)}(\xi_1)\diamond\varpi_{f^{-1}(B)}(\xi_2).\end{aligned}$$

(iv)
$$\begin{aligned}\varpi_{f^{-1}(B)}(\xi_1\xi_2) &= \varpi_B(f(\xi_1\xi_2))\\ &= \varpi_B(f(\xi_1)f(\xi_2))\\ &\leq \varpi_B(f(\xi_2))\\ &\leq \varpi_{f^{-1}(B)}(\xi_2).\end{aligned}$$

Therefore, $f^{-1}(B)$ is a PFNLI of NR

Similarly, $f^{-1}(B)$ is a PFNRI.

So, $f^{-1}(B)$ is a PFNI of NR. □

4. Conclusions

This article defines the intrinsic product of two PFNIs and demonstrates that this product constitutes a subset of their intersection. The concept of characteristic functions is introduced, along with several results pertaining to the characteristic functions of Pythagorean fuzzy normed ideals. Additionally, significant results are provided concerning the epimorphism of Pythagorean fuzzy normed ideals. This methodology is anticipated to contribute significantly to the domain of Fermatean fuzzy set theories by broadening and generalizing key characteristics and algebraic frameworks.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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