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Research Article

Closeness Centrality Weight of Graphs Under Some Graph Operations

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Abstract. Many complex systems exhibit a natural hierarchy in which elements can be ordered according to a notion of influence with closeness centrality being one of the three well-known centrality measures used in social network analysis, determining the importance of vertices in a network, a core task in each network application. It describes the relative importance of a single vertex within a network or graph by finding the average proximity of that vertex to all others in that graph. In this paper, we derive formulae for the closeness centrality weight of the graph resulting from the operations between some graph families namely, join graph, Cartesian product of graph, shadow graph, corona graph, lexicographic product, disjunction graph and total graph.

Keywords. Closeness centrality, Closeness centrality weight

Mathematics Subject Classification (2020). 05C12, 05C99

1. Introduction

Nowadays, social networks have taken a very important place in our personal and professional lives. The growing significance of social networks has prompted researchers to delve deeper into their dynamics, exploring how communication and sharing function within these networks (Landherr *et al*. [10], and Parisutham [13]). In 1977, Freeman [6] introduced three centrality measures to assess the importance of nodes based on both their local and global connectivity. These definitions were primarily designed for undirected and unweighted networks. Centrality studies help determine an individual's role in a social network, including their impact on the flow and dissemination of information (Rosselló and Valiente [14]). Centrality measures are used to rank entities in a social network based on their position, allowing for a better understanding of each individual's role within the network (Fouad and Rego [5]). Many centrality indices rely on the shortest paths between pairs of entities, counting the number of paths that pass through a specific vertex, including Degree, Closeness, Betweenness, and Eigenvector Centrality (Nirmala *et al*. [12]). Closeness centrality is used to measure the importance of a vertex within a network. It quantifies how close a vertex is to all other vertices in the graph, providing insights into the vertex's influence and connectivity. A vertex with high closeness centrality can quickly interact with other vertices in the network. Vertices with high closeness centrality are strategically positioned within the network, making them key players in processes like communication, dissemination of information, and network resilience. Totally, the closeness centrality is a crucial measure in graph theory that provides valuable insights into the connectivity and influence of vertices within a network. Its applications span across various domains, making it a versatile and widely used tool in network analysis. Namely, *Social Networks*: Identifying influential individuals who can efficiently spread information or social influence. *Transportation Networks*: Finding crucial hubs that facilitate efficient transportation and connectivity. *Biological Networks*: Locating important proteins that are central to cellular processes. *Communication Networks*: Enhancing network design by optimizing the placement of servers or routers for efficient communication.

2. Preliminaries

Let G be a simple, connected, finite and undirected graph with vertex set $V(G)$ and edge set $E(G)$. We denote the number of vertices and edges of G by $n = |V|$ and $m = |E|$, respectively. If $u,v \in V(G)$ are adjacent then we denote it by $u \sim v$ (Balakrishnan and Ranganathan [2]), the degree of v is denoted by $d(v)$ or $d_G(v)$ and is defined as the number of edges incident with v, the distance between u and v is denoted by $d(u,v)$ or $d_G(u,v)$ and is defined as the length of the shortest path connecting u and v in G. If $e(= uv) \in E(G)$, then $d(e)$ or $d_G(e)$ denotes the degree of e and $d(e) = d(u)+d(v)-2$ (Harary [7]). The distance $d(e, f)$ or $d_G(e, f)$ between two edges e and f is defined as the distance between the corresponding vertices in line graph of G (Varma *et al*. [15]). The vertices of the line graph $L(G)$ are the edges of G with two vertices of the line graph adjacent whenever the corresponding edges of G are adjacent (Harary [7]). The vertices and edges of a graph are called elements. The elements of a graph are neighbors if they are either incident or adjacent. The total graph $T(G)$ has a vertex set $V(G) \cup E(G)$, and two vertices of $T(G)$ are adjacent whenever they are neighbors in G (Harary [7]). The neighborhood of $u \in V(G)$ is the set $N(u)$ consisting of all vertices v which are adjacent with u. The closed neighborhood is $N[u] = N(u) \cup \{u\}$ (Harary [7]). The open edge neighborhood set $N(e)$ of $e \in E(G)$ is the set of all edges adjacent to e. The edge neighborhood graph $N_e(G)$ of G is the graph with the vertex set $E \cup S$ where S is the set of all open edge neighborhood sets of edges of G, with two vertices u,v in $N_e(G)$ adjacent if $u \in E(G)$ and v is an open edge neighborhood set containing u (Kulli [9]). A fan graph F_n, $n \geq 2$ is obtained by joining all the vertices of P_n to a further vertex, called center (Meena and Vaithilingam [11]). Let G_1 and G_2 be disjoint graphs. The join of G_1 and G_2 is the graph $G = G_1 + G_2$ having the vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$ (Eballe *et al*. [4]). The cartesian product of G_1

and G_2 is $G = G_1 \times G_2$, with the vertex set $V(G) = V(G_1) \times V(G_2)$, where two distinct vertices (u,v) and (x,y) of $G_1 \times G_2$ are adjacent if either $u = x$ and $vy \in E(G_2)$ or $v = y$ and $ux \in E(G_1)$ (Eballe *et al*. [4]). The shadow graph $S(G)$ of G is the graph obtained by taking two copies of G, say G_1 and G_2, and then joining each $v \in V(G_1)$ to the neighbors of $v' \in V(G_2)$, where v' is the vertex in $V(G_2)$ corresponding to v (Alfeche *et al*. [1]). The corona of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is defined to be the the graph obtained by taking one copy of G_1 which has presumably n vertices and n copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. For convenience, we denote the ith copy of G_2 in the corona $G_1 \circ G_2$ by G_2^i for $i \in V(G_1)$ (Eballe *et al*. [4]). The lexicographic product G of two graphs G_1 and G_2, denoted by $G_1[G_2]$ has vertex set $V(G) = V(G_1) \times V(G_2)$, where any two distinct vertices (u,v) and (x,y) of $G_1[G_2]$ are adjacent if either u is adjacent with x in G_1 or $u = x$ and v is adjacent with y in G_2 (Eballe *et al*. [4]). The disjunction of graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is the graph with $V(G_1 \vee G_2) = V(G_1) \times V(G_2)$ and $(a,x)(b,y) \in E(G_1 \vee G_2)$ if and only if $ab \in E(G_1)$ or $xy \in E(G_2)$ (Alfeche *et al*. [1]). The closeness centrality measure is one of the three classic centrality indices at the node or vertex level. If order of G is n and if $u \in V(G)$, then the closeness centrality of u is given by $\mathcal{C}_G(u) = \frac{n-1}{\tau_G(u)}$, where $\tau_G(u) = \sum_{x \in V(G)} d_G(u,x)$ (Eballe and Cabahung, Jr. [3]).

Theorem 2.1 ([3]). (i) *If $G = P_n$ of order $n \geq 2$, then*

$$\mathcal{C}_G(u_i) = \begin{cases} \frac{n-1}{\binom{n}{2}}, & \text{if } u_i \text{ is a pendant vertex,} \\ \frac{n-1}{\binom{i}{2}+\binom{n+1-i}{2}}, & \text{if } u_i \text{ is a nonpendant vertex.} \end{cases}$$

(ii) *If $G = C_n$ of order n, then*

$$\mathcal{C}_G(u) = \begin{cases} \frac{4}{n+1}, & \text{if } n \text{ is odd,} \\ \frac{4(n-1)}{n^2}, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2.2 ([4]). (i) *Let G and H be graphs of orders m and n respectively, with G is connected. Then, the closeness centralities of the vertices u and $(u,h) \in V(G \circ H)$ are given by the following expressions*

$$\mathcal{C}_{G\circ H}(u) = \frac{mn+m-1}{(n+1)\tau_G(u)+mn} \quad \text{and} \quad \mathcal{C}_{G\circ H}(u,h) = \frac{mn+m-1}{(n+1)\tau_G(u)-d_H(h)+2mn+m-2}.$$

(ii) *Let G and H be nontrivial connected graphs of orders m and n, respectively. Then, the closeness centrality of any vertex $(u,v) \in V(G[H])$ is given by*

$$\mathcal{C}_{G[H]}(u,v) = \frac{mn-1}{n\tau_G(u)-d_H(v)+2n-2}.$$

Theorem 2.3 ([8]). *Let G and H be the connected nontrivial graphs of order m and n respectively, and let $(x,p) \in V(G \vee H)$. Then*

$$\mathcal{C}_{G\vee H}(x,p) = \frac{mn-1}{2mn+d_G(x)d_H(p)-nd_G(x)-md_H(p)-2}.$$

Motivated by the above definition, in this paper we study the closeness centrality weight of graphs under some operations.

3. Main Results

Definition 3.1. The closeness centrality weight $\mathcal{C}W(G)$ of G, is defined as $\sum_{u\in V(G)} f(u)$, where $f:V(G)\to(0,1]$ is a function and $f(u)=\mathcal{C}_G(u)$.

Theorem 3.2. *Let G be a graph of order n and $v\in V(G)$. Then,*

$$\tau_G(v)\le\frac{n(n-1)}{2}.$$

Proof. For any $v,u\in V(G)$, $d(v,u)\le n-1$. If there exists a vertex $u\in V(G)$ such that $d(v,u)=n-1$, then there exist $n-2$ number of vertices in $V(G)$ say $v_1,v_2,\dots,v_{n-2}$ such that $d(v,v_i)=i$, with $1\le i\le n-2$. So,

$$\begin{aligned}\tau_G(v)&=\sum_{u\in V(G)} d(v,u)\\&=(n-2)+(n-1)+\ldots+2+1\\&=\frac{n(n-1)}{2}.\end{aligned}$$

□

Theorem 3.3. *Let G_1 and G_2 be connected graphs of order p and n respectively. Then, the closeness centrality weight of G_1+G_2 is given by*

$$\mathcal{C}W(G_1+G_2)=\frac{(p+n-1)[p\tau_{G_1+G_2}(x_i)+n\tau_{G_1+G_2}(y_j)]}{\tau_{G_1+G_2}(y_j)\tau_{G_1+G_2}(x_i)},$$

where $x_i\in V(G_1+G_2)$, $1\le i\le p$ and $y_j\in V(G_1+G_2)$, $1\le j\le n$.

Proof. Let G_1 and G_2 be the graphs with $V(G_1)=\{x_1,x_2,\dots,x_p\}$ and $V(G_2)=\{y_1,y_2,\dots,y_n\}$. If $x_i\in V(G_1+G_2)$, $1\le i\le p$, then $\tau_{G_1+G_2}(x_i)$ can be computed as

$$\begin{aligned}\tau_{G_1+G_2}(x_i)&=\sum_{v\in V(G_1+G_2)} d(x_i,v)\\&=\sum_{v\in N_{(G_1+G_2)}[x_i]} d(x_i,v)+\sum_{v\in V(G_1+G_2)\setminus N_{(G_1+G_2)}[x_i]} d(x_i,v)\\&=(n+d_{G_1}(x_i))+2(p-1-d_{G_1}(x_i))\\&=2p+n-2-d_{G_1}(x_i).\end{aligned}$$

Hence,

$$\mathcal{C}_{G_1+G_2}(x_i)=\frac{p+n-1}{2p+n-2-d_{G_1}(x_i)}.$$

Similarly, for $y_j\in V(G_1+G_2)$, $1\le j\le n$, then $\tau_{G_1+G_2}(y_j)$ can be computed as

$$\begin{aligned}\tau_{G_1+G_2}(y_j)&=\sum_{u\in V(G_1+G_2)} d(y_j,u)\\&=\sum_{u\in N_{(G_1+G_2)}[y_j]} d(y_j,u)+\sum_{u\in V(G_1+G_2)\setminus N_{(G_1+G_2)}[y_j]} d(y_j,u)\\&=(p+d_{G_2}(y_j))+2(n-1-d_{G_2}(y_j))\\&=2n+p-2-d_{G_2}(y_j).\end{aligned}$$

Hence,

$$\mathcal{C}_{G_1+G_2}(y_j) = \frac{p+n-1}{2n+p-2-d_{G_2}(y_j)}.$$

Therefore,

$$\mathcal{C}W(G_1+G_2) = \sum_{i=1}^{p} \frac{p+n-1}{2p+n-2-d_{G_1}(x_i)} + \sum_{j=1}^{n} \frac{p+n-1}{2n+p-2-d_{G_2}(y_j)}. \qquad \square$$

Theorem 3.4. *Let G_1 and G_2 be connected nontrivial graphs of order p and n respectively. Then, the closeness centrality weight of $G_1 \times G_2$ is given by*

$$\mathcal{C}W(G_1 \times G_2) = \sum_{(x,y)\in V(G_1\times G_2)} \left[\frac{pn-1}{n\tau_{G_1}(x)+p\tau_{G_2}(y)}\right].$$

Proof. Let $(x,y) \in V(G_1 \times G_2)$, then $\tau_{G_1\times G_2}(x,y)$ can be computed as

$$\begin{aligned}
\tau_{G_1\times G_2}(x,y) &= \sum_{(a,b)\in V(G_1\times G_2)} d((x,y),(a,b)) \\
&= \sum_{(a,b)\in V(G_1\times G_2)} [d_{G_1}(x,a)+d_{G_2}(y,b)] \\
&= n\sum_{a\in V(G_1)} d_{G_1}(x,a) + p\sum_{b\in V(G_2)} d_{G_2}(y,b) \\
&= n\tau_{G_1}(x)+p\tau_{G_2}(y).
\end{aligned}$$

Therefore,

$$\mathcal{C}W(G_1 \times G_2) = \sum_{(x,y)\in V(G_1\times G_2)} \left[\frac{pn-1}{n\tau_{G_1}(x)+p\tau_{G_2}(y)}\right]. \qquad \square$$

Theorem 3.4 can be applied to the grid graph $P_n \times P_k$ and the toroidal graph $C_n \times C_k$, producing the following two corollaries.

Corollary 3.5. (i) *Let $P_n = (x_1,x_2,\dots,x_n)$ and $P_k = (y_1,y_2,\dots,y_k)$ be two nontrivial paths of order n and k, respectively. Then, the closeness centrality weight of the (grid) graph $P_n \times P_k$ is given by*

$$\mathcal{C}W(P_n \times P_k) = \sum_{(x_i,y_j)\in V(P_n\times P_k)} \frac{nk-1}{k\tau_{P_n}(x_i)+n\tau_{P_k}(y_j)}.$$

(ii) *Let $C_n = (x_1,x_2,\dots,x_n,x_1)$ and $C_k = (y_1,y_2,\dots,y_k,y_1)$ be two cycles of order n and k, respectively. Then, the closeness centrality weight of the (toroidal) graph $C_n \times C_k$ is given by*

$$\mathcal{C}W(C_n \times C_k) = \sum_{(x_i,y_j)\in V(C_n\times C_k)} \frac{nk-1}{k\tau_{C_n}(x_i)+n\tau_{C_k}(y_j)}.$$

Theorem 3.6. *The closeness centrality weight of the shadow graph, $n \geq 3$ is given by*

$$\mathcal{C}W(S(G)) = 2\left[\sum_{i=1}^{n} \frac{2n-1}{2\tau_{G_1}(v_i)+2}\right].$$

Proof. Let $v \in V(S(G))$, then $\tau_{S(G)}(v)$ can be computed as follows

$$\tau_{S(G)}(v) = \sum_{w\in V(S(G))} d_{S(G)}(v,w)$$

$$= \sum_{w\in V(G_1)} d_{S(G)}(v,w) + \sum_{w^1\in V(G_2)} d_{S(G)}(v,w^1)$$
$$= \sum_{w\in V(G_1)} d_{G_1}(v,w) + \sum_{w^1\in V(G_2)-\{v^1\}} d_{S(G)}(v,w^1) + d_{S(G)}(v,v^1)$$
$$= \tau_{G_1}(v) + \sum_{u\in V(G_1)-\{v\}} d_{G_1}(v,u) + 2$$
$$= 2\tau_{G_1}(v) + 2.$$

Therefore,

$$\mathcal{C}W(S(G)) = 2\left[\sum_{i=1}^{n} \frac{2n-1}{2\tau_{G_1}(v_i)+2}\right]. \qquad \square$$

Proposition 3.7. (i) *Let G_1 and G_2 be connected graphs of orders m and n, respectively. Then, the closeness centrality weight of $G_1 \circ G_2$ is given by*

$$\mathcal{C}W(G_1\circ G_2) = \sum_{(u,h)\in V(G_1\circ G_2)} \frac{mn+m-1}{(n+1)\tau_{G_1}(u)+2mn+m-2-d_{G_2}(h)} + \sum_{u\in V(G_1)} \frac{mn+m-1}{(n+1)\tau_{G_1}(u)+mn}.$$

(ii) *Let G_1 and G_2 be the nontrivial connected graphs of orders p and n, respectively. Then, the closeness centrality weight of $G_1[G_2]$ is given by*

$$\mathcal{C}W(G_1[G_2]) = \sum_{(u,v)\in V(G_1[G_2])} \frac{pn-1}{n\tau_{G_1}(u)-d_{G_2}(v)+2n-2}.$$

Proof. We attain the above results from Theorem 2.2 and by the definition of closeness centrality weight of the graphs. $\square$

(iii) *Let G_1 and G_2 be the nontrivial connected graphs of orders p and n respectively. Then closeness centrality weight of $G_1 \vee G_2$ is given by*

$$\mathcal{C}W(G_1\vee G_2) = \sum_{(s,t)\in V(G_1\vee G_2)} \frac{pn-1}{2pn-2+d_{G_1}(s)d_{G_2}(t)-n(d_{G_1}(s))-p(d_{G_2}(t))}.$$

Proof. We attain the above result from Theorem 2.3 and by the definition of closeness centrality weight of the graphs. $\square$

Observation 3.8. (i) *Since $L(K_{1,n}) = K_n$, $n \ge 2$, the closeness centrality weight of line graph of star graph $K_{1,n}$ is given by $\mathcal{C}W(L(K_{1,n})) = n$.*

(ii) *Since $L(C_n) \cong C_n$, the closeness centrality weight of cycle graph and closeness centrality weight of line graph of cycle graph are equal.*

(iii) *Since $L(C_3) \cong L(K_{1,3})$, the closeness centrality weight of line graph of cycle graph with 3 vertices and the closeness centrality weight of line graph of star graph $K_{1,3}$ are equal.*

Theorem 3.9. *Let $G = K_n$ be a complete graph. Then, the closeness centrality weight of the total graph of complete graph is given by*

$$\mathcal{C}W(T(G)) = \frac{(n+1)(n+2)}{2}.$$

Proof. The total graph of complete graph has $\frac{n(n+1)}{2}$ vertices, $\frac{n(n^2-1)}{2}$ edges and it is $2(n-1)$-regular graph.

Let $v \in V(G)$, then

$$\tau_{T(G)}(v) = (2n-2)1 + \left(\frac{n(n+1)}{2} - 2(n-1) - 1\right)2 = n^2 - n.$$

Let $u \in E(G)$, then

$$\tau_{T(G)}(u) = (2)1 + (2n-4)1 + \left(\frac{n(n+1)}{2} - 2 - (2n-4) - 1\right)2 = n^2 - n.$$

Then,

$$\begin{aligned} \mathcal{C}W(T(G)) &= \sum_{u \in V(T(G))} \frac{\frac{(n+1)(n+2)}{2}}{\tau_G(u)} \\ &= \frac{(n+1)(n+2)}{2} \frac{n}{n^2-n} + \frac{(n+1)(n+2)}{2} \frac{n(n-1)}{2(n^2-n)}. \end{aligned}$$

Therefore,

$$\mathcal{C}W(T(G)) = \frac{(n+1)(n+2)}{2}. \qquad \square$$

Theorem 3.10. *Let $G = K_{1,n}$ be a star graph. Then, the closeness centrality weight of the total graph of star graph is given by*

$$\mathcal{C}W(T(G)) = \frac{7n^3 + 3n^2 - 5n + 1}{6n^2 - 5n + 1}.$$

Proof. Let $u \in V(T(G))$. If u corresponds to the edge of G, then

$$\tau_{T(G)}(u) = (n-1)1 + (2)1 + (n-1)2 = 3n - 1.$$

If u corresponds to the central vertex of G then $\tau_{T(G)}(u) = 2n$. If u corresponds to the pendent vertex of G, then

$$\tau_{T(G)}(u) = (1)1 + (1)1 + (n-1)2 + (n-1)2 = 4n - 2.$$

Then,

$$\mathcal{C}_{T(G)}(u) = \begin{cases} \frac{2n}{3n-1}, & \textit{if } u \textit{ is an edges of } G, \\ \frac{2n}{4n-2}, & \textit{if } u \textit{ is pendant vertex of } G, \\ 1, & \textit{if } u \textit{ is a central vertex of } G. \end{cases}$$

Therefore,

$$\mathcal{C}W(T(G)) = \frac{7n^3 + 3n^2 - 5n + 1}{6n^2 - 5n + 1}. \qquad \square$$

Theorem 3.11. *Let $G = C_n$ be a cycle graph of order n. Then, the closeness centrality weight of the total graph of cycle graph is given by*

$$\mathcal{C}W(T(G)) = \begin{cases} \frac{8n^2-4n}{4\tau_G(v)+n+1}, & \textit{if } n \textit{ is odd}, \\ \frac{8n^2-4n}{4\tau_G(v)+n}, & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. We consider the following cases:

Case 1. Let n be odd and $u \in V(T(G))$ with corresponds to a vertex of G. Then, the number of vertices n is given distance $d_{T(G)}(u,v) \le \text{diam}(T(G))$ is given in Table 1.

Table 1. Number of vertices of $T(G)$ corresponds to $V(G)$ and $E(G)$

With $d_{T(G)}(u,v)$	Number of vertices of $T(G)$ corresponds to	
	vertex of G	edge of G
1	2	2
2	2	2
3	2	2
4	2	2
$\vdots$	$\vdots$	$\vdots$
$\frac{n-1}{2}$	2	2
$\frac{n+1}{2}$	0	1

We get

$$\tau_{T(G)}(u) = 2\tau_G(u) + \frac{n+1}{2}.$$

Similarly for $u \in V(T(G))$ corresponding to an edge of G.

Case 2. Let n be even and $u \in V(T(G))$ with corresponds to a vertex of G. Then, the number of vertices n is given distance $d_{T(G)}(u,v) \le \text{diam}(T(G))$ is given in Table 2.

Table 2. Number of vertices of $T(G)$ corresponds to $V(G)$ and $E(G)$

With $d_{T(G)}(u,v)$	Number of vertices of $T(G)$ corresponds to	
	vertex of G	edge of G
1	2	2
2	2	2
3	2	2
4	2	2
$\vdots$	$\vdots$	$\vdots$
$\frac{n}{2}$	1	2

We get

$$\tau_{T(G)}(u) = 2\tau_G(u) + \frac{n}{2}.$$

Similarly for $u \in V(T(G))$ corresponding to an edge of G.

Then,

$$\mathcal{C}_{T(G)}(u) = \begin{cases} \frac{2n-1}{2\tau_G(u)+\frac{n+1}{2}}, & \text{if } n \text{ is odd,} \\ \frac{2n-1}{2\tau_G(u)+\frac{n}{2}}, & \text{if } n \text{ is even.} \end{cases}$$

Therefore,

$$\mathcal{C}W(T(G)) = \begin{cases} \frac{8n^2-4n}{4\tau_G(v)+n+1}, & \text{if } n \text{ is odd,} \\ \frac{8n^2-4n}{4\tau_G(v)+n}, & \text{if } n \text{ is even.} \end{cases}$$ □

Theorem 3.12. *Let G be a graph of order n and size m. If $v_i \in V(G)$ and $u_i \in E(G)$, then closeness centrality weight of total graph is given by*

$$\mathcal{C}W(T(G)) \geq n\left(\frac{n+m-1}{\tau_{(G)}(u)+m(1+\operatorname{diam}(G))}\right) + \frac{n(n-1)}{2}\left(\frac{n+m-1}{(m+n)(1+\operatorname{diam}(G))}\right).$$

Proof. Let $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(G) = \{u_1, u_2, u_3, \dots, u_m\}$, then $V(T(G)) = V(G) \cup E(G)$.

Case 1. If $v_i \in V(G)$, we have

$$\begin{aligned} \tau_{T(G)}(v_i) &= \sum_{x \in V(T(G))} d_{T(G)}(v_i, x) \\ &= \sum_{v_j \in V(G)} d_{T(G)}(v_i, v_j) + \sum_{u_j \in E(G)} d_{T(G)}(v_i, u_j) \\ &= A + B. \end{aligned}$$

Now,

$$A = \sum_{v_j \in V(G)} d_{T(G)}(v_i, v_j) = \tau_G(v_i).$$

To calculate B, consider $u_j = a_j b_j \in E(G)$, then

$$B = \sum_{a_j b_j \in E(G)} [1 + \min\{d_G(v_i, a_j), d_G(v_i, b_j)\}].$$

Since $d(v_i, v_j) \leq \operatorname{diam}(G)$, for all $1 \leq i, j \leq n$. We have

$$\begin{aligned} \tau_{T(G)}(v_i) &= \tau_G(v_i) + \sum (1 + \operatorname{diam}(G)) \\ &= \tau_G(v_i) + m + m(\operatorname{diam}(G)) \\ &= \tau_G(v_i) + m(1 + \operatorname{diam}(G)). \end{aligned}$$

Case 2. If $u_i = x_i y_i \in E(G)$, we have

$$\begin{aligned} \tau_{T(G)}(u_i) &= \sum_{x \in V(T(G))} d_{T(G)}(u_i, x) \\ &= \sum_{v_j \in V(G)} d_{T(G)}(u_i, v_j) + \sum_{u_j \in E(G)} d_{T(G)}(u_i, u_j) \\ &= M + N. \end{aligned}$$

Now,

$$\begin{aligned} M &= \sum_{v_j \in V(G)} d_{T(G)}(u_i, v_j) \\ &= \sum_{v_j \in V(G)} [1 + \min\{d_G(x_i, v_j), d_G(y_i, v_j)\}] \\ &\leq \sum_{v_j \in V(G)} (1 + \operatorname{diam}(G)) \\ &= n + n(\operatorname{diam}(G)). \end{aligned}$$

To calculate N, consider $u_j = a_j b_j \in E(G)$, then

$$\begin{aligned} N &= \sum_{u_j \in E(G)} d_{T(G)}(u_i, u_j) \\ &= \sum_{a_j b_j \in E(G)} [1 + \min\{(d_G(x_i, a_j), d_G(x_i, b_j), d_G(y_i, a_j), d_G(y_i, b_j))\}] \\ &\le \sum_{a_j b_j \in E(G)} 1 + \operatorname{diam}(G) \\ &= m(1 + \operatorname{diam}(G)). \end{aligned}$$

Hence,

$$\begin{aligned} \tau_{T(G)}(u_i) &\le n + n(\operatorname{diam}(G)) + m + m(\operatorname{diam}(G)) \\ &= (m+n)(1 + \operatorname{diam}(G)). \end{aligned}$$

From the above cases, we get

$$\tau_{T(G)}(u) \le \begin{cases} (m+n)(1+\operatorname{diam}(G)), & \text{if } u \in E(G), \\ \tau_G(u) + m(1 + \operatorname{diam}(G)), & \text{if } u \in V(G). \end{cases}$$

Therefore,

$$\mathcal{C}W(T(G)) \ge n\left(\frac{n+m-1}{\tau_{(G)}(u) + m(1+\operatorname{diam}(G))}\right) + \frac{n(n-1)}{2}\left(\frac{n+m-1}{(m+n)(1+\operatorname{diam}(G))}\right). \qquad \square$$

4. Conclusion

The study re-examined the concept of closeness centrality weight of a graph and to determine the formulas for the closeness centrality weight of join graph, cartesian product of graph, shadow graph, corona graph, lexicographic product of graph, disjunction of graph, and for the total graph. The parameter can be studied further for other graphs under some other graph operations.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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